Hadron Spectroscopy Ia

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Part I: introduction and basic methods

- Motivation
- Two point functions
- Irreducible representations
- Smearing
- Generalized eigenvalue/variational method
- "Distillation"

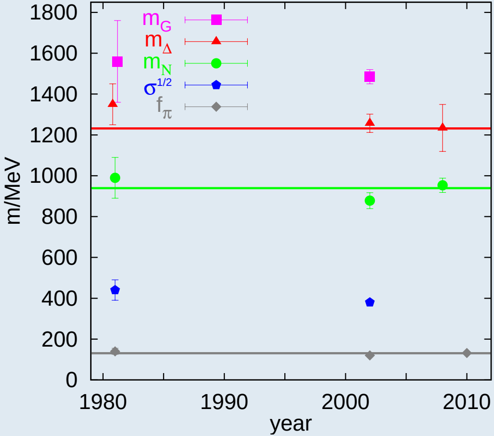
Part II: all-to-all and stochastic methods

- Low mode averaging
- Stochastic all-to-all methods
- Variance reduction techniques
- The one-end-trick
- Thinning the estimates: grid noise and distillation

Many examples and more details by Christian Lang and David Richards!

Some history

Spectroscopy is (almost) the easiest thing one can do.

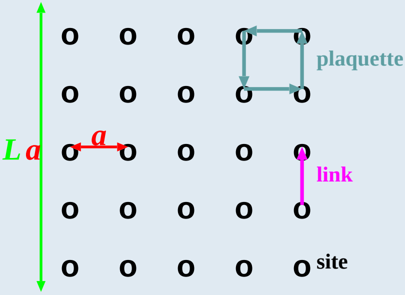


H Hamber, G Parisi, PRL 47 (81) 1792.

CP-PACS: S Aoki et al, PRD 67 (03) 034503.

BMWc: S Dürr et al, Science 322 (08) 1224, arXiv:0906.3599.

Lattice QCD



typical values:

$a^{-1} = 1.5$–4 GeV, $La = 1.5$–6 fm

continuum limit: $a \to 0$, La fixed

infinite volume: $La \to \infty$

$$\langle O \rangle_U = \frac{1}{Z} \int [dU]\,[dq][d\bar{q}]\; O[U] e^{-S[U,q,\bar{q}]}$$

"Measurement": average over a *representative* ensemble of gluon configurations $\{U_i\}$ with probability $P(U_i) \propto \int [dq][d\bar{q}] e^{-S[U,q,\bar{q}]}$

$$\langle O \rangle = \frac{1}{N} \sum_{i=1}^{N} O(U_i) + \Delta O \qquad \Delta O \propto \frac{1}{\sqrt{N}} \xrightarrow{N \to \infty} 0$$

Input: Some discretization of $\mathcal{L}_{\mathcal{QCD}} = -\frac{1}{16\pi\alpha_L} FF + \bar{q}^f(\not{D} + m_f)q^f$

$$M_\Omega = a m_\Omega \quad \longrightarrow \quad a$$

$$2M_K^2 - M_\pi^2 = a^2\left(2m_K^2 - m_\pi^2\right) \quad \longrightarrow \quad m_s$$

$$M_\pi^2 = a^2 m_\pi^2 \quad \longrightarrow \quad m_u \approx m_d$$

etc.

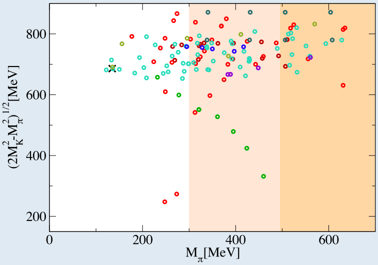
Output: hadron masses, matrix elements, decay constants, etc...

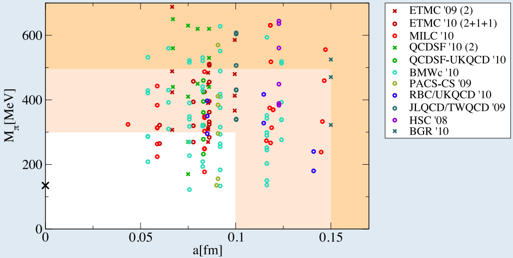
Extrapolations:

1. $a \to 0$: functional form known.
2. $L \to \infty$: harmless but often computationally expensive.
3. $m_q^{\text{latt}} \to m_q^{\text{phys}}$: chiral perturbation theory (χPT) but m_q^{latt} must be sufficiently small to start with.

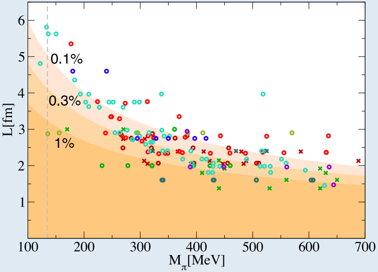
   ($a^{-1}M_\pi = m_\pi^{\text{latt}} \leq m_\pi^{\text{phys}}$ has only very recently been realized.)

Landscape of current lattice simulations

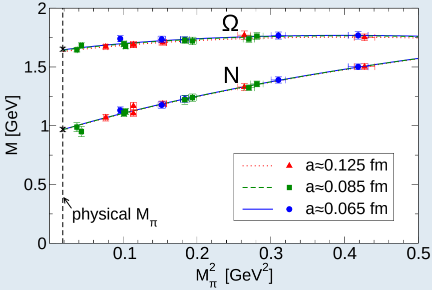


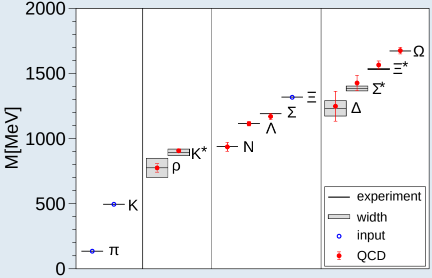


Figures taken from
C Hoelbling,
PoS (LATTICE2010) 011,
arXiv:1102.0410.



Chiral extrapolation and the continuum spectrum





BMWc: S Dürr et al, Science 322 (08) 1224.

Similar results by PACS-CS and HPQCD/MILC

Problems:

- Excited states decay strongly (in experiment and on the lattice when sea quark masses become light).
- $\rightarrow$ Difficulties to differentiate resonances from the continuum of multi-particle scattering states.
- Decay channels affect the position of the real part of the pole (mass).

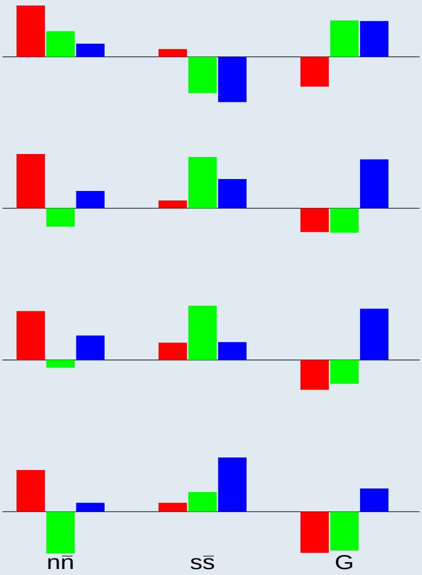
When sea quarks are light (as in experiment) in particular the scalar meson sector may become distorted by tetraquarks/molecules.

Scalar mesons $J^{PC} = 0^{++}$ in experiment

$I \neq 0$	$I = 0$	
$\kappa(800\ ??)$	$\sigma(600\ ??)$	$\pi\pi$, $q\bar{q}q\bar{q}$ tetraquark or $\pi\pi$ FSI ?
$a_0^-(980)$ $a_0^0(980)$ $a_0^+(980)$	$f_0(980)$	$K\overline{K}$, $q\bar{s}\bar{q}s$ tetraquark ?
$K_0^*(1430)$ $K_0^{*+}(1430)$	$f_0(1370)$	$u\bar{u} + d\bar{d} - 2s\bar{s}$?
$a_0^-(1450)$ $a_0^0(1450)$ $a_0^-(1450)$	$f_0(1500)$	$u\bar{u} + d\bar{d} + s\bar{s}$?
$K_0^{*-}(1430)$ $K_0^*(1430)$	$f_0(1710)$	glueball ?

Mixing? Coupling to $\pi\pi$, KK, $\eta\eta$?

$f_0(1370)$
$f_0(1500)$
$f_0(1710)$



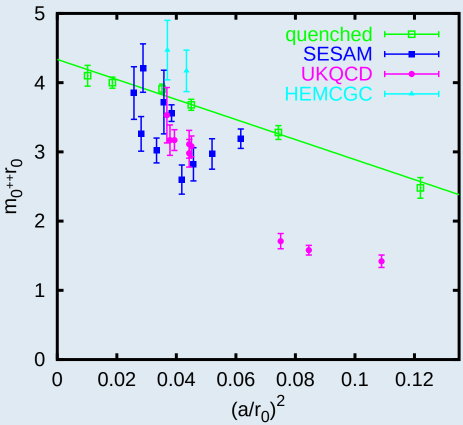
C Amsler, F Close, PRD 53 (96) 295

L Burakovsky, P Page, PRD 59 (99) 014022, 079902(E)

W-J Lee, D Weingarten, arXiv:hep-lat/980502

F Close, A Kirk, EPJC 21 (01) 531

This all uses ancient lattice input



UKQCD quenched

`GB et al, PLB 309 (93) 378; B Lucini, M Teper, JHEP 0106 (01) 050.`

SESAM/TχL $n_f = 2$ Wilson

`GB et al, PRD 62 (00) 054503.`

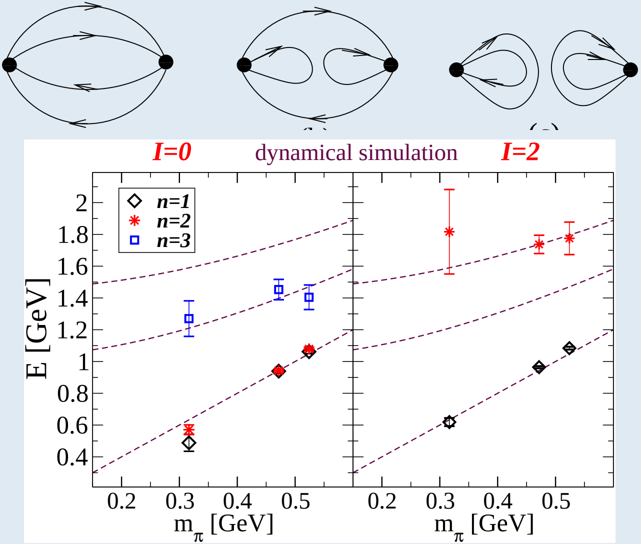
UKQCD $n_f = 2$ clover

`A Hart, M Teper, PRD 65 (02) 034502.`

HEMCGC $n_f \approx 2$ staggered

`K Bitar et al, PRD 44 (91) 2090.`

What about light scalar mesons/tetraquarks/molecules?



S Prelovsek et al, PRD 82 (10) 094507. (also recent $I = 2$ work by HSC: Dudek et al, PRD 83 (11) 071504.)

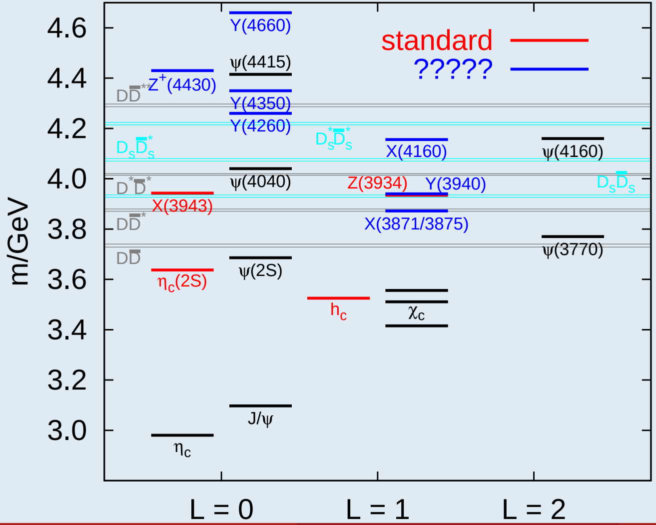
Heavy hadron spectroscopy: exciting times

First time in > 20 years: several new narrow(ish) resonances!

- ★ Υ D wave(s), η_b
- ★ B_c
- ★ η_c', h_c
- ★ $X(3872), X(3943), Y(4260), \ldots$
- ★ $D_{s0}^*(2317)$, $D_{s1}^*(2460)$, $\ldots$
- ★ > 10 new charmed baryons

1974 – 1977: 10 $c\bar{c}$ resonances, 1978 – 2001: 0 $c\bar{c}$'s

2002 – 2008: $\leq$ 12 new $c\bar{c}$'s found by BaBar, Belle, CLEO-c, CDF, D0



new detectors

higher luminosity

new channels:

B decays

$\gamma\gamma$

$\psi\psi$-production

gg in $p\bar{p}$ collisions.

$c\bar{q}q\bar{c}$ in $c\bar{c}$?

$cg\bar{c}$ hybrids ?

Some definitions

Lattice spacing: a. Sometimes (not considered here): $a_s \neq a_t$ (anisotropy).

Lattice sites $x = na$, $n = (n_\mu)$, $\mu \in \{1, 2, 3, 4\}$, $n_\mu \in \{0, 1, \ldots, L_\mu - 1\}$.
Usually: $L_s := L_1 = L_2 = L_3 \leq L_t = L_4$. # of 3-volume sites: $V_3 = L_s^3$,
4-volume sites: $V = V_s L_t$.

Gauge links $U_{x,\mu} = U^\dagger_{x+a\hat{\mu},-\mu} \in \mathrm{SU}(3)$ have toroidal boundary conditions:
$U_{x+L_\nu a\hat{\nu},\mu} = U_{x,\mu}$.

Fermion fields q_x are Grassmann variables and are antisymmetric in time:
$q_{x+L_t a\hat{4}} = -q_x$, $q_{x+L_s a\hat{\imath}} = q_x$

Lattice momenta (L_μ even): $p_\mu = \frac{2\pi}{La} \times \left\{-\frac{L_\mu}{2} + 1, -\frac{L_\mu}{2} + 2, \ldots, \frac{L_\mu}{2}\right\}$.

Largest momentum (in $3+1$ dimensions): $|p_{\max}| = \sqrt{\sum_\mu p^2_{\max,\mu}} = \frac{\sqrt{4}\,\pi}{a}$.

Operators

- Glueball: $\hat{O}_G^{\mathbf{p}} = \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \sum_i N_i \,\text{spatial Wilson loop}(i)_{\mathbf{x}}$.
- Meson: $\hat{O}_M^{\mathbf{p}} = \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \left(\bar{q}^1 \Gamma_A D\Phi q^2\right)_{\mathbf{x}}$.
  $\Gamma_A = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}$ acts on the Dirac spinor space. (n_1 is the least significant bit of $A \in \{0, \cdots, 15\}$: $\Gamma_0 = \mathbb{1}$, $\Gamma_8 = \gamma_4$, $\Gamma_{15} = \gamma_5$ etc.)
  D contains derivatives and other gauge covariant transporters.
  Φ is a smearing function. (D and Φ do not act on the spin index.)
- Baryon (example):
  $\hat{O}_{N^\pm,\alpha}^{\mathbf{p}} = \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \left[\epsilon_{abc} P_\pm \Phi_1 u_{\alpha,a} \left(u_b^T C\gamma_5 \Phi_2 d_c\right)\right]_{\mathbf{x}}$
  This Fermion contains an open Dirac spin index α.
  $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_4)$ is a parity projector (for $\mathbf{p} \neq \mathbf{0}$ see `F Lee, D Leinweber, NPPS 73 (99) 258`).
  $C = i\gamma_2\gamma_4 = i\Gamma_{10}$ is the charge conjugation matrix.

The spectral decomposition

Two point Green (correlation) function:

$$C(t) = \langle O(t+t_0)O^\dagger(t_0)\rangle$$

We will often exploit the translational invariance of expectation values:

$$\langle O_x\rangle_U = \langle O_0\rangle_U = \frac{1}{V}\sum_x \langle O_x\rangle_U\,.$$

From now on we choose $t_0 = 0$ and assume $L_t = \infty$.

Spectral decomposition:

$$C(t) = \sum_n \langle 0|\hat{O}|n\rangle e^{-E_n t}\langle n|\hat{O}^\dagger|0\rangle = \sum_n c_n e^{-E_n t}\,,$$

where $c_n = |\langle n|\hat{O}|0\rangle|^2 \geq 0$.

This holds for any t for actions with point and link reflection positivity (e.g. Wilson) and for $t \geq t_{\min}$ for less local actions ($t/a \geq 1$ for clover).

For different operators $\tilde{O}$ and O, $c_n = \langle 0|\hat{\tilde{O}}|n\rangle\langle n|\hat{O}^\dagger|0\rangle$ can be negative.

Effective masses

Effective "mass".

$$E_{\text{eff}}(t) = -a^{-1} \ln \frac{C(t)}{C(t+a)} \approx -\frac{d}{dt} \ln C(t) \stackrel{t\to\infty}{\longrightarrow} E_1 + \frac{c_2}{c_1} \Delta E e^{-\Delta E t} + \cdots,$$

where $\Delta E = E_2 - E_1$.

E_{eff} approaches the ground state energy E_1 within the sector of physical states with overlap to $O^\dagger|0\rangle$ exponentially in t.

The coefficients c_n depend on the choice of the operator O, e.g., on the smearing function Φ.

If $c_1 \gg c_i$ $(i > 1)$ then the effective mass will plateau at smaller t-values where the signal is less noisy.

Example: meson

$$C_M^{\mathbf{p}}(t) = \langle O_M^{\mathbf{p}}(t) O_M^{\mathbf{p}\dagger}(0)\rangle$$
$$= \sum_{\mathbf{x},\mathbf{y}} e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})} \left\langle \left(\bar{q}^1\Gamma q^2\right)_x \left(\bar{q}^1\Gamma q^2\right)_y^\dagger \right\rangle ,$$

where $x = (\mathbf{x}, t)$, $y = (\mathbf{y}, 0)$. For simplicity, derivatives D and smearing functions Φ are omitted.

We compute $(\bar{q}^1\Gamma q^2)^\dagger = -q^{2\dagger}\Gamma^\dagger\gamma_4 q^1 = \pm\bar{q}^2\Gamma q^1$ (Γ: $\gamma_4\Gamma = \mp\Gamma^\dagger\gamma_4$), rename $\mathbf{x}_{\mathrm{new}} = \mathbf{x} - \mathbf{y}$ and set $\mathbf{y} = \mathbf{0}$ (translational invariance):

$$C_M^{\mathbf{p}}(t) = \pm V_3 \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \langle \bar{q}_x^1\Gamma q_x^2 \bar{q}_0^2\Gamma q_0^1\rangle$$
$$= \mp V_3 \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \left\langle \mathrm{Tr}\left[\Gamma(q_x^2\bar{q}_0^2)\Gamma(q_0^1\bar{q}_x^1)\right]\right\rangle ,$$

where the trace is over spin and colour (Wick contraction).

Propagator

Fermionic action: $\sum_f \bar{q}^f M^f[U]q^f$. This means $\langle q^f_x \bar{q}^f_y\rangle_q = {M^f}^{-1}(x|y)$ where the subscript q denotes the Fermionic expectation value.
Due to γ_5-Hermiticity, $M^\dagger = \gamma_5 M \gamma_5$, we can rewrite,
$\langle q_0 \bar{q}_x\rangle_F = M^{-1}(0|x) = \gamma_5[M^{-1}(x|0)]^\dagger\gamma_5$, and obtain,

$$C^{\mathbf{p}}_M(t) = \mp V_3 \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \left\langle \mathrm{Tr}\left[{M^2}^{-1}(x|0)\Gamma\gamma_5[{M^1}^{-1}(x|0)]^\dagger\gamma_5\Gamma\right]\right\rangle_U ,$$

where the remaining expectation value is over gauge configurations.
The above expression is particularly simple for the π ($\Gamma = \gamma_5$, i.e. $\Gamma\gamma_5 = \mathbb{1}$).

For $q = q^1 = q^2$ (flavour-/iso-singlet) additional contractions occur and $M^{-1}(x|x)$ is needed.
Otherwise only the 12 columns of M^{-1} starting from $y = 0$ are required, the “point-to-all” propagator $S(x|0)^{\alpha\beta}_{ab} = M^{-1}(x|0)^{\alpha\beta}_{ab}$.

Normalization and poles

The usual normalization convention is, $\langle n|m\rangle = 2E_m\delta_{mn}$.
If we normalize O so that $C(0) = 1$ this implies $C(t) = \sum_n \frac{c_n}{2E_n} e^{-E_n t}$ where $\sum_n c_n = 1$.
What do masses have to do with poles? We create a $J = 0$ mesons by means of a local quark bilinear. Then $(L_s = L_t = \infty)$,

$$C(x) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{ik.x} \left(\sum_j \frac{c_j}{\hat{k}^2 + m_j^2} \right) ,$$

where $\hat{k}_\mu \approx \frac{2}{a} \sin \frac{ak_\mu}{2}$ (This is the simplest latticized and not necessarily correct ansatz.), $\hat{k}^2 = \sum_\mu \hat{k}_\mu^2$.
Masses were extracted from fits to $C(x)$ in

M Chu, J Grandy, S Huang, PRD 48 (93) 3340;
D Leinweber, PRD 51 (95) 6369;
C Allton, S. Capitani, NPB 526 (98) 463;
L Levkova, C DeTar, PRD 83 (11) 074504.

Problems with these point-to-point correlators:
no smearing possible, no exponential suppression of excitations.

But: better signal over noise. The usual position sum includes contributions from large Euclidean distances (little signal but noise).

The momentum projected correlation function reads,

$$C^{\mathbf{p}}(t) = \int \frac{d^3x\, d^4k}{(2\pi)^4} \frac{c_1 e^{ikt} e^{i\mathbf{x}(\mathbf{k}-\mathbf{p})}}{\hat{k}^2 + m_1^2} + \cdots = \int \frac{dk_4}{2\pi} \frac{c_1 e^{ik_4 t}}{\hat{k}_4^2 + \hat{\mathbf{p}}^2 + m_1^2} + \cdots$$

$$= \sum_j \frac{c_j}{2E_j} e^{-E_j t} \quad \text{where} \quad \sinh \frac{aE_j}{2} = \frac{a}{2}\sqrt{\hat{\mathbf{p}}^2 + m_j^2} \quad (E = -ik_4)\,.$$

This dispersion relation agrees with the continuum relation up to $\mathcal{O}(a^2)$ terms.
Note that the spectrum is real in Euclidean space (no decay widths) and, on a finite lattice, discrete.

In QCD with sea quarks, quark and antiquark numbers are not separately conserved.

States can be classified according to isospin I, strangeness S, charm C etc. and their momentum $\mathbf{p}$. Integer spin J particles are called "mesons", (containing an even number of quarks minus antiquarks) and the Fermions are called "baryons" (containing an odd number of quarks minus antiquarks).

Finally, there are the $\mathrm{O}(3) \otimes \mathcal{C}$ J^{PC} quantum numbers in the continuum. At $\mathbf{p} = 0$ the particles form $(2J + 1)$-dimensional mass-degenerate polarization multiplets.

On the cubic lattice the infinite dimensional $\mathrm{SO}(3)$ is broken down to its 24-element cubic $\mathrm{O_h}$ subgroup. This contains only five Bosonic irreps or three Fermionic irreps.

O_h Bosonic irreps

irrep.	dimension	continuum J's
A_1	1	0,4,...
A_2	1	3,...
E	2	2,4,...
T_1	3	1,3,4,...
T_2	3	2,3,4,...

J	O_h rep.	dimensions
0	A_1	1
1	T_1	3
2	$E,\ T_2$	2+3
3	$A_2,\ T_1,\ T_2$	1+3+3
4	$A_1,\ E,\ T_1,\ T_2$	1+2+3+3
...	...	...

O_h' Fermionic irreps

irrep.	dimension	continuum J's
G_1	2	$\frac{1}{2},\ \frac{7}{2},\ \ldots$
G_2	2	$\frac{5}{2},\ \frac{7}{2},\ \ldots$
H	4	$\frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},\ \ldots$

J	O_h' rep.	dimensions
$\frac{1}{2}$	G_1	2
$\frac{3}{2}$	H	4
$\frac{5}{2}$	$G_2,\ H$	2+4
$\frac{7}{2}$	$G_1,\ G_2,\ H$	2+2+4
...	...	...

$$A_1 \otimes G_1 = G_1\,, \quad E \otimes G_1 = H\,, \quad T_1 \otimes G_1 = G_1 \oplus H\,,$$
$$A_1 \otimes H = H\,, \quad E \otimes H = G_1 \oplus G_2 \oplus H\,, \quad T_1 \otimes H = G_1 \oplus G_2 \oplus H \oplus H\,.$$

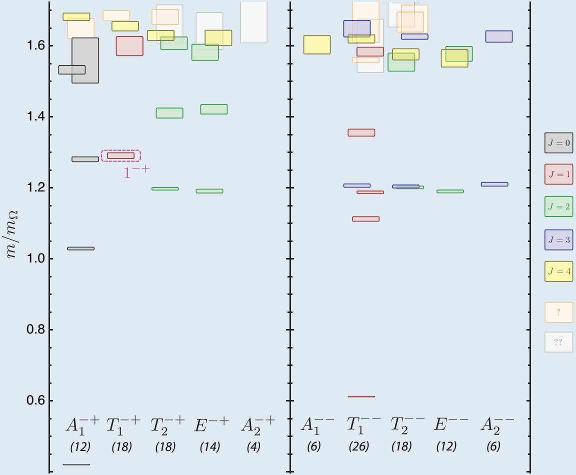
Some group theory references

- Glueballs/mesons: B Berg, A. Billoire, NPB 221 (83) 109; J Mandula, G Zweig, J Govaerts, NPB 228 (83) 91.
- Baryons: J Mandula, E Shpiz, NPB 232 (84) 180; LHPC: S Basak et al, PRD 72 (05) 074501.
- Staggered: F Gliozzi, NPB 204 (82) 419; M Golterman, J Smit, NPB 255 (85) 328; M Golterman, NPB 278 (86) 417.

Problem: continuum J^{PC} is not unique but there are hints: couplings of different operators to particular mass eigenstates and expected degeneracies in the continuum limit.

Example: light meson masses

HSC: J Dudek et al, PRL 103 (09) 262001.



Effective masses will converge faster if the ground state overlap c_1 of an operator $\hat{O}$ is maximized.

Ground state wavefunctions have no nodes and are smooth.

$\rightarrow$ employ extended operators $\bar{q}^1_{\mathbf{x}}\Gamma\Phi_{\mathbf{x},\mathbf{z}}q^2_{\mathbf{z}}$ where the smearing function Φ mimics a wavefunction.

This was done

1. Using Coulomb gauge fixing, e.g.: T DeGrand, M Hecht, PLB 275 (92) 435; T Draper, C McNeile, NPPS 34 (94) 453; BMWc: S Dürr et al, Science 322 (08) 1224.
2. Employing gauge covariant iterative procedures.
   Wuppertal=Gauss smearing: S Güsken et al, PLB 227 (89) 266.
   Jacobi smearing: UKQCD: C Allton et al, PRD 47 (93) 5128.
3. Using more general gauge covariant basis vectors.
   Distillation: HSP: M Peardon et al, PRD 80 (09) 054506.

Wuppertal smearing

Define a covariant lattice Laplacian in $d = 3$ spatial dimensions, acting on a scalar or vector field $\psi_{\mathbf{y}}$:

$$a^2 \left(\nabla^2 \psi\right)_{\mathbf{x}} = -2d\psi_{\mathbf{x}} + \sum_{j=\pm 1}^{\pm 3} \overline{U}_{\mathbf{x},j} \psi_{\mathbf{x}+a\hat{\jmath}} \,.$$

It is advisable (SESAM: GB et al, NPPS 140 (05) 609) to use a smeared covariant transporter $\overline{U}$ instead of U.
Wuppertal smearing amounts to iteratively replacing:

$$\psi^{(n+1)} = \psi^{(n)} + \frac{\delta}{1+2d\delta}\, a^2 \nabla^2 \psi^{(n)} \,.$$

$\delta = 0.3$ is a reasonable value for the free parameter. The (arbitrary) normalization convention is chosen to avoid numerical overflows for large iteration counts n. This can be employed checkerboard or "all-at-once".

We introduce a fictitious time $t = n\Delta t$. Now:

$$\frac{\partial \psi(t)}{\partial t} \approx \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = \kappa \nabla^2 \psi(t) \quad \text{with} \quad \kappa = \frac{a^2}{\Delta t} \frac{\delta}{1+2d\delta}\,.$$

This diffusion equation is formally solved by,

$$\psi(t) \approx e^{\kappa t \nabla^2} \psi(0)\,.$$

Starting from $\psi_{\mathbf{x}}(0) = \delta_{\mathbf{x0}}$ this gives a Gauss packet with the rms width of $\psi^\dagger \psi$:

$$\Delta r = d\sqrt{\kappa t} = da\sqrt{\frac{\delta}{1+2d\delta}\, n}\,.$$

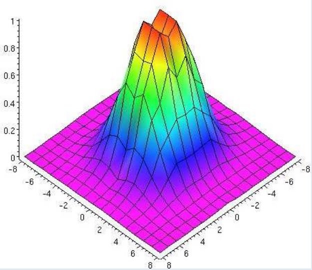
The diffusion speed is maximal for $\delta \to \infty$ ($\kappa \to a^2/(2d\Delta t)$) while the resulting wavefunction is more continuum-like for $\delta \to 0$ ($\kappa \to 0$).

APE smearing

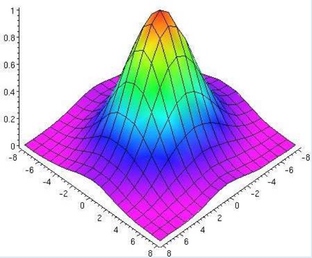
$$U_{\mathbf{x},i}^{(n+1)} = P_{\mathrm{SU}(3)}\left(\alpha\, U_{\mathbf{x},i}^{(n)} + \sum_{|j|\neq i} U_{\mathbf{x},j}^{(n)} U_{\mathbf{x}+a\hat{\jmath},i}^{(n)} U_{\mathbf{x}+a\hat{\imath},j}^{(n)\dagger}\right)$$

M Teper, PL 183B (87) 345; APE: M Albanese et al, PL 192B (87) 163

$(\alpha = 2.5,\ n_{\mathrm{APE}} = 15,\ \delta = 0.3,\ n_{\mathrm{Wup}} = 100)$ C Ehmann, GB, PoS (LAT2007) 094



No APE



With APE

Other link smearing prescriptions work just as well, e.g.

- spatial HYP A Hasenfratz, F Knechtli, PRD 64 (01) 034504
- and/or Stout C Morningstar, M Peardon, PRD 69 (04) 054501.

Jacobi smearing: iteratively expand the exponential of

$$\psi^{(n)} = e^{\kappa n \Delta t \nabla^2} \psi^{(0)} \,.$$

to $O(\kappa^n)$.

Due to radius $\propto \sqrt{n_{\rm iter}}$ iterative smearing can take a long time at small lattice spacings and may not be the optimal solution of the problem.

The point-to-all propagator $S(x|0)$ (12×12 spin-colour matrix) is obtained by solving

$$\sum_{x,\alpha,a} M(z|x)_{ca}^{\gamma\alpha} S(x|0)_{ab}^{\alpha\beta} = \delta_{z0}\delta_{cb}\delta_{\gamma\beta}$$

for all 12 $\beta \in \{1,2,3,4\}$ and $b \in \{1,2,3\}$ δ-sources.
It can be smeared at the sink with a smearing function $\Phi^{(n)}$, corresponding to n Wuppertal smearing iterations. Φ only depends on the position and colour but commutes with Γ-matrices.

Each source smearing requires a new solve of:

$$\sum_{x,\alpha,a} M(z|x)_{ca}^{\gamma\alpha} S^{\Phi}(x|0)_{ab}^{\alpha\beta} = (\Phi\delta_{\mathbf{0},b})(z|0)_{cb}\delta_{\gamma\beta}\,,$$

where $\delta_{\mathbf{0},b}$ denotes a colour vector with only one entry, at position 0 and colour b.

The Wuppertal and Jacobi smearing operators

1. are gauge covariant and singlets under O_h, charge and parity transformations,
2. are translationally invariant in space,
3. are Hermitian and commute with Γs,
4. Wuppertal has the property: $\Phi^{(n_1)}\Phi^{(n_2)} = \Phi^{(n_1+n_2)}$.

Properties 1–2 mean that smearing can be added, without affecting the irrep, gauge invariance or momentum projection.

Property 3 means that Wick-contractions remain the same.

Properties 3–4 mean that within mesons (not containing derivatives) smearing can freely be distributed between the quarks: in the equal mass case evenly, to minimize the computer time. Source smearing can be performed on the heavier quark, whose propagator is cheaper to generate. This is also statistically favourable. (At the (non-momentum-projected) source things are not symmetric with respect to the quarks).

For baryonic operators (without derivatives) there exist in general two independent smearing "positions".