Lattice gravity

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4d QG regularized by CDT

Main goal (at least in 80ties) for QG

- Obtain the background geometry ($\langle g_{\mu\nu} \rangle$) we observe
- Study the fluctuations around the background geometry

What lattice gravity (CDT) offers:

- A non-perturbative QFT definition of QG
- A background independent formulation
- An emergent background geometry ($\langle g_{\mu\nu} \rangle$)
- The possibility to study the quantum fluctuations around this emergent background geometry.

Problems to confront for a lattice theory

(1) How to face the non-renormalizability of quantum gravity (this is a problem for any field theory of quantum gravity, not only lattice theories)

(2) Provide evidence of a continuum limit (where the continuum field theory has the desired properties)

(3) If rotation is performed to Euclidean signature, how does one deal with the unboundedness of the Euclidean Einstein-Hilbert action?

(4) If there exists no continuum field theory of gravity, can a lattice theory be of any use?

Effective QFT of gravity

We believe gravity exists as an effective QFT for $E^2 \ll 1/G$.

True for other non-renormalizable theories

Weak interactions	$\mathcal{L} = \bar{\psi}\partial\psi + G_F \bar{\psi}(\cdot)\psi\bar{\psi}(\cdot)\psi$
Nonlinear sigma model	$\mathcal{L} = (\partial\pi)^2 + \frac{1}{F_\pi^2}\frac{(\pi\partial\pi)^2}{1-\pi^2/F_\pi^2}$

Good for $E^2 \ll 1/G_F$ and $E^2 \ll F_\pi^2$.

Effective QFT of gravity

Lowest order quantum correction to the gravitational potential of a point particle:

$$\frac{G}{r} \to \frac{G(r)}{r}, \quad G(r) = G\left(1 - \omega \frac{G}{r^2} + \cdots\right), \quad \omega = \frac{167}{30\pi}.$$

The gravitational coupling constant becomes scale dependent and transferring from distance to energy we have

$$G(E) = G(1 - \omega G E^2 + \cdots) \approx \frac{G}{1 + \omega G E^2}.$$

Effective QFT of the electric charge

Same calculation in QED

$$\frac{e^2}{r} \to \frac{e^2(r)}{r}, \quad e^2(r) = e^2 \left(1 - \frac{e^2}{6\pi^2} \ln(m\, r) + \cdots \right), \quad m\, r \ll 1.$$

The electric charge is also scale dependent and has a Landau pole

$$e^2(E) = e^2 \left(1 + \frac{e^2}{6\pi^2} \ln(E/m) + \cdots \right) \approx \frac{e^2}{1 - \frac{e^2}{6\pi^2} \ln(E/m)}.$$

$$GE^2 \ll 1 \quad \rightarrow \quad G(E)E^2 \ll 1$$

BUT

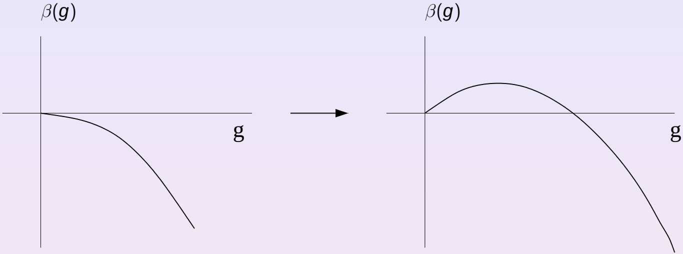
$$G(E)E^2 < 1 \quad (\ll 1\ ?).$$

Suddenly seems as if quantum gravity has become an (almost) reliable quantum theory at all energy scales.

The behavior can be described the β function for QG. For the dimensionless coupling constant $\tilde{G}(E) = G(E)E^2$

$$E\frac{d\tilde{G}}{dE} = \beta(\tilde{G}), \qquad \beta(\tilde{G}) = 2\tilde{G} - 2\omega\tilde{G}^2.$$

Two fixed points ($\beta(\tilde{G}) = 0$): $\tilde{G} = 0$ and $\tilde{G} = 1/\omega$.

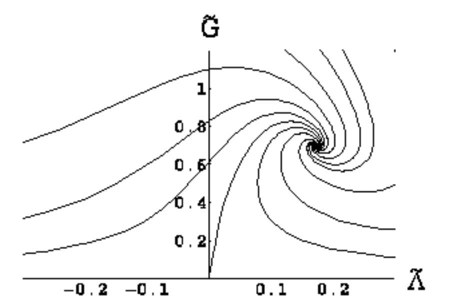


Generic situation for asymptotic free theories in d dimensions, extended to $d+\varepsilon$ dimensions.

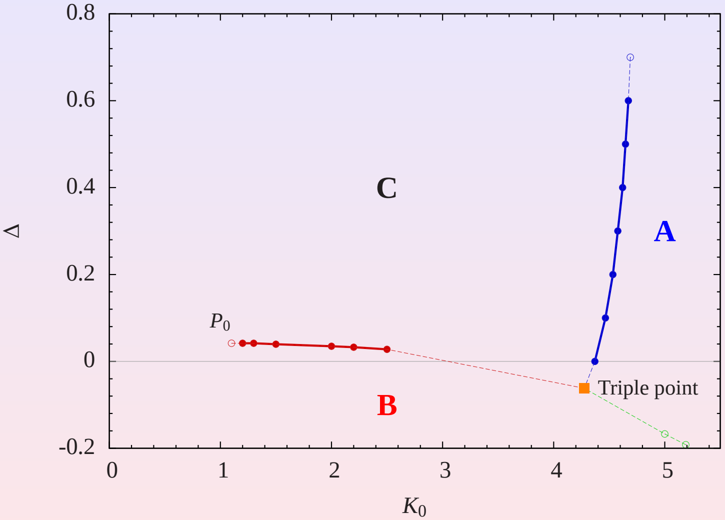
$$\beta(g) \to \varepsilon g + \beta(g)$$

The four-Fermi action, the nonlinear sigma model and QG are all renormalizable theories in 2d, with a negative β-function and have a $2+\varepsilon$ expansion. For QG first explored by Kawai et al.

Alternatively one can use the exact renormalization group approach. (Reuter et al., Litim, ...). Philosophy: asymptotic safety (Weinberg).



(2) Continuum limit ?



Defining the continuum limit in lattice field theory

Let the lattice coordinate be $x_n = a\,n$, a being the lattice spacing and $\mathcal{O}(x_n)$ an observable.

$$-\log\langle\mathcal{O}(x_n)\mathcal{O}(x_m)\rangle \sim |n-m|/\xi(g_0) + o(|n-m|).$$

$$\xi(g_0) \propto \frac{1}{|g_0 - g_0^c|^\nu}, \qquad \boxed{a(g_0) \propto |g_0 - g_0^c|^\nu}.$$

$$m_{ph} a(g_0) = 1/\xi(g_0), \quad e^{-|n-m|/\xi(g_0)} = e^{-m_{ph}|x_n - x_m|}$$

$\langle\mathcal{O}(x_n)\mathcal{O}(y_m)\rangle$ falls off exponentially like $e^{-m_{ph}|x_n-y_m|}$ for $g_0 \to g_0^c$ when $|x_n - y_m|$, but not $|n-m|$, is kept fixed in the limit $g_0 \to g_0^c$.

How to define the equivalent of $\langle \mathcal{O}(x_n)\mathcal{O}(y_m)\rangle$ in a diffeomorphism invariant theory

$$\langle \phi\phi(R)\rangle \equiv \int \mathcal{D}[g_{\mu\nu}]\, e^{-S[g_{\mu\nu}]} \times$$
$$\iint \sqrt{g(x)}\sqrt{g(y)}\; \langle \phi(x)\phi(y)\rangle^{[g_{\mu\nu}]}_{matter}\; \delta(R - d_{g_{\mu\nu}}(x,y)).$$

$\langle \phi(x)\phi(y)\rangle^{[g_{\mu\nu}]}_{matter}$ denotes the correlator of the matter fields calculated for a fixed geometry, defined by the metric $g_{\mu\nu}(x)$.

It works in 2d Euclidean QG (Liouville gravity)

(3) Unboundedness of the Euclidean action

Already the discussion about continuum limit of the lattice theories hinted a rotation to Euclidean signature. The Einstein-Hilbert action is unbounded from below, caused by the conformal factor:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$S[g,\Lambda,G] = -\frac{1}{16\pi G}\int d^4\xi\sqrt{g}\Big(R-2\Lambda\Big).$$

$$S[\tilde{g},\Lambda,G] = -\frac{1}{16\pi G}\int d^4\xi\sqrt{g}\Big(\Omega^2 R + 6\partial^\mu\Omega\partial_\mu\Omega - 2\Lambda\Omega^4\Big).$$

How is this dealt with ?.

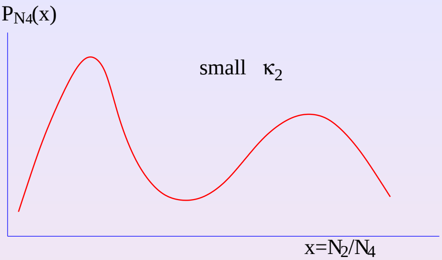
Using the lattice regularization called dynamical triangulations (DT) the Euclidean action is bounded for a fixed lattice spacing a and a fixed four-volume $V_4 = N_4 a^4$. However, for $a \to 0$ the unboundedness re-emerges.

$$S[T] = -\kappa_2 N_2(T) + \kappa_4 N_4(T), \quad c_1 < \frac{N_2}{N_4} \, (= x) < c_2.$$

The unbounded configurations corresponds to $x \approx c_2$. But are they important in the non-perturbative path integral ?

$$Z = \sum_T e^{-S[T]} = \sum_{N_4} e^{-k_4 N_4} \sum_{N_2} \mathcal{N}(N_2, N_4) e^{\kappa_2 N_2}$$

$$\mathcal{N}(N_2, N_4)\, e^{\kappa_2 N_2} = P_{N_4}(x), \quad \sum_x P_{N_4}(x) = f(N_4)\, e^{\kappa_4^c(\kappa_2) N_4}$$

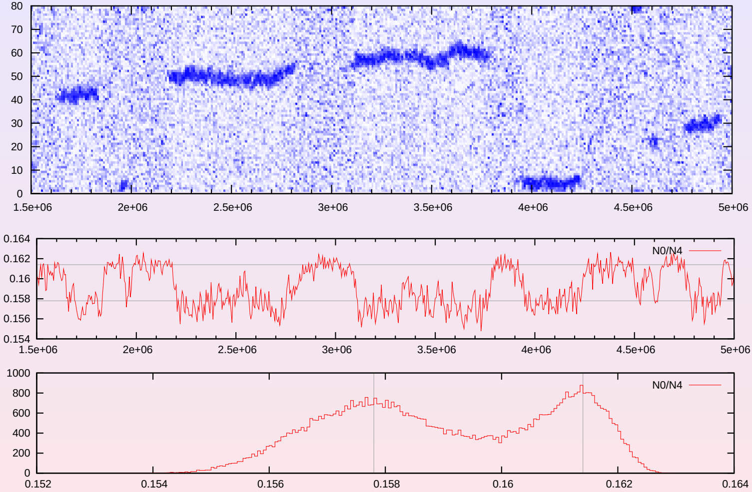


$$P_{N_4}(x) \approx A\, e^{N_4\left(\kappa_4^c - \alpha(x-x_0)^2\right)} + \tilde{A}\, e^{N_4\left(\tilde{\kappa}_4^c - \tilde{\alpha}(x-\tilde{x}_0)^2\right)}.$$

$$k_2 \to \kappa_2 + \Delta\kappa_2, \quad \kappa_4^c \to k_4^c + \Delta\kappa_2 x_0, \quad \tilde{\kappa}_4^c \to \tilde{k}_4^c + \Delta\kappa_2 \tilde{x}_0$$

Phase transition when $\kappa_4^c = \tilde{\kappa}_4^c$.

The A-C transition



Do we know examples of such entropy driven phase transitions? Yes, the Kosterlitz-Thouless transition in the XY model. This Abelian 2d spin model has vortices with energy

$$E = \kappa \ln(R/a)$$

Saturating the partition function by single vortex configurations:

$$Z \equiv e^{-F/k_B T} = \sum_{\text{spin configurations}} e^{-E[\text{spin}]/k_B T} \approx \left(\frac{R}{a}\right)^2 e^{-[\kappa \ln(R/a)]/k_B T}.$$

$S = k_B \ln(\text{number of configurations})$ has the same functional form as the vortex energy. Thus

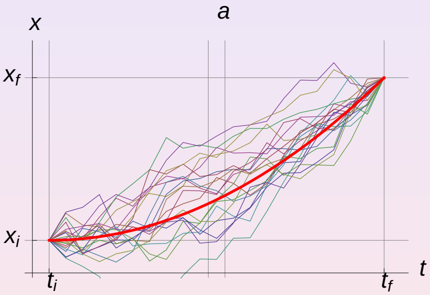
$$F = E - ST = (\kappa - 2k_B T) \ln(R/a)$$

Examples

- Lattice compact $U(1)$ gauge theory in 3 dimensions has confinement for all values of the coupling constant, due to lattice monopoles. It describes perfectly the non-perturbative physics of the Georgi-Glashow model, i.e. the physics below the scale of Higgs and the W-particle. The formula for the string tension is the same expressed in terms of lattice monopoles masses and continuum monopole masses.
- Lattice compact $U(1)$ gauge theory in 4 dimensions at the phase transition point describes the low energy physics of certain broken $\mathcal{N} = 1, 2$ supersymmetric field theories. In fact, one can use the supersymmetric symmetry breaking technology of Seiberg et al. scale matching to “post-dict” (unfortunately) the lattice critical exponents.

Basic tool: The path integral

Text-book example: non-relativistic particle in one dimension.



$$x(t) = \langle x(t) \rangle + y(t)$$
$$\langle |y| \rangle \propto \sqrt{\hbar/m\omega}$$

In QG we want $\langle x(t) \rangle$

$$\langle |y| \rangle \propto \sqrt{\hbar G}$$

Transition amplitude as a weighted sum over all possible trajectories. On the plot: time is discretized in steps *a*, trajectories are piecewise linear.

In a continuum limit $a \to 0$

$$G(\mathbf{x}_\mathrm{i}, \mathbf{x}_\mathrm{f}, t) := \int_{\text{trajectories: } \mathbf{x}_\mathrm{i} \to \mathbf{x}_\mathrm{f}} e^{iS[\mathbf{x}(t)]}$$

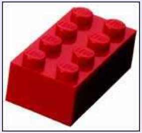
where $S[\mathbf{x}(t)]$ is a classical action.

The QG amplitude between the two geometric states

$$G(\mathbf{g}_\mathrm{i}, \mathbf{g}_\mathrm{f}, t) := \int_{\text{geometries: } \mathbf{g}_\mathrm{i} \to \mathbf{g}_\mathrm{f}} e^{iS[\mathbf{g}_{\mu\nu}(t')]}$$

To define this path integral we need a geometric cut-off a and a definition of the class of geometries entering.

showcasing piecewise linear geometries via building blocks:



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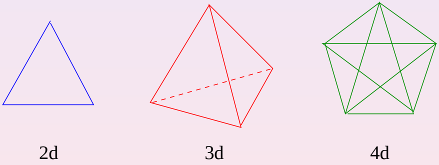
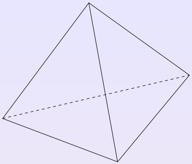
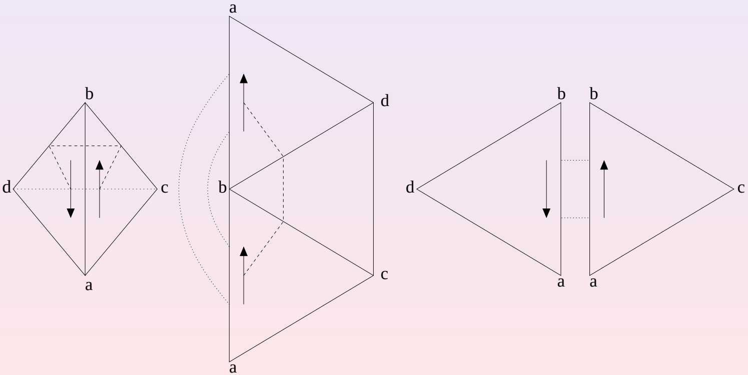


Piecewise linear geometry is defined without coordinates

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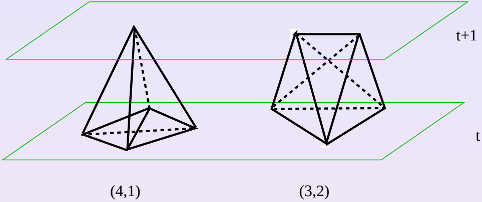
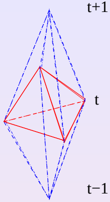




CDT slicing in proper time. Topology of space preserved.

$$a_t^2 = -\alpha a_s^2, \qquad iS_L[\alpha] = -S_E[-\alpha]$$

$$S_E[-\alpha] = -(\kappa_0 + 6\Delta) N_0 + \kappa_4 \left(N_4^{(2,3)} + N_4^{(1,4)} \right) + \Delta \left(N_4^{(2,3)} + 2 N_4^{(1,4)} \right)$$

The way this comes about:

$$S_E(\Lambda, G) = -\frac{1}{16\pi G a^2} \sum_{\sigma^{d-2} \in T_{d-2}} \left(2\varepsilon_{\sigma^{d-2}} V_{\sigma^{d-2}} - 2\Lambda V(\sigma^{d-2}) \right),$$

Introducing dimensionless quantities:

$$\mathcal{V}(\sigma^{d-2}) = a^{-d} V(\sigma^{d-2}), \quad \mathcal{V}_{\sigma^{d-2}} a^{2-d}, \quad \kappa = \frac{a^{d-2}}{16\pi G}, \quad \lambda = \frac{2\Lambda a^d}{16\pi G},$$

and the action becomes

$$S_E(\lambda, \kappa; T) = -\sum_{\sigma^{d-2} \in T_{d-2}} \left(\kappa\, 2\varepsilon_{\sigma^{d-2}} \mathcal{V}_{\sigma^{d-2}} - \lambda \mathcal{V}(\sigma^{d-2}) \right)$$

Using building blocks the action can be expressed in terms of the number of building blocks (and the number of subsimplices)

$$G(\mathbf{g}_\mathrm{i}, \mathbf{g}_\mathrm{f}, t) := \int_{\text{geometries: } \mathbf{g}_\mathrm{i} \to \mathbf{g}_\mathrm{f}} \mathrm{e}^{iS[\mathbf{g}_{\mu\nu}(t')]}$$

$$= \lim_{a \to 0} \sum_{T: T_i^{(3)} \to T_f^{(3)}} \frac{1}{C_T} \; \mathrm{e}^{iS_T}$$

$$G_E(\mathbf{g}_\mathrm{i}, \mathbf{g}_\mathrm{f}, t, \kappa_0, \kappa_4, \Delta) = \lim_{a \to 0} \sum_{T: T_i^{(3)} \to T_f^{(3)}} \frac{1}{C_T} \; \mathrm{e}^{-S_E[T]}$$

$$\langle x_f | \mathrm{e}^{i\hat{H}t} | x_i \rangle \to \langle x_f | \mathrm{e}^{-\hat{H}\tau} | x_i \rangle$$

Scaling in the IR limit?

$$Z(\kappa_0, \kappa_4) = \sum_{N_4} \mathrm{e}^{-\kappa_4 N_4} Z_{N_4}(\kappa_0),$$

where $Z_{N_4}(\kappa_0)$ is the partition function for a fixed number N_4 of four-simplices (we ignore Δ for simplicity), namely,

$$Z_{N_4}(\kappa_0) = \mathrm{e}^{k_4^c N_4} f(N_4, \kappa_0), \quad Z(\kappa_0, \kappa_4) = \sum_{N_4} \mathrm{e}^{-(\kappa_4 - \kappa_4^c) N_4} f(N_4, \kappa_0)$$

We want to consider the limit $N_4 \to \infty$, and fine-tune $\kappa_4 \to \kappa_4^c$ for fixed κ_0. We expect the physical cosmological constant Λ to be defined by the approach to the critical point according to

$$\kappa_4 = \kappa_4^c + \frac{\Lambda}{16\pi G} a^4, \quad (\kappa_4 - \kappa_4^c)\, N_4 = \frac{\Lambda}{16\pi G} V_4, \quad V_4 = N_4 a^4,$$

How can one imagine obtaining an interesting continuum behavior as a function of κ_0? Assume $f(N_4, \kappa_0)$ has the form (numerical evidence)

$$f(N_4, \kappa_0) = e^{k_1(\kappa_0)\sqrt{N_4}}, \qquad \left\langle e^{-\frac{1}{G}\int_{V_4}\sqrt{g}R} \right\rangle = e^{c\frac{\sqrt{V_4}}{G}}.$$

$$Z(\kappa_4, \kappa_0) = \sum_{N_4} e^{-(\kappa_4 - \kappa_4^c)N_4 + k_1(\kappa_0)\sqrt{N_4}}.$$

Search for κ_0^c with $k_1(\kappa_0^c) = 0$, with the approach to this point governed by

$$k_1(\kappa_0) \propto \frac{a^2}{G}, \quad \text{i.e.} \quad k_1(\kappa_0)\sqrt{N_4} \propto \frac{\sqrt{V_4}}{G}.$$

$$Z(\kappa_4, \kappa_0) \approx \exp\left(\frac{k_1^2(\kappa_0)}{4(\kappa_4 - \kappa_4^c)}\right) = \exp\left(\frac{c}{G\Lambda}\right),$$

as one would naïvely expect from Einstein's equations, with the partition function being dominated by a typical instanton contribution, for a suitable constant c.

UV scaling limit?

If we are close to the UV fixed point, we know that G will not be constant when we change scale, but $\hat{G}(a)$ will. Writing $G(a) = a^2 \hat{G}(a) \approx a^2 \hat{G}^*$,

$$\kappa_4 - \kappa_4^c = \frac{\Lambda}{G(a)} a^4 \approx \frac{\Lambda}{\hat{G}^*} a^2,$$

$$k_1(\kappa_0^c) = \frac{a^2}{G(a)} \approx \frac{1}{\hat{G}^*}.$$

The first of these relations now looks two-dimensional because of the anomalous scaling of $G(a)$! Nevertheless, the expectation value of the four-volume is still finite:

$$\langle V_4 \rangle = \langle N_4 \rangle \, a^4 \propto \frac{\kappa_1^2(\kappa_0^c)}{(\kappa_4 - \kappa_4^c)^2} \, a^4$$

Relation to asymptotic freedom

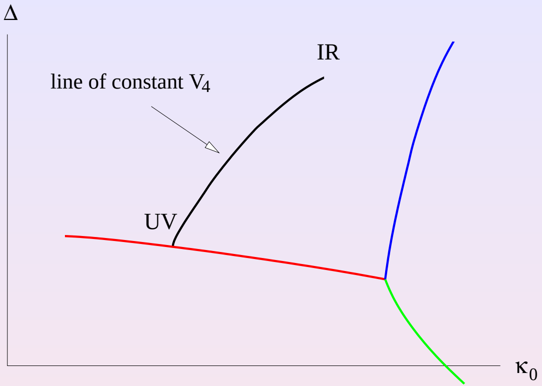
Assume now that we have a fixed point for gravity. The gravitational coupling constant is dimensionful, and we can write for the bare coupling constant

$$G(a) = a^2 \hat{G}(a), \quad a\frac{d\hat{G}}{da} = -\beta(\hat{G}), \quad \beta(\hat{G}) = 2\hat{G} - c\hat{G}^3 + \cdots .$$

The putative non-Gaussian fixed point corresponds to $\hat{G} \to \hat{G}^*$, i.e. $G(a) \to \hat{G}^* a^2$. In our case it is tempting to identify our dimensionless constant k_1 with $1/\hat{G}$, up to the constant of proportionality. Close to the UV fixed point we have

$$\hat{G}(a) = \hat{G}^* - Ka^{\tilde{c}}, \quad k_1 = k_1^* + Ka^{\tilde{c}}, \qquad \tilde{c} = -\beta'(\hat{G}^*).$$

Usually one relates the lattice spacing near the fixed point to the bare coupling constants with the help of some correlation length ξ.

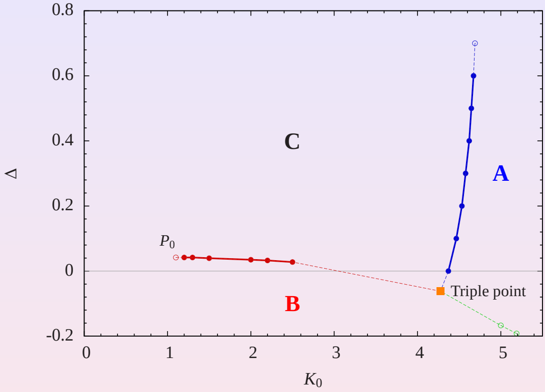


Consider $V_4 = N_4 a^4$ as fixed. It requires the fine-tuning of coupling constants.

$$k_1(N_4) = k_1^c - \tilde{K} N_4^{-\tilde{c}/4}.$$

How to determine $k_1(N_4)$?

Phase diagram of CDT



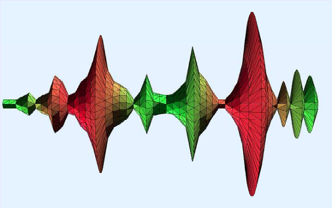
Lifshitz-like diagram....

Phase C: constant magnetization (constant 4d geometry)

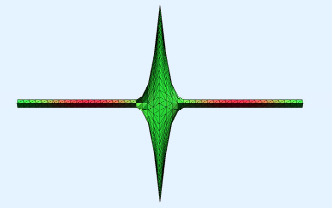
Phase B: zero magnetization (no 4d geometry)

Phase A: oscillating magnetization (conformal mode ?)

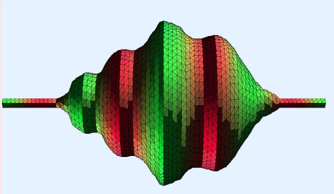
Volume distribution in (imaginary) time



- **Phase A**. The universe "oscillating" in time direction. The oscillation maybe reflecting the dominance of the conformal mode.

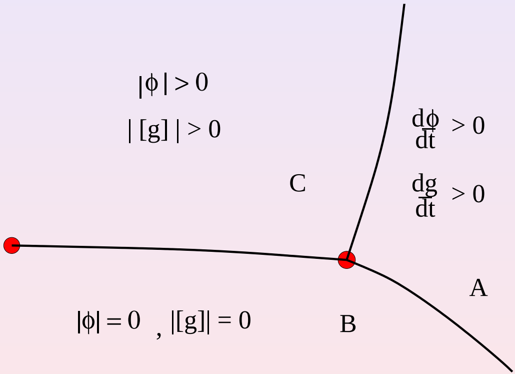


- **Phase B**. Compactification into a 3d Euclidean DT. Only minimal extension in the time direction.

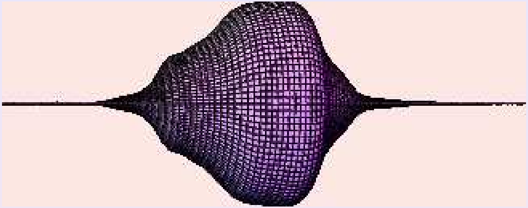


- **Phase C**. Extended de Sitter phase. $d_H = 4$.

$$S_{\text{Lifshitz}}[\phi] = \int d^D x \left(\mu_i (\partial_i \phi)^2 + (\Delta\phi)^2 + \cdots + \nu\phi^2 + \phi^4 + \cdots \right)$$



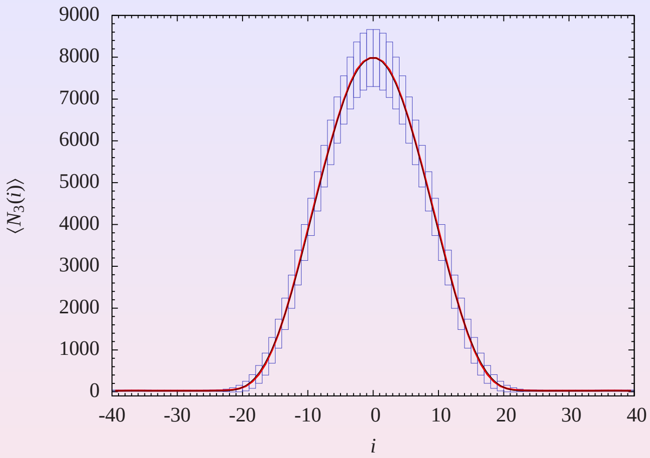
Snapshot of a typical configuration



A typical configuration. Distribution of a spatial volume $N_3(t)$ as a function of (imaginary) time t.

Quantum fluctuation around a semiclassical background?

Configuration consists of a “stalk” of the cut-off size and a “blob”. Center of the blob can shift. We fix the “center of mass” to be at zero time.



$$\langle N_3(i)\rangle \propto N_4^{3/4} \cos^3\left(\frac{i}{s_0 N_4^{1/4}}\right)$$

Minisuperspace model

The semiclassical distribution can be obtained from the minisuperspace effective action of Hartle and Hawking

$$S_{eff} = \frac{1}{24\pi G} \int dt \sqrt{g_{tt}} \left(\frac{g^{tt} \dot{V_3}^2(t)}{V_3(t)} + k_2 V_3^{1/3}(t) - \lambda V_3(t) \right),$$

The discretization of this action is (and we have reconstructed it from the date (the 3-volume–3-volume correlations))

$$S_{discr} = k_1 \sum_i \left(\frac{(N_3(i+1) - N_3(i))^2}{N_3(i)} + \tilde{k}_2 N_3^{1/3}(i) - \tilde{\lambda} N_3(i) \right),$$

$$G = \frac{a^2}{k_1} \frac{\sqrt{C_4}\, s_0^2}{3\sqrt{6}}.$$

Quantum fluctuations

The classical solution to the minisuperspace action is

$$\sqrt{g_{tt}}\; V_3^{cl}(t) = V_4\, \frac{3}{4B} \cos^3\left(\frac{t}{B}\right)$$

where $\tau = \sqrt{g_{tt}}\; t$, $V_4 = 8\pi^2 R^4/3$ and $\sqrt{g_{tt}} = R/B$.

Writing $V_3(t) = V_3^{cl}(t) + x(t)$ we can expand the action around this solution

$$S(V_3) = S(V_3^{cl}) + \frac{1}{18\pi G}\frac{B}{V_4} \int \mathrm{d}t\; x(t)\hat{H}x(t).$$

where the Hermitian operator $\hat{H}$ is:

$$\hat{H} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\cos^3(t/B)}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{4}{B^2\cos^5(t/B)},$$

In the quadratic approximation the volume fluctuations are:

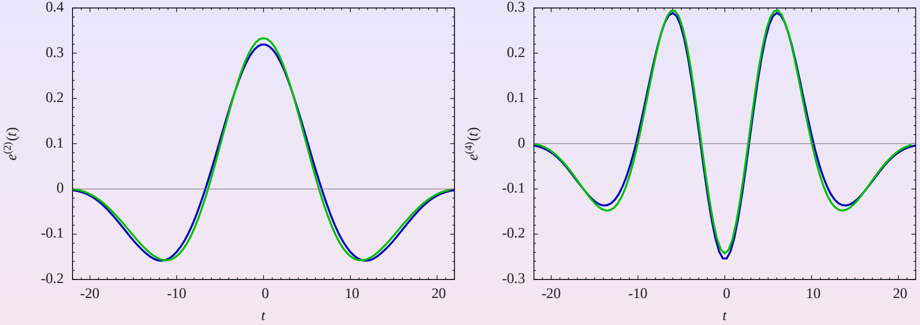
$$C(t,t') := \langle x(t)x(t')\rangle \sim \hat{H}^{-1}(t,t').$$

$\hat{C}$ and $\hat{H}$ have the same eigenfunctions.

$C(t,t')$ can be measured as

$$C(i,i') = \Big\langle \Big(N_3(i) - \langle N_3(i)\rangle\Big)\Big(N_3(i') - \langle N_3(i')\rangle\Big)\Big\rangle,$$

and its eigenfunctions can be found and compared to the ones calculated from $\hat{H}$.



No parameters are put in ! (expect $t_i/B = i/s_0 N_4^{1/4}$)

We conclude that the quadratic approximation to the minisuperspace action describes the measured quantum fluctuations well.

Size of our Quantum universe

For a specific value of the bare coupling constants ($\kappa_0 = 2.2, \Delta = 0.6$) we have high-statistics measurements for N_4 ranging from 45.500 to 362.000 four-simplices.

Largest universe corresponds to approx. 10^4 hyper-cubes.

We have $G = \text{const.}\ a^2/k_1$ and we have measured k_1.

$$G \approx 0.23 a^2, \qquad \ell_P \approx 0.48 a, \quad \ell_P \equiv \sqrt{G}.$$

From $V_4 = 8\pi^2 R^4/3 = C_4 N_4 a^4$, we obtain that

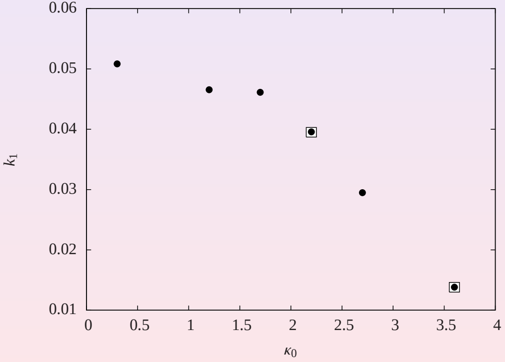
$$R = 3.1 a$$

The linear size πR of the quantum de Sitter universes studied here lies in the range of 12-21 ℓ_P for the N_4 used.

Trans-Planckian ?

$$\ell_P = \sqrt{G} \propto \frac{a}{\sqrt{k_1(\kappa_0,\Delta)}} \quad \text{i.e.} \quad k_1(\kappa_0,\Delta) \to 0.$$

BUT IS IT POSSIBLE ?



Summary and perspectives

- We have obtained the (Euclidean) minisuperspace action from first principles. (The self-organized de Sitter space)
- We have an effective field theory of (something we call) QG down to a few Planck scales.

- Investigate a possible UV fixed point (points, the B-C line). Possibly Hořava-Lifshitz gravity.
- couple matter to the system and investigate cosmological implications.
- Measure the wave function of the universe

$$\langle x|\, e^{-t\hat{H}} |y\rangle \rightarrow \Psi_0(y)\Psi_0(x)\, e^{-tE_0}$$

$$Z = \sum_{\phi} e^{-S[\phi]},$$

$$\langle \mathcal{O}(\phi) \rangle = Z^{-1} \sum_{\phi} \mathcal{O}(\phi)\, e^{-S[\phi]}.$$

The purpose of the Monte Carlo simulations is to generate a sequence of statistically independent field configurations $\phi(n)$, $n = 1, \ldots, N$ with the probability distribution

$$P(\phi(n)) = Z^{-1} e^{-S[\phi(n)]}.$$

Then

$$\langle \mathcal{O}(\phi) \rangle_N = \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}(\phi(n)).$$

serves as an *estimator* of the expectation value and one has

$$\langle \mathcal{O}(\phi) \rangle_N \to \langle \mathcal{O}(\phi) \rangle \quad \text{for} \quad N \to \infty.$$

In a Monte Carlo simulation a change $\phi \to \phi'$ of the field configuration is usually generated by a stochastic process $\mathcal{T}$, a Markow chain. The field will perform a random walk in the space of field configurations with a transition function, or transition probability, $\mathcal{T}(\phi \to \phi')$. Thus, if we after a certain number n of steps (changes of the field configuration) have arrived at a field configuration $\phi(n)$, $\mathcal{T}(\phi(n) \to \phi(n+1))$ is the probability of changing $\phi(n)$ to $\phi(n+1)$ in the next step. We have

$$\sum_{\phi'} \mathcal{T}(\phi \to \phi') = 1 \quad \text{for all } \phi.$$

The transition probability should be chosen such that

(i) Any field configuration ϕ can be reached in a finite number of steps (ergodicity)

(ii) The probability distribution of field configurations converges, as the number of steps goes to infinity, to the Boltzmann distribution.

The convergence of the Markov chain is usually ensured by choosing $\mathcal{T}$ to satisfy the so-called rule of detailed balance

$$P(\phi)\,\mathcal{T}(\phi \to \phi') = P(\phi')\mathcal{T}(\phi' \to \phi).$$

Thus $\mathcal{T}(\phi \to \phi') = \mathcal{T}(\phi' \to \phi) = 0$ or

$$\frac{\mathcal{T}(\phi \to \phi')}{\mathcal{T}(\phi' \to \phi)} = \frac{P(\phi')}{P(\phi)}.$$

Usually decomposes the transition probability $\mathcal{T}(\phi \to \phi')$ into a selection probability $g(\phi \to \phi')$ and an acceptance ratio $A(\phi \to \phi')$. We then have:

$$\frac{P(\phi')}{P(\phi)} = \frac{\mathcal{T}(\phi \to \phi')}{\mathcal{T}(\phi' \to \phi)} = \frac{g(\phi \to \phi')A(\phi \to \phi')}{g(\phi' \to \phi)A(\phi' \to \phi)}.$$

The selection probability $g(\phi \to \phi')$ is now designed to select the configurations ϕ, ϕ' where $\mathcal{T}(\phi \to \phi')$ is different from zero and assign a weight of our own choice to the transition $\phi \to \phi'$. The acceptance ratio $A(\phi \to \phi')$ should then be chosen to ensure detailed balance. A general choice, used in many Monte Carlo simulations, is the so-called Metropolis algorithm:

$$\begin{aligned} A(\phi \to \phi') &= \min\left(1, \frac{g(\phi' \to \phi)}{g(\phi \to \phi')} \frac{P(\phi')}{P(\phi)}\right), \\ A(\phi' \to \phi) &= \min\left(1, \frac{g(\phi \to \phi')}{g(\phi' \to \phi)} \frac{P(\phi)}{P(\phi')}\right). \end{aligned}$$

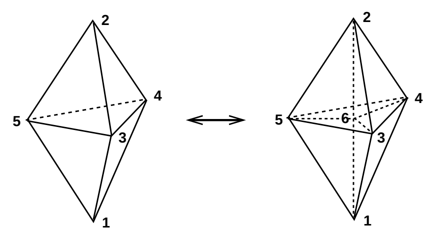
Unlabeled versus labeled triangulations:

$$Z = \sum_{T} \frac{1}{C(T)} \, \mathrm{e}^{-S(T)} = \sum_{T_l} \frac{1}{N_0(T_l)!} \, \mathrm{e}^{-S(T_l)}$$

Two labeled triangulations will be identical if a mapping (a relabeling) $i \to \sigma(i)$ maps neighbors to neigbors.

In the computer we use labeled triangulations.

We fix topology. We need moves to get ergodically arround in the class of d-dimensions triangulations of a fixed topology. A minimal set of moves are the Pachner moves. (here for CDT in the case of three dimensions)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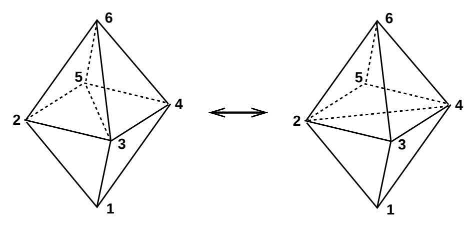
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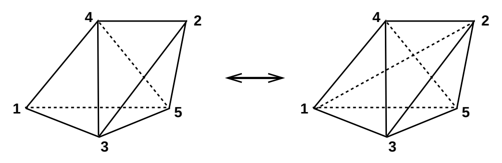
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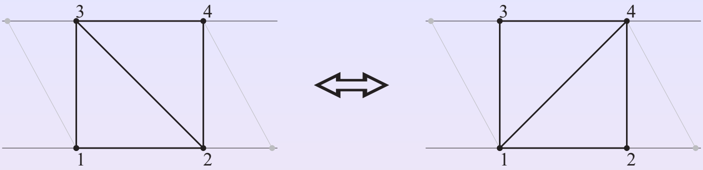

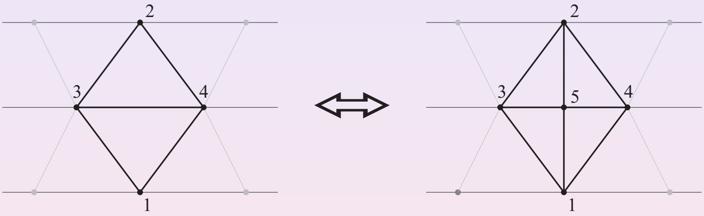

In 2d $\int \sqrt{g} R = 2\pi\chi$, i.e. a topological invariant for fixed topology. Thus $S[T] = \lambda N_2(T) = \lambda(2N_0 - \chi)$, and

$$P(T) = \frac{1}{Z} \frac{1}{N_0(T)!} \, e^{-2\lambda N_0(T)}.$$

The (2,4) move: If the old vertices are labeled from 1 to N_0, we assign the label $N_0 + 1$ to the new vertex (and new links and new triangles are defined by pairs and triples of vertex labels, the triples also defining the correct orientation). Given the labeled triangulation T_{N_0} we can in this way get to N_0 labeled triangulations T_{N_0+1} by chosing different vertices and performing the (2,4)-move. We define the selection probability $g(T_{N_0} \to T_{N_0+1})$ to be the same for all triangulations T_{N_0+1} which can be reached in this way and to be zero for all other labeled T_{N_0+1} triangulations. Thus for the labeled triangulations which can be reached we have:

$$g(T_{N_0} \to T_{N_0+1}) = \frac{1}{N_0},$$

and we implement this in the computer program by choosing randomly with uniform probability a vertex in T_{N_0}.

The (4,2) move: Given a labeled triangulation T_{N_0+1} we perform the inverse move by the following procedure: we select the vertex labeled $N_0 + 1$. Let us assume it is of order 4. Then we delete it from the list, in this way creating a labeled triangulation T_{N_0}. If the vertex labeled $N_0 + 1$ is not of order 4 we do not perform the move. Thus, for the triangulations T_{N_0+1} where the move can be performed we can only reach one triangulation T_{N_0} and the selection probability $g(T_{N_0+1} \to T_{N_0})$ defined by this procedure is one.

Finally we choose the acceptance ratios $A(T \to T')$ in accordance with the Metropolis algorithm

$$A(T_{N_0} \to T_{N_0+1}) = \min\left(1, \frac{N_0}{N_0+1}\, e^{-2\lambda}\right),$$
$$A(T_{N_0+1} \to T_{N_0}) = \min\left(1, \frac{N_0+1}{N_0}\, e^{2\lambda}\right).$$

Data structure: natural pointer oriented

Label the vertices of (a 2d) triangulation by i, $i = 1, \ldots, n$. Denote the neighbors to vertex i by $k(i, m)$, $m = 1, \ldots, o(i)$, ordered cyclicly in accordance with the orientation, and where $o(i)$ denotes the order of vertex i