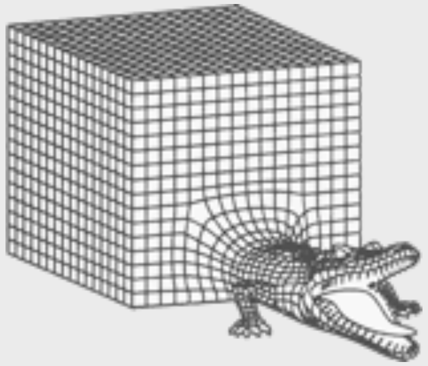
A lecture on excited hadrons

Christian B. Lang

Institut f. Physik / FB Theoretische Physik
Universität Graz

Overview

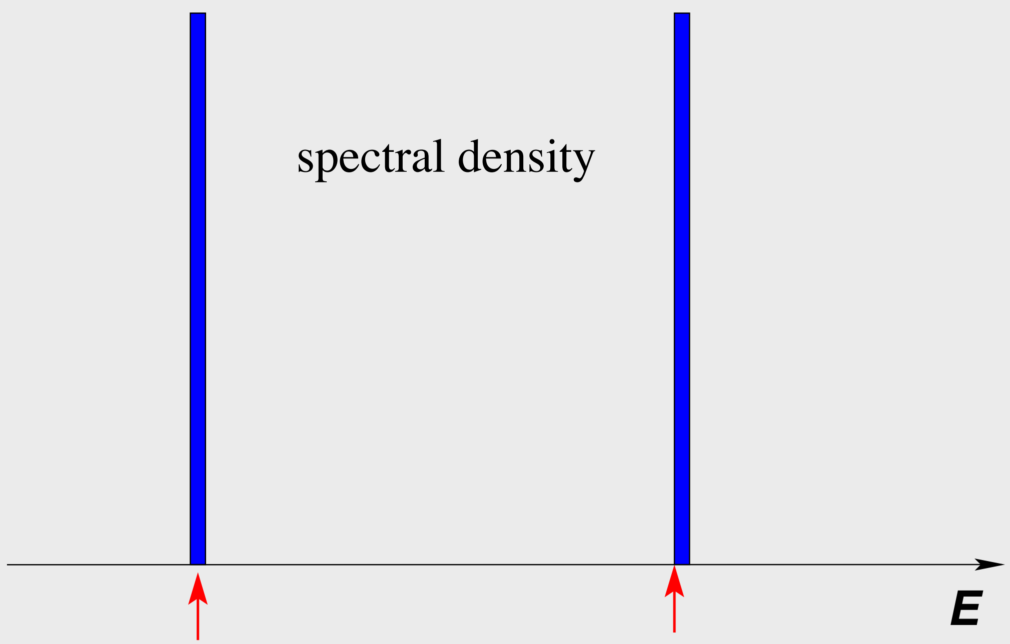


1. Motivation: Why bother?
2. What do we need?
3. Example 1: Hadron excitations
4. Example 2: Rho decay

Motivation

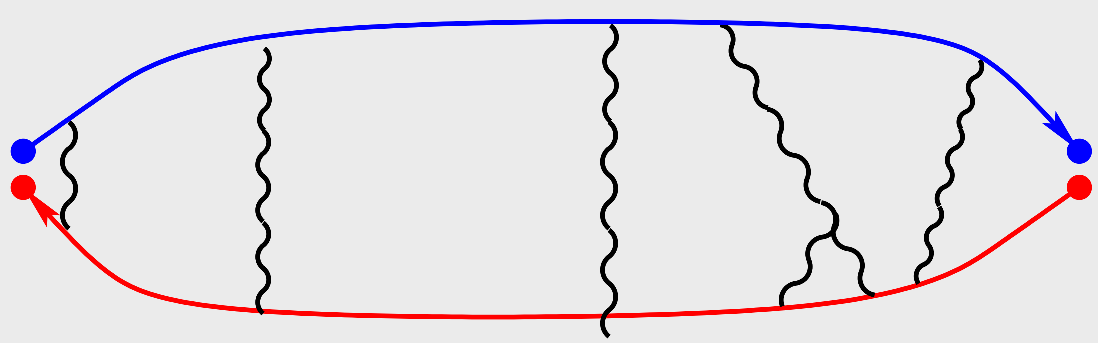
- Only p and π are stable (under strong interactions).
- Even lowest 'states' in other channels decay (ρ, N*,...) hadronically.
- For many 'particles' the classification is uncertain (multiplet, 'molecular' bound state, glueball)

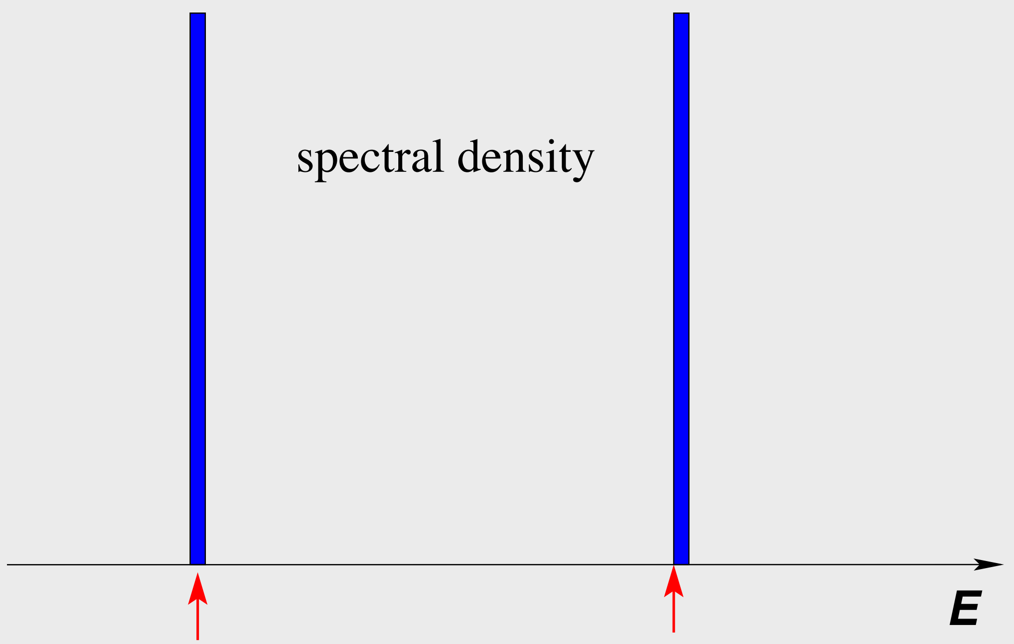
Hadron propagators and spectral function



Small volume

Hadron propagators and spectral function





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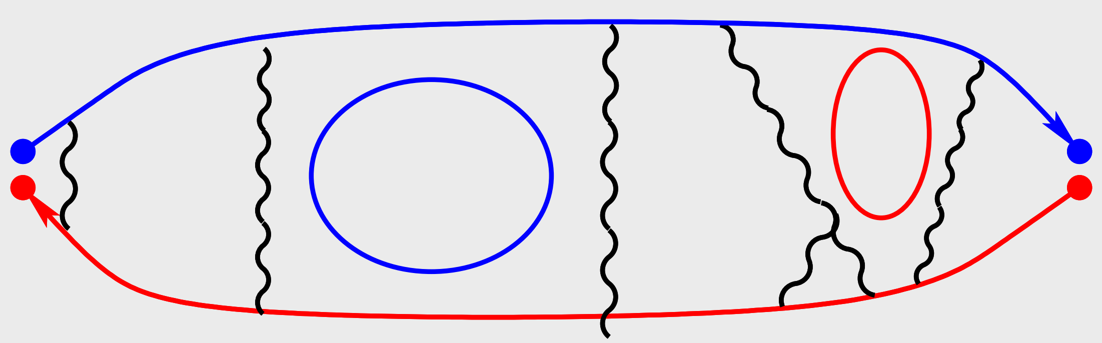

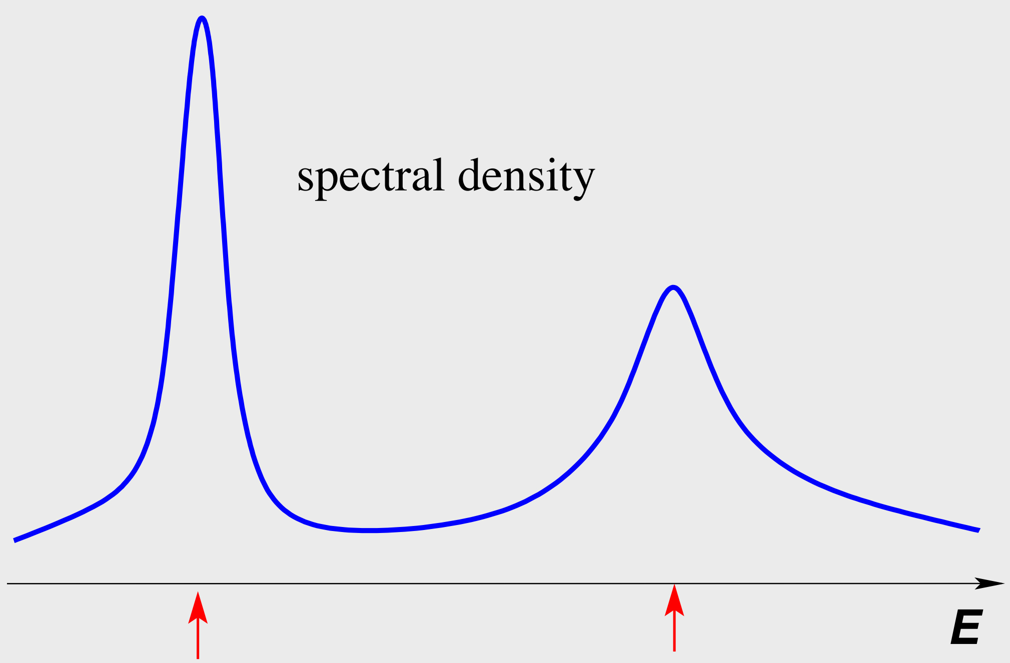

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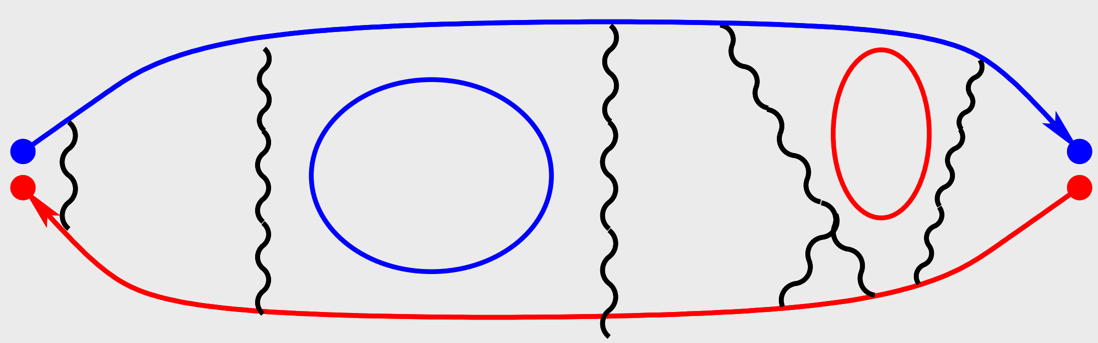

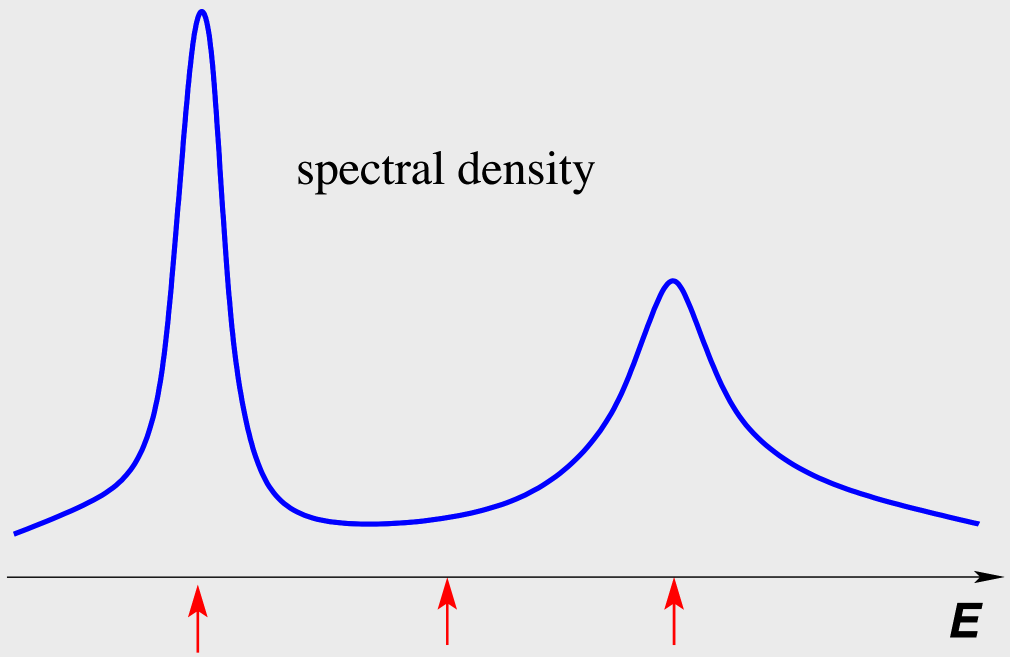

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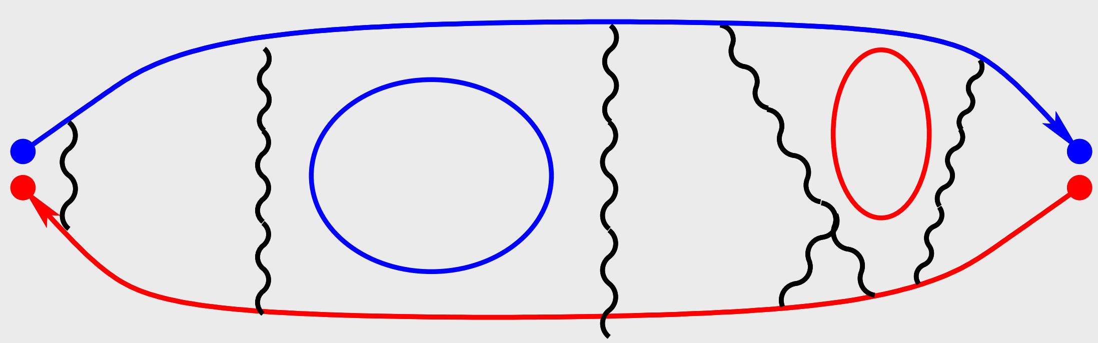
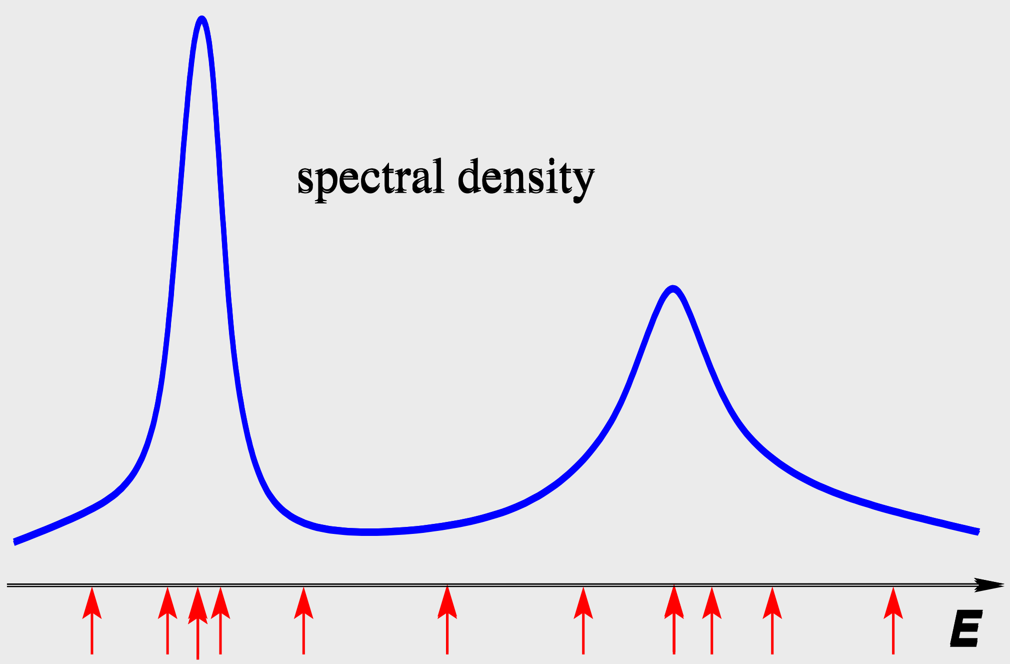


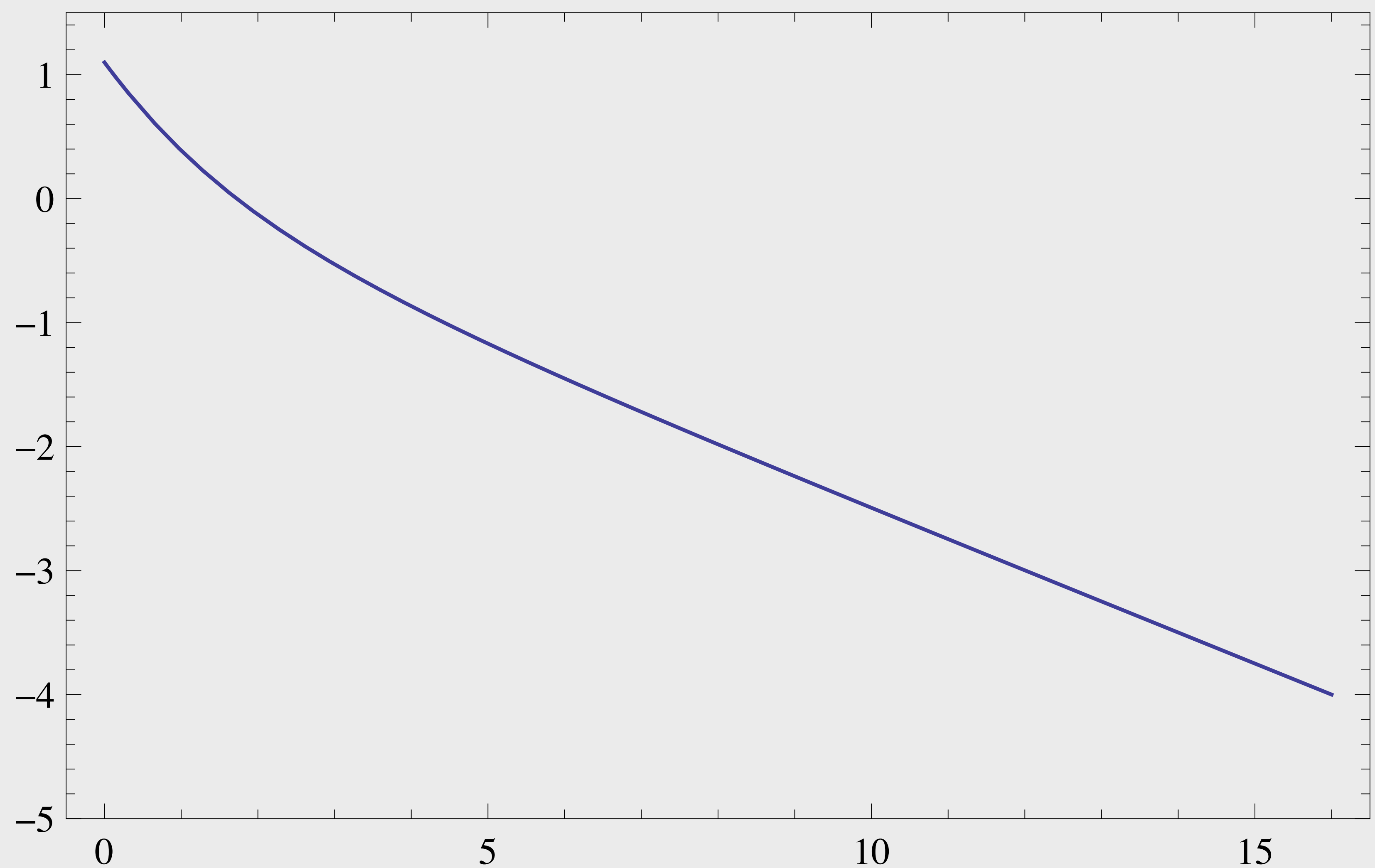


Large volume

Example for hadron correlation function

$$\langle X_i(t) X_j^\dagger(0)\rangle = C_{ij}(t) = \sum_n \langle 0|X_i|n\rangle e^{-E_n t}\langle n|X_j^\dagger|0\rangle$$

$$= a_1 e^{-E_1 t} + a_2 e^{-E_2 t} + a_3 e^{-E_3 t} \ldots$$



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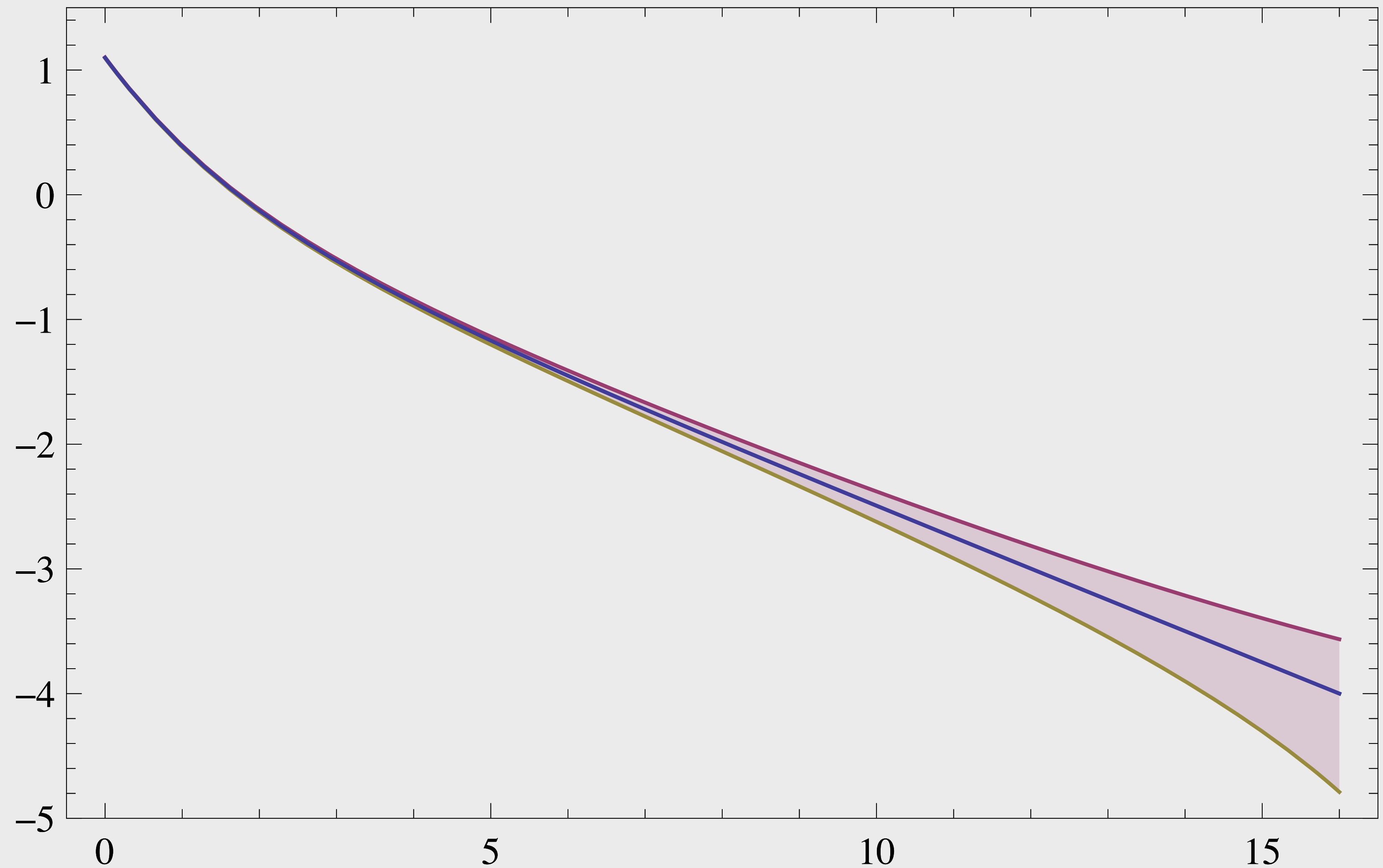
+noise



Hadron propagators and spectral function

- Finite volume: Energy levels are discrete
- Energy values: masses of hadrons
- Dynamical quarks: hadronic intermediate

states, more levels expected

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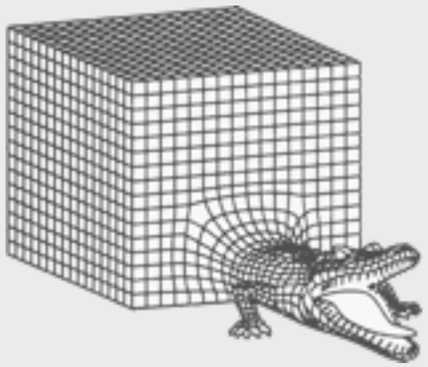
states, more levels expected

How to extract several energy levels from correlation functions?

How to interpret the (hopefully) observed values?

Overview



1. Motivation: Why bother?
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What do we need?

- Gauge configurations (with dynamical quarks)
- Quark propagators
- Hadron interpolators and propagators
- A method to extract higher energy levels
- Interpretation of the obtained energy levels

How to get energy levels...

A fit to several exponentials is usually unstable!

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- Bayesian analysis (stepwise reduction of exponential with biased estimators): minimize

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where ϕ is a stabilizing function(prior)

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- Maximum entropy method

Sasaki (05)

- Variational method (Michael, Lüscher/Wolff)

Burch (03/06)
Basak (05)

Disentangle the states

- Use several interpolators X_i
- Compute all cross-correlations

$$C_{ij}(t) = \langle X_i(t) X_j^\dagger(0) \rangle$$

- Solve the generalized eigenvalue problem:

$$C(t)\, u^{(n)} = \lambda^{(n)}\, C(t_0)\, u^{(n)}$$

- The eigenvalues give the energy levels (masses):

$$\lambda^{(n)}(t) \propto e^{-t\, E_n} \left(1 + \mathcal{O}(e^{-t \Delta E_n})\right)$$

- The eigenvectors are "fingerprints" of the state and allow to identify the "composition" of the state

Disentangle the states

"Variational method" (Lüscher/Wolff; Michael)

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Lüscher, Wolff: NPB339(90)222
Michael, NPB259(85)58
See also Blossier et al., JHEP0904(09)094

Hadron operators

We need several hadron interpolators to allow a good representation of the hadronic states!

- Several Dirac structures, e.g

Pion $\overline{u}\gamma_5 d,\ \overline{u}\gamma_t\gamma_5 d,\ \ldots$

$$N^{(i)} = \epsilon_{abc}\,\Gamma_1^{(i)}\,u_a\,\left(u_b^T\,\Gamma_2^{(i)}\,d_c - d_b^T\,\Gamma_2^{(i)}\,u_c\right)$$

$$\Delta_\mu = \epsilon_{abc}u_a(u_b^T C\gamma_\mu u_c)$$

	$\Gamma_1^{(i)}$	$\Gamma_2^{(i)}$
$i=1$	1	$C\gamma_5$
$i=2$	γ_5	C
$i=3$	i	$C\gamma_4\gamma_5$

(projected to definite parity)

Quark sources

Different quark source shapes:

- Point
- Wall
- Stochastic
- Separable sources (see: distillation)

- Spatially smeared quarks (Jacobi smearing)

- Derivative sources

$$S_0 = \delta(m - m_0)\, \delta_{\alpha\alpha_0}\, \delta_{a a_0}$$

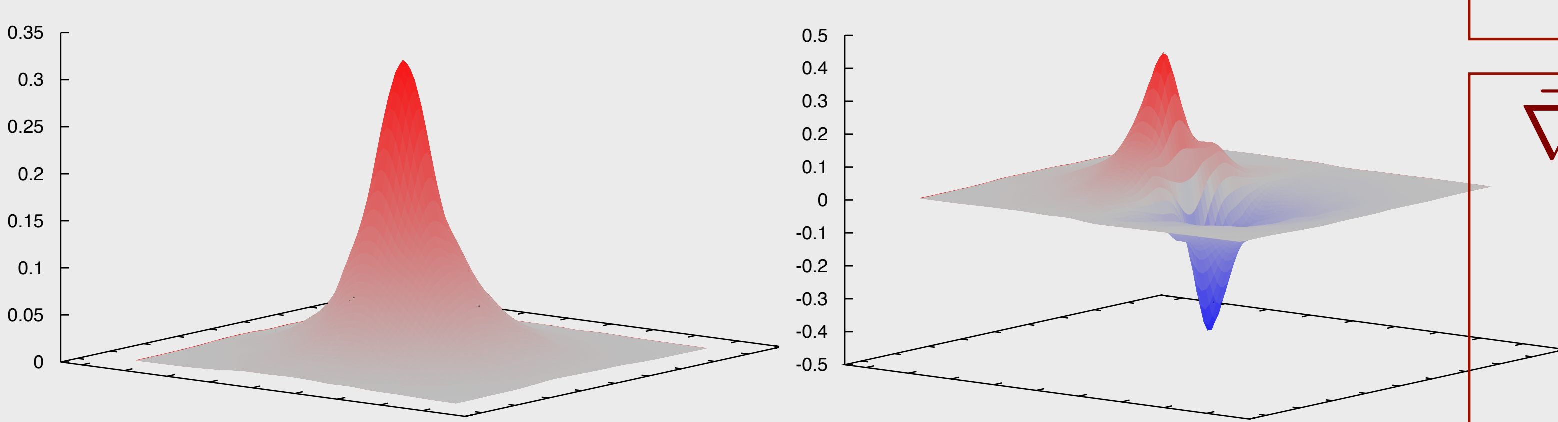
$$G = D^{-1}\, S_0 \to D\, G = S_0$$

$$S = \sum_{n=0}^{N} \kappa^n\, H^n S_0$$

$$H(\vec{n},\vec{m}\,) = \sum_{j=1}^{3} \Big[U_j(\vec{n},0)\, \delta(\vec{n}+\hat{j}\,,\vec{m}\,) + U_j(\vec{n}-\hat{j}\,,0)^\dagger\, \delta(\vec{n}-\hat{j}\,,\vec{m}\,) \Big]$$

$$\vec{\nabla}_i(\vec{x},\vec{y}) = U_i(\vec{x},0)\delta_{\vec{x}+\hat{i},\vec{y}} - U_i(\vec{x}-\hat{i},0)^\dagger \delta_{\vec{x}-\hat{i},\vec{y}}$$

$$S_{\partial_i} = \vec{\nabla}_i S$$



Separable sources

LapH smearing and distillation

Peardon et al. PRD80(09)054506

e.g. meson:

$$\bar{u}_x S^\dagger(x,x') D_{x',y'} \Gamma S(y',y) d_y$$

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$$\langle M(0) M(t) \rangle = \sum_{ijkn} \langle \bar{u}\, g_i\, g_i^\dagger\, D\Gamma\, g_j\, g_j^\dagger\, d\, \bar{d}\, g_k\, g_k^\dagger\, D\Gamma\, g_n\, g_n^\dagger\, u \rangle$$

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$$= \sum_{ijkn} \phi_{ij}(0)\, \tau_{jk}(0,t)\, \phi_{kn}(t)\, \tau_{ni}(t,0)$$

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$$= \sum_{ijkn} \phi_{ij}(0) \, \tau_{jk}(0,t) \, \phi_{kn}(t) \, \tau_{ni}(t,0)$$

Laplacian Heaviside smearing

Perambulator: Propagator from source i to sink j

Distillation operator: Spectral representation in terms of eigenvectors of the 3D Laplacian

$$S(x,y) = \sum_i^N c_i g_i(x) g_i^\dagger(x')$$

e.g. spectral representation of Gaussian $S(x,x') = \exp(\sigma \vec{\nabla}^2)$

or, for $c_i = 1,\ N = 3N_s^3 \quad \rightarrow \quad S(x,x') = \delta(x - x')$

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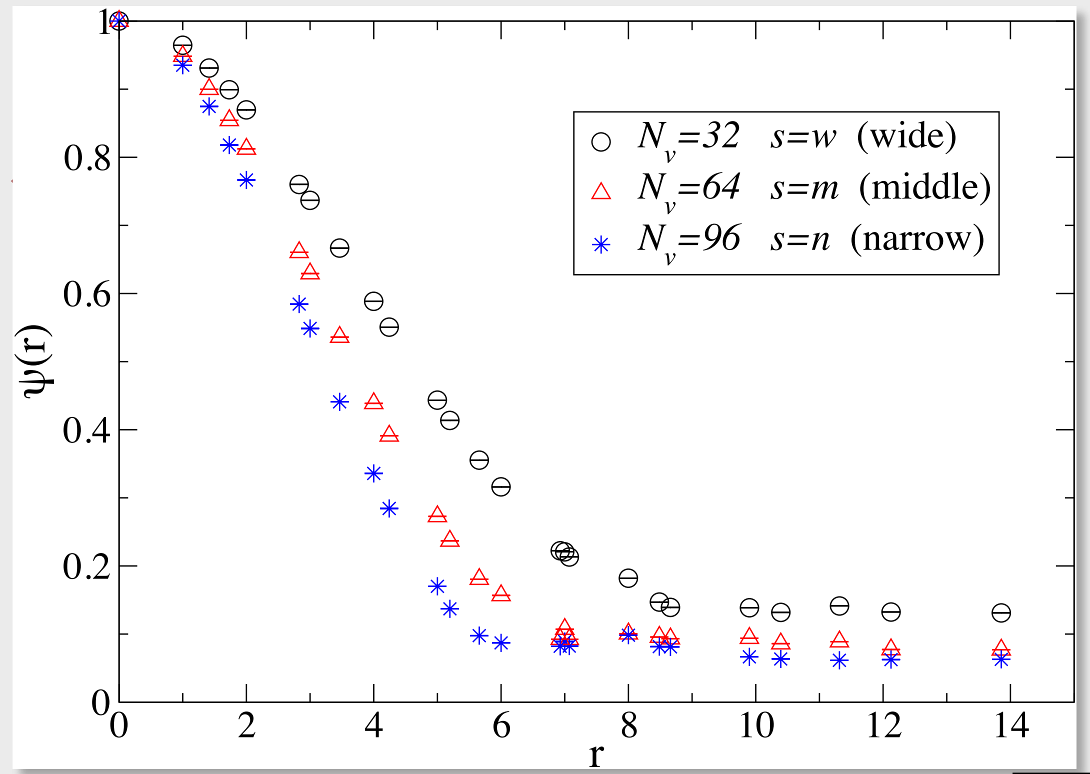
or, for $c_i = 1,\ N = 32,\ 64,\ 96$



Plus/minus

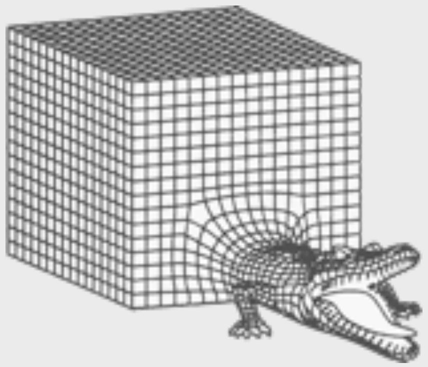
- All hadron-hadron correlators (and 3-point functions) can be constructed from the perambulators.

- High flexibility for interpolator structure: $\Gamma, \ \vec{\nabla}_i, \ \exp(\mathrm{i}\,\vec{p}\cdot\vec{x})$

- Needs many ($N x N_T$) Dirac operator inversions (perambulators)!

- Volume scaling! Stochastic dilution?

Overview



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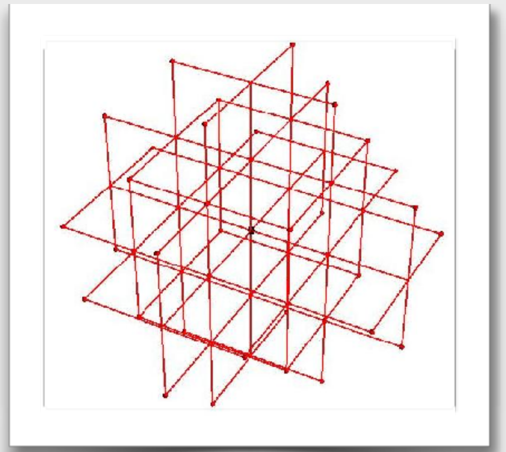
Example 1: Baryons and mesons

Gattringer et al. PRD 79 (2009) 054501
Engel et al. PRD 82 (2010) 034505

Simulation with 2 sea quarks:



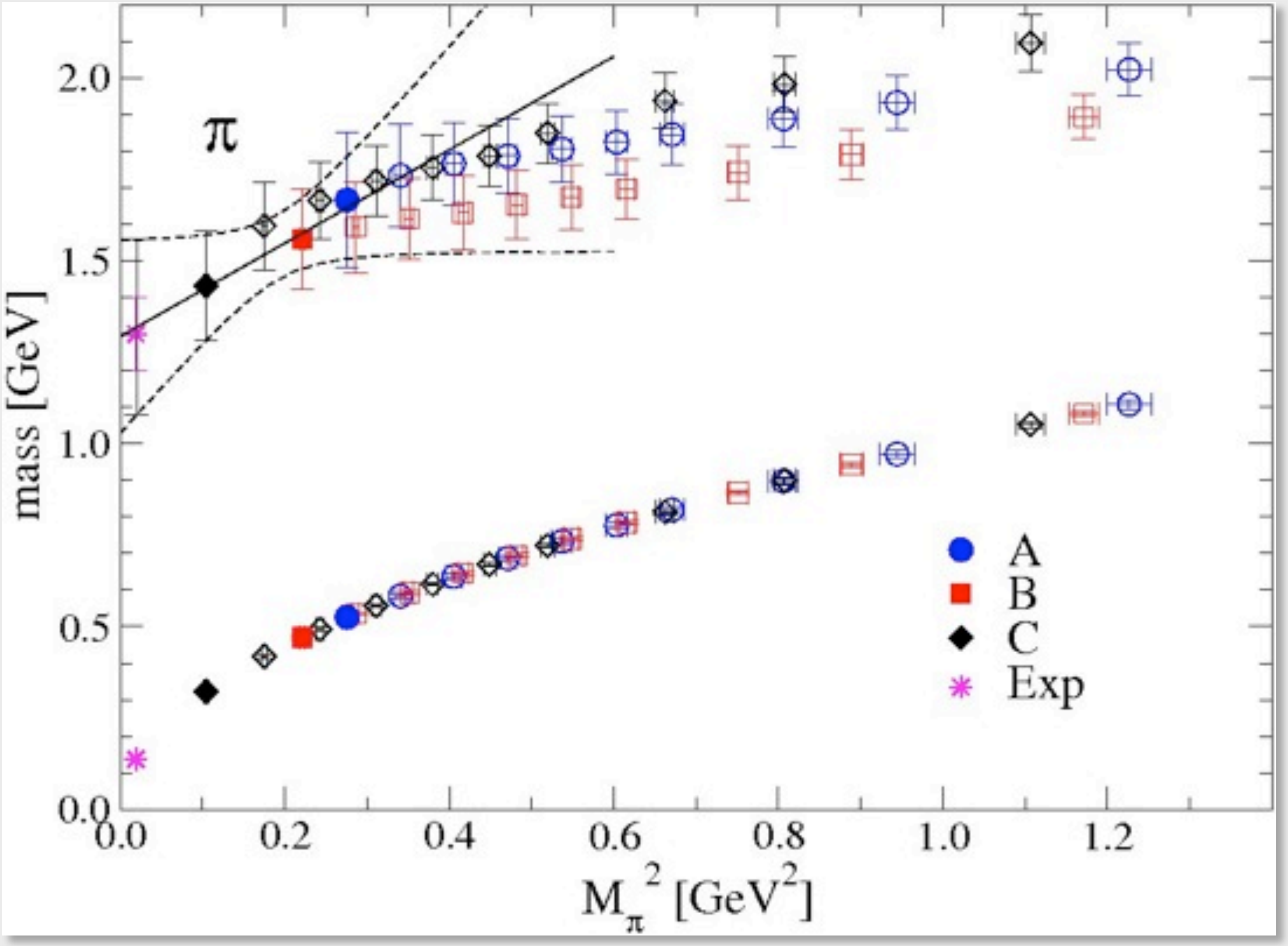
(Gattringer, PRD63(2001)114501)

- Chirally improved (approximate GW) action + stout smearing
- Lüscher-Weisz gauge action
- HMC: Hasenbusch preconditioning (2 pseudofermions), chron. inverter, mixed prec. inverter
- 3(7) ensembles of 200-300 configurations
- 16^3x32 (size 2.4 fm)
- Pion masses 260..540 MeV

see, e.g., also other collab.s:
Edwards et al., arXiv:1104.5152
and citations in the review
Lin, arXiv:1106.1608

0^{-+}: π(140), π' (1300)

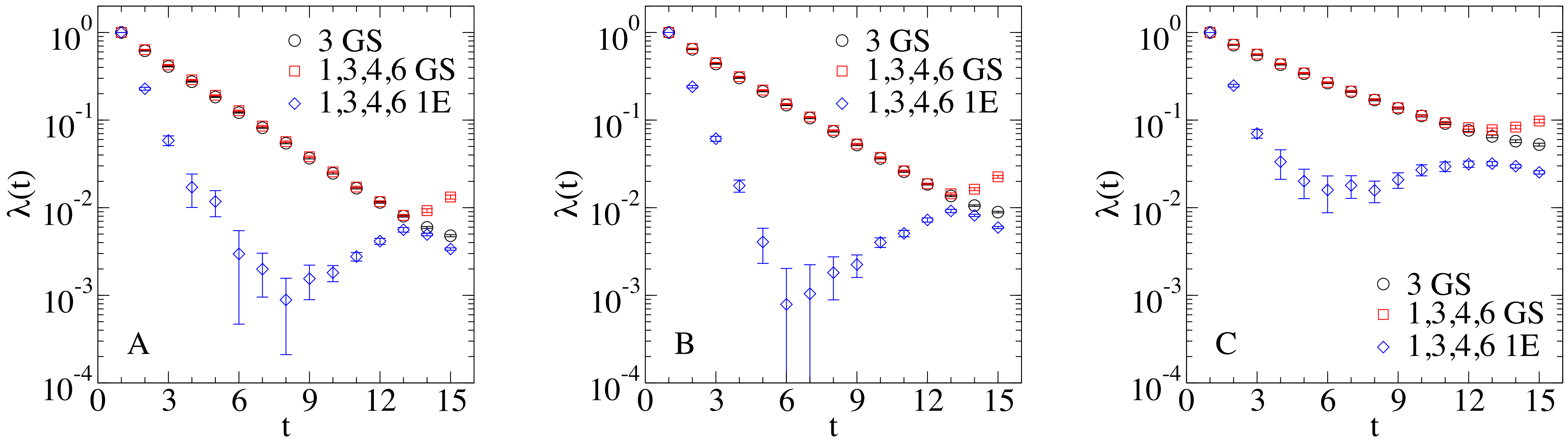
Ground state and excited pion state, including partially quenched data



Engel et al. PRD 82 (2010) 034505

Attention: Signals from the future!

Multi-operator (variational) analysis at small pion masses:
the back-running pion limits the observation range for the excited state!

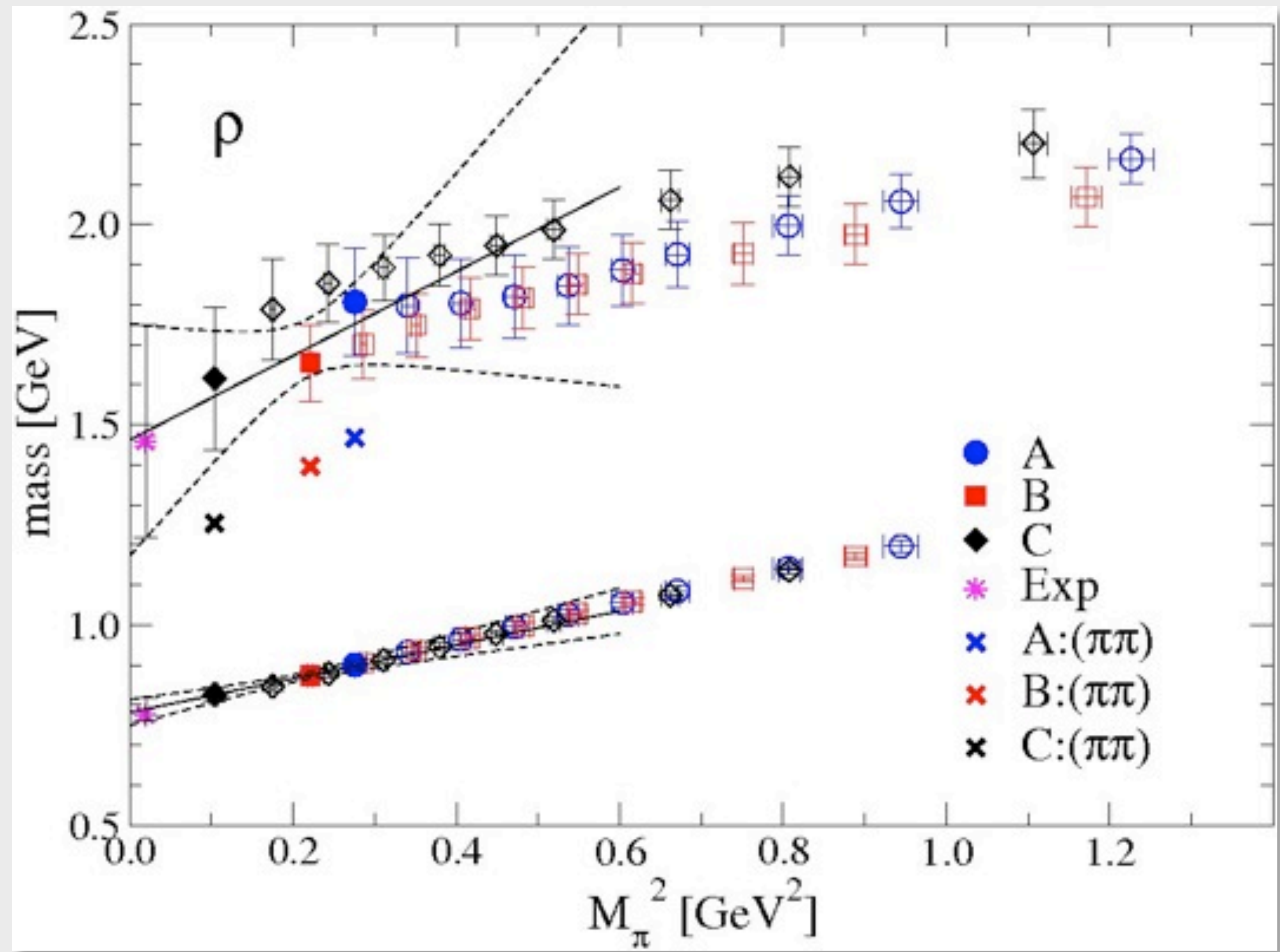


Similar effects for combinations of mesons running forward and backward, cf. Prelovsek et al., Phys. Rev. D82 (2010) 094507

Possible cures: larger time-size, modified boundary conditions, ...

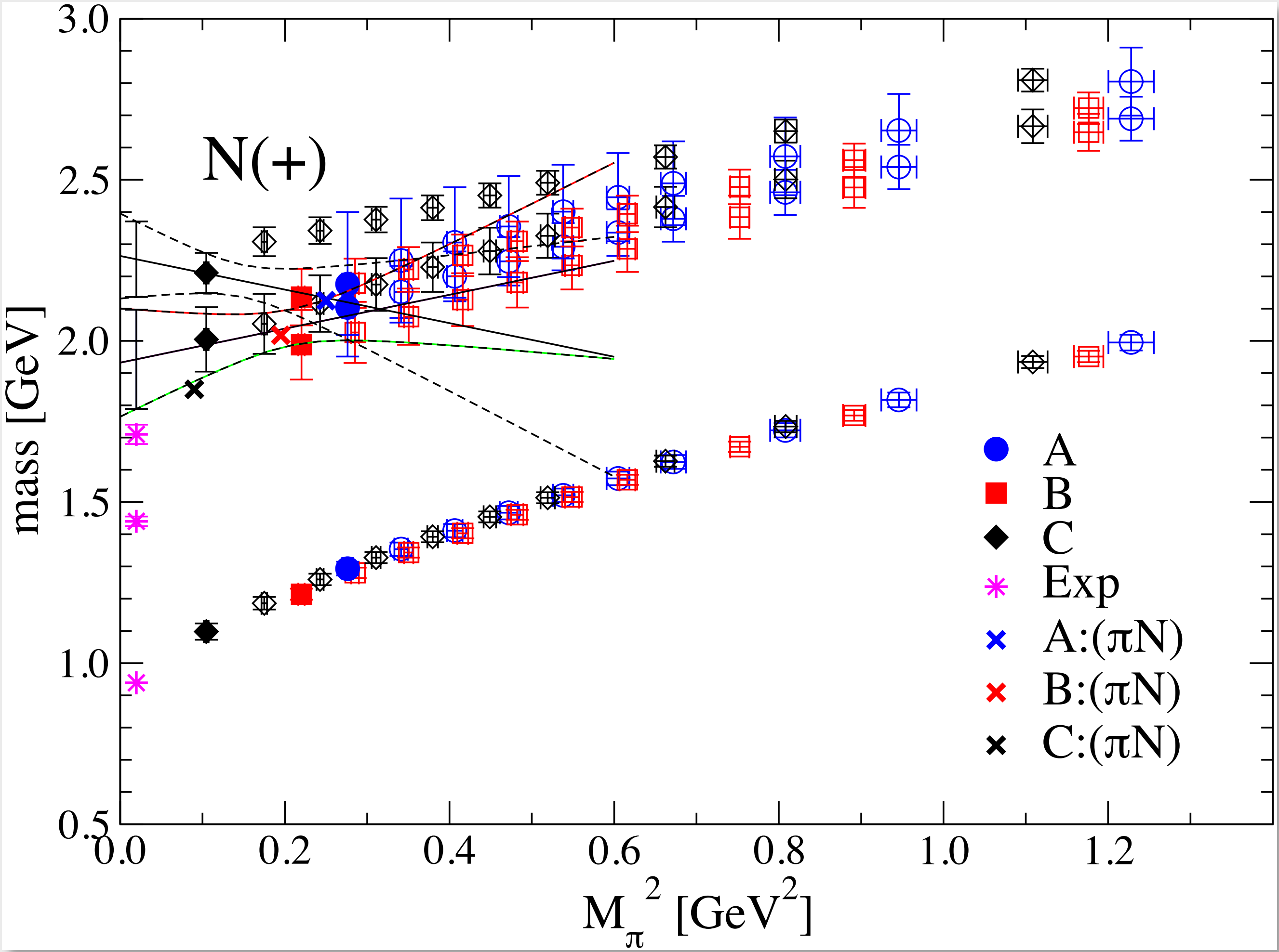
1^{--}: ρ(770), ρ ‘(1450)

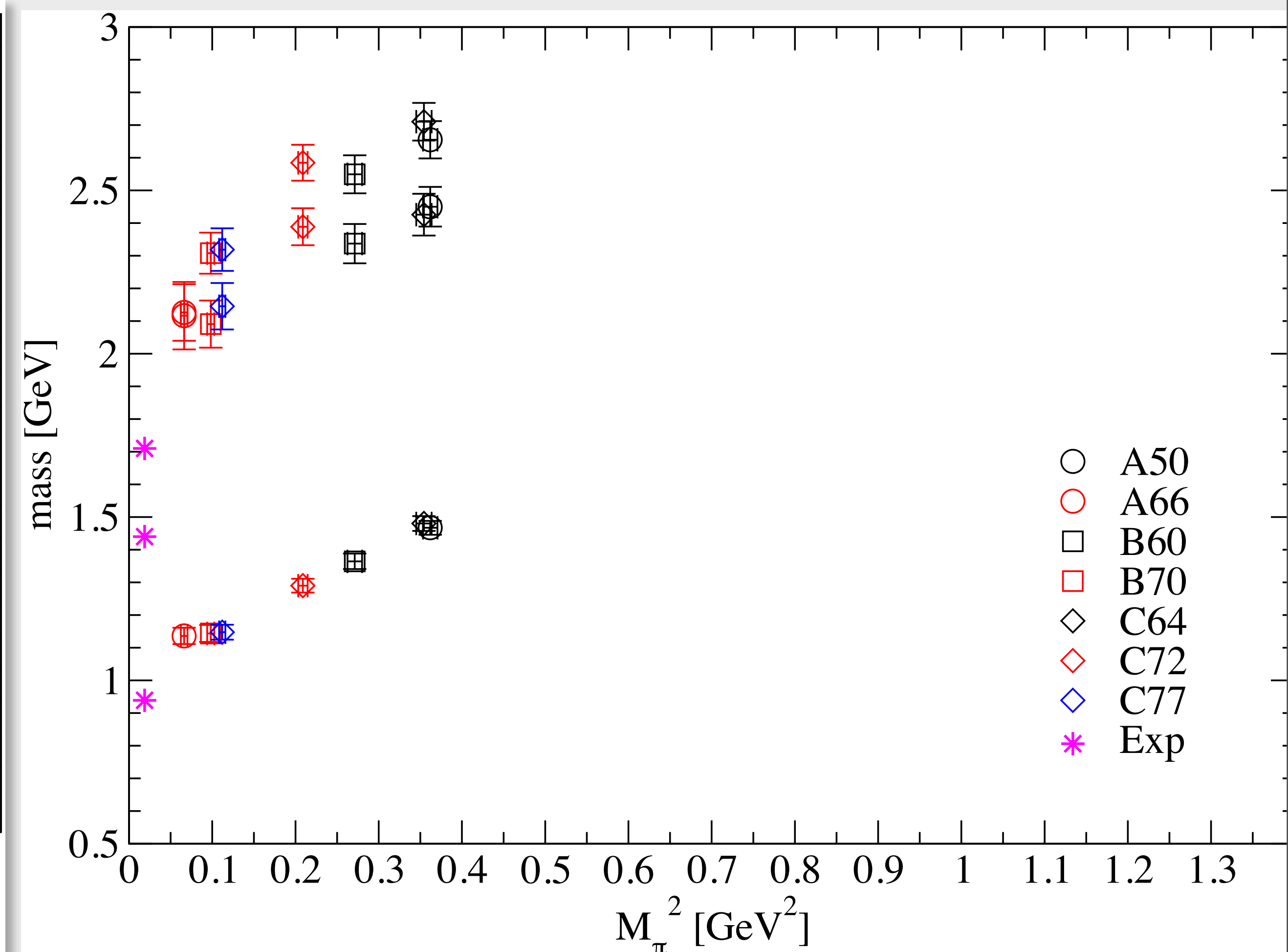
- No decay yet (p-wave)
- More contamination with higher excitations, thus t_0=2 is preferable.
- Optimal combination chosen for each data set.
- 2nd excitation ρ(1720) signal is seen for some combinations of interpolators
- Challenge: Where is the ππ state?



Engel et al. PRD 82 (2010) 034505

$1/2^+$: N(940), N(1440), N(1710)





Similar to quenched results! Two excitations (higher one vague), too high up! Challenge: Roper?

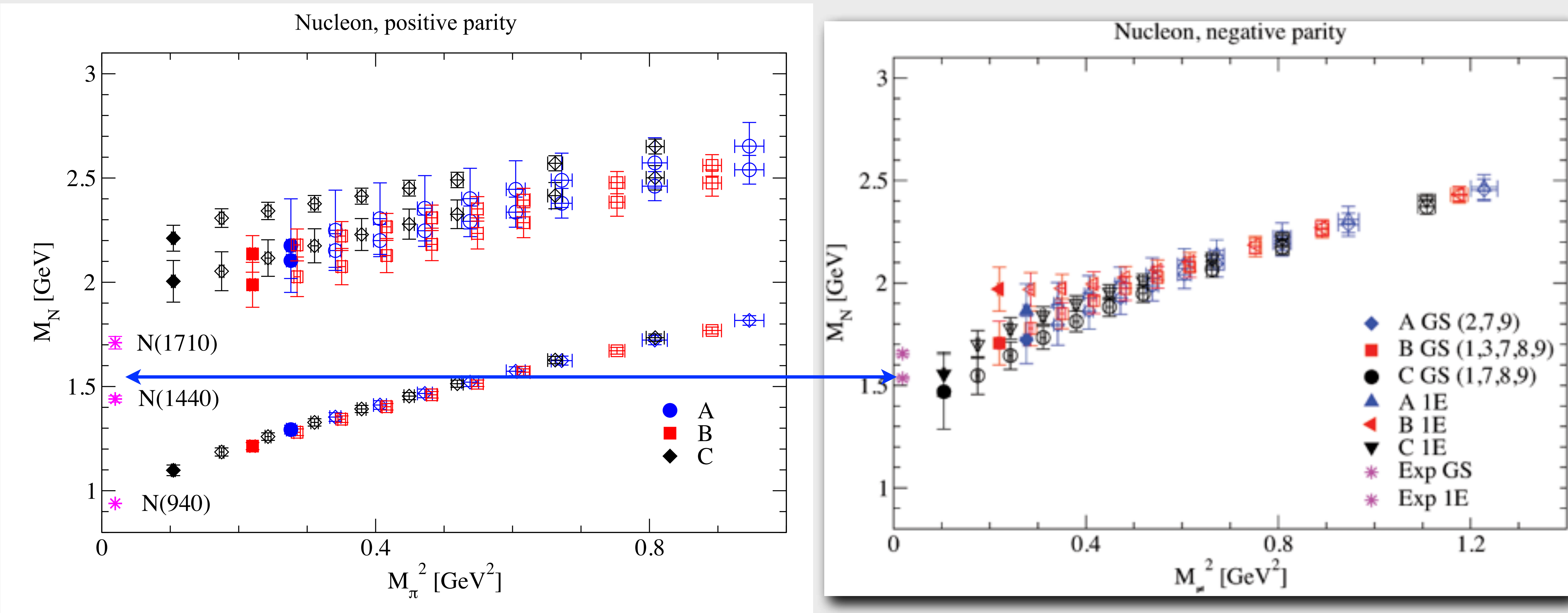
(cf. Mahbub et al. arXiv:1011.5724v1 ?)

Preliminary: Only dynamical points, common set of interpolators, results compatible

Engel et al. PRD 82 (2010) 034505,
Engel et al., prelim.

The Roper puzzle

Level crossing (from + - + - to + + - -)?



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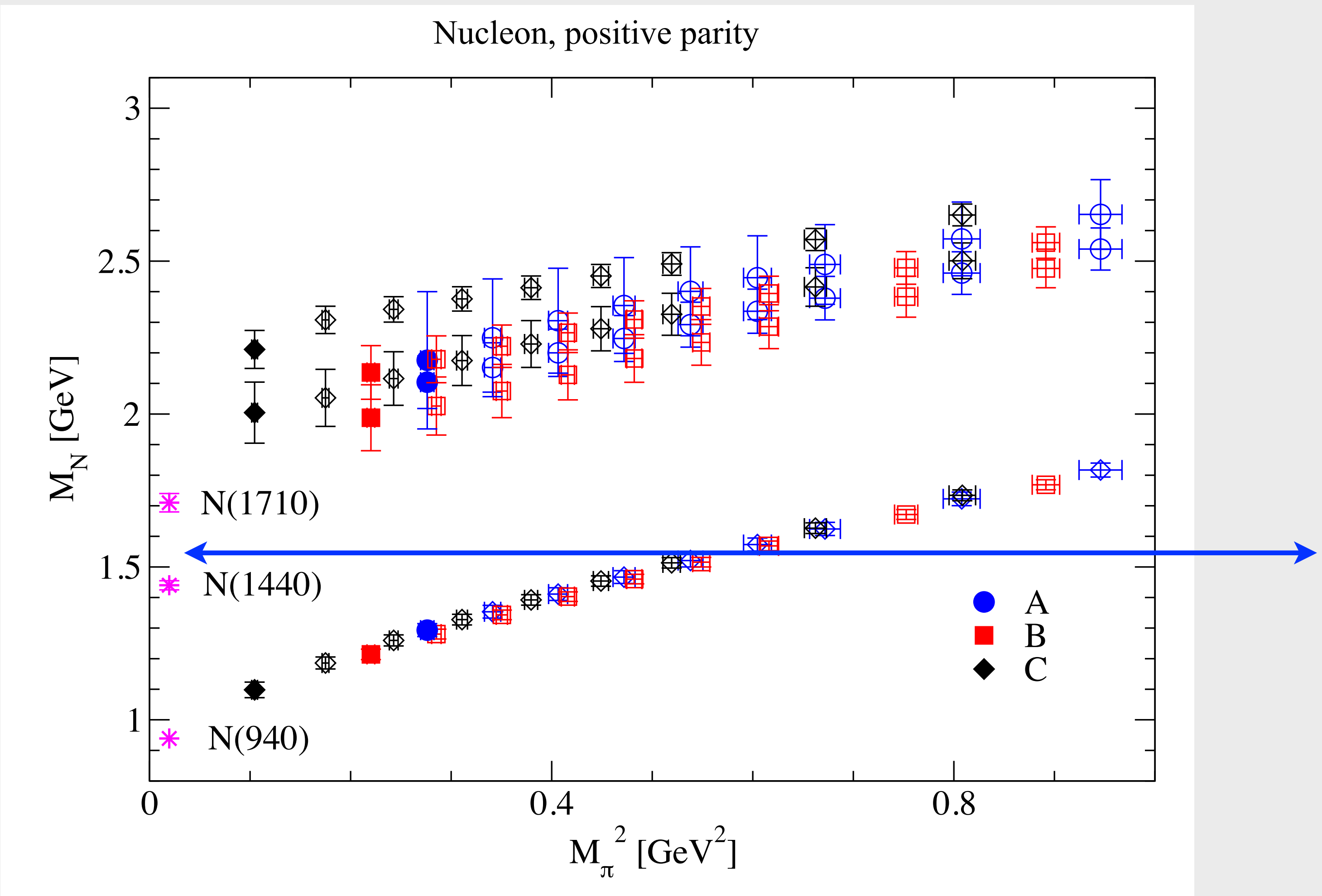

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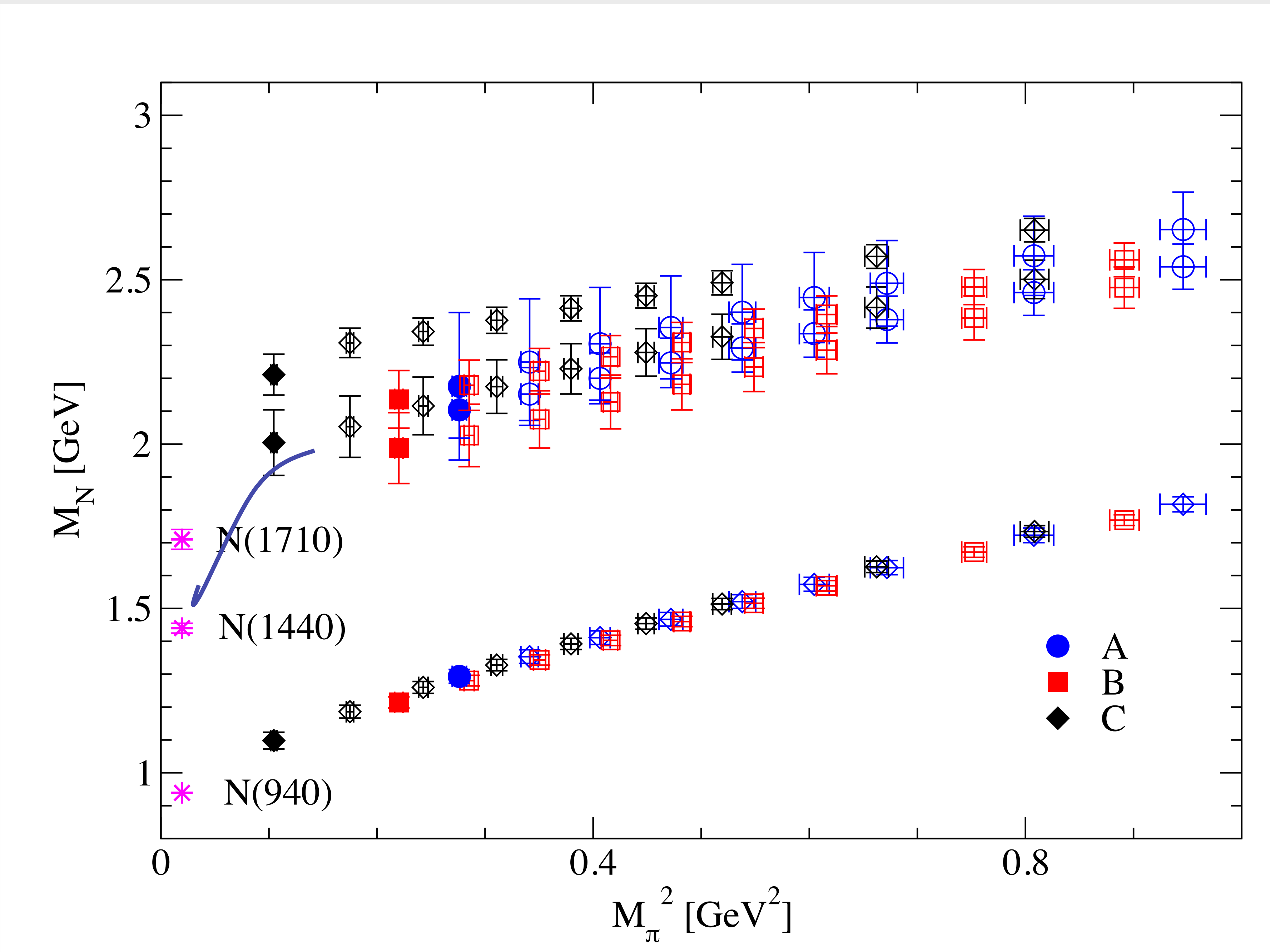
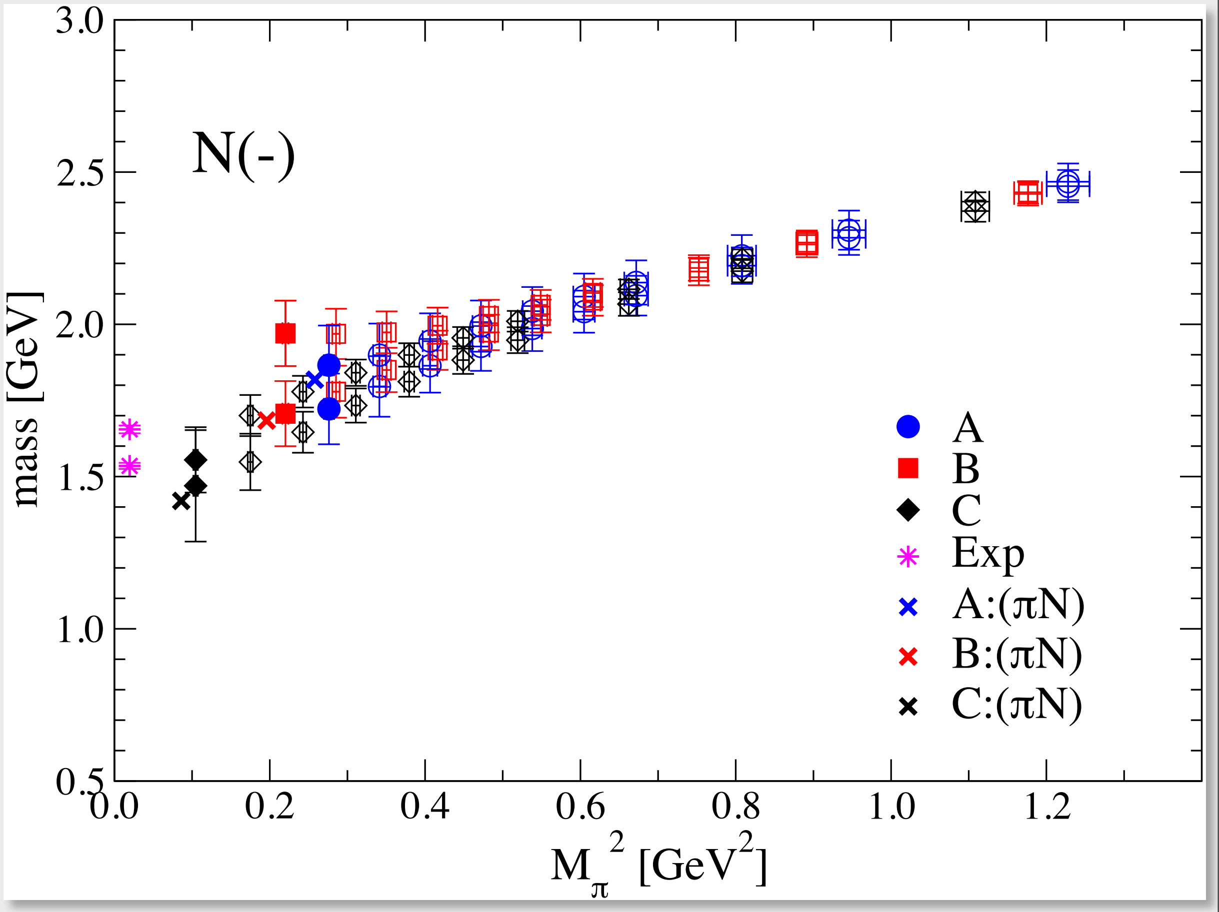


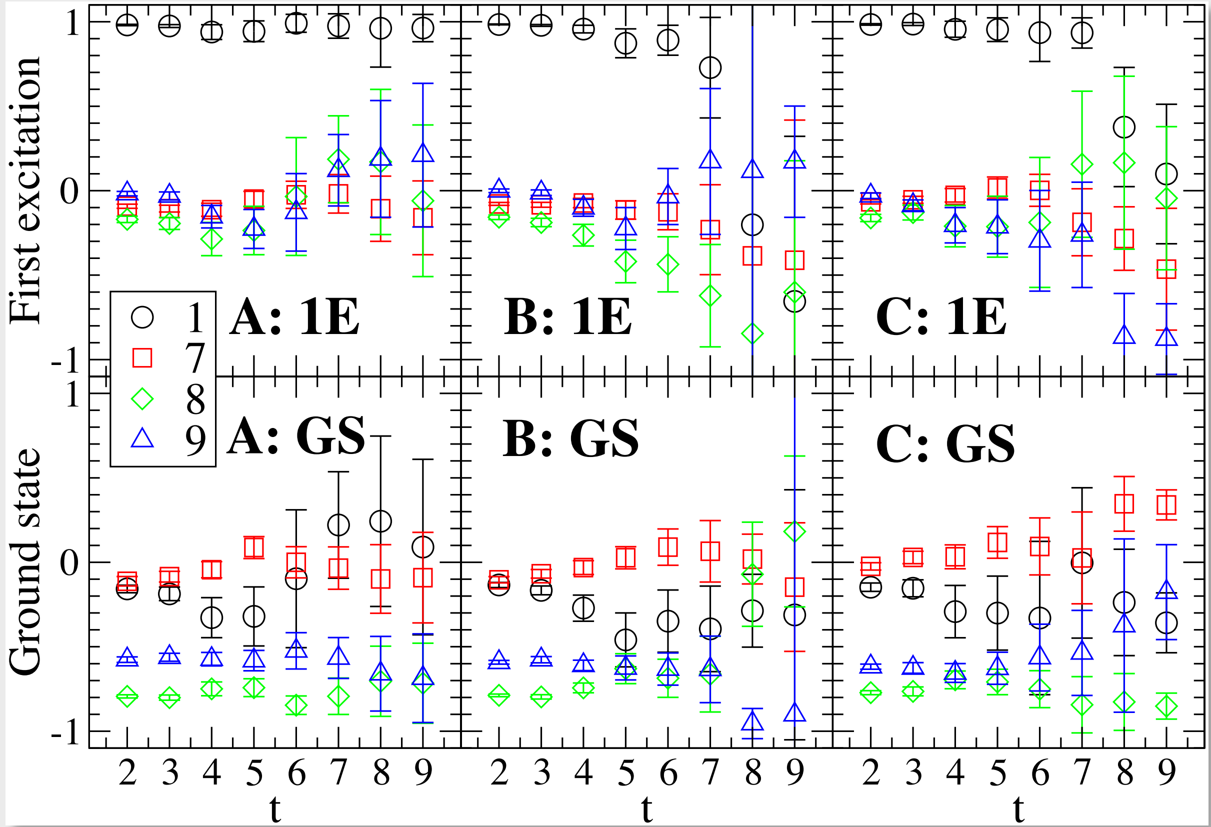
1/2⁻: N(1535), N(1650)

Two states seen, but not clearly resolvable; lower level dominated by χ_2

Challenge: Is one level a πN in s-wave signal?

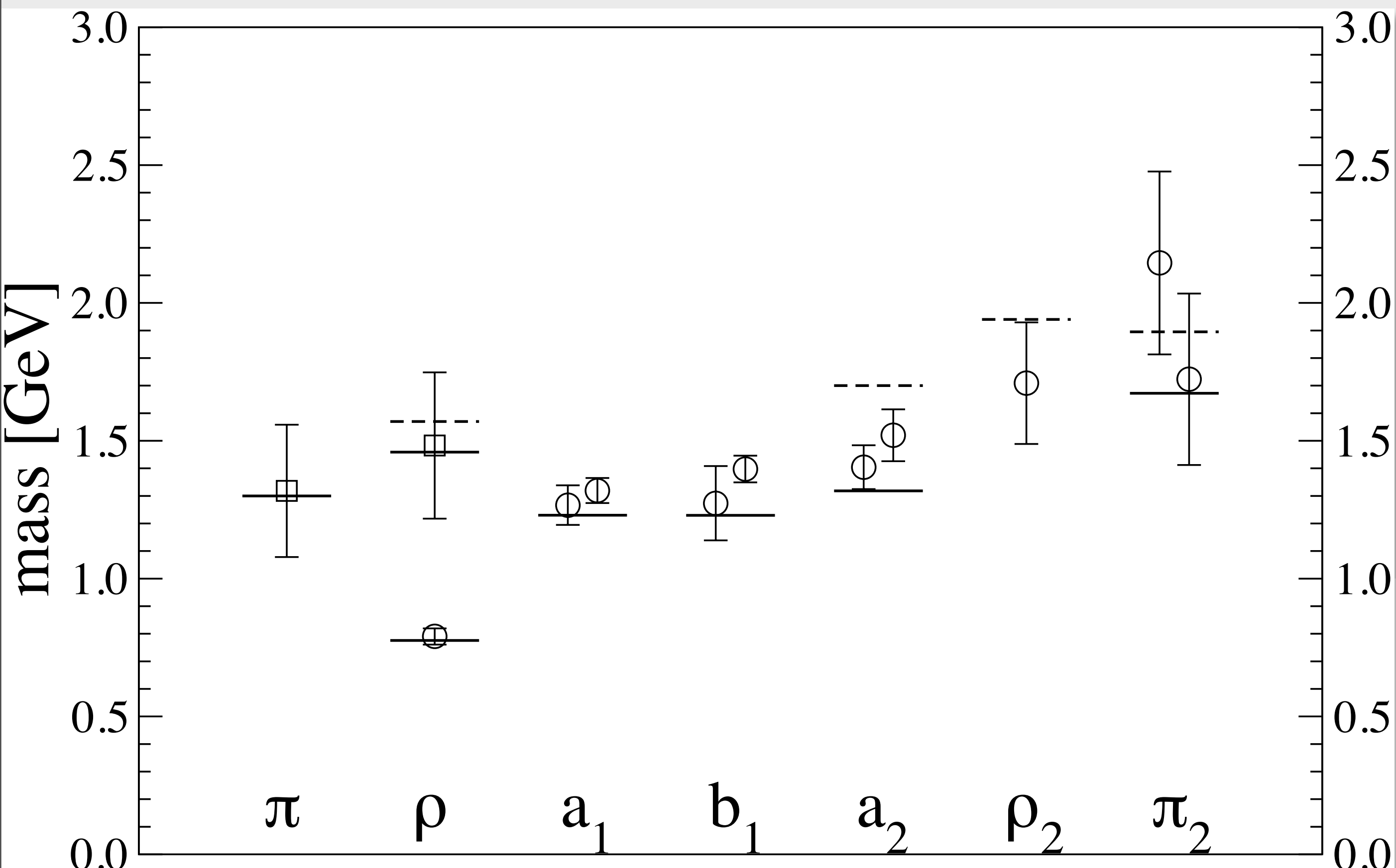
(pro/con: eigenvectors are stable for A,B,C: no level crossing, no change of splitting towards higher valence masses? But: g_A?)

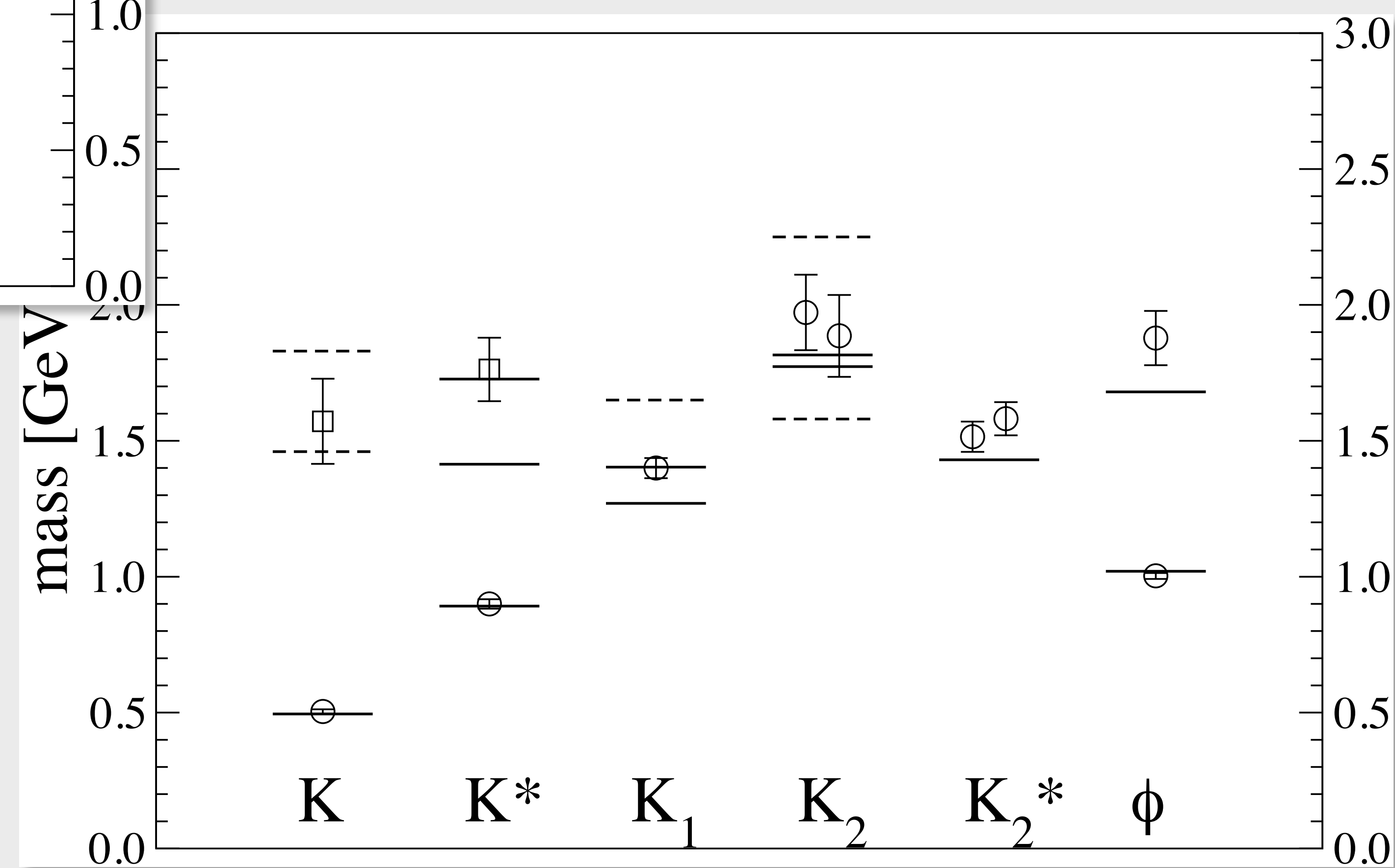




Engel et al. PRD 82 (2010) 034505

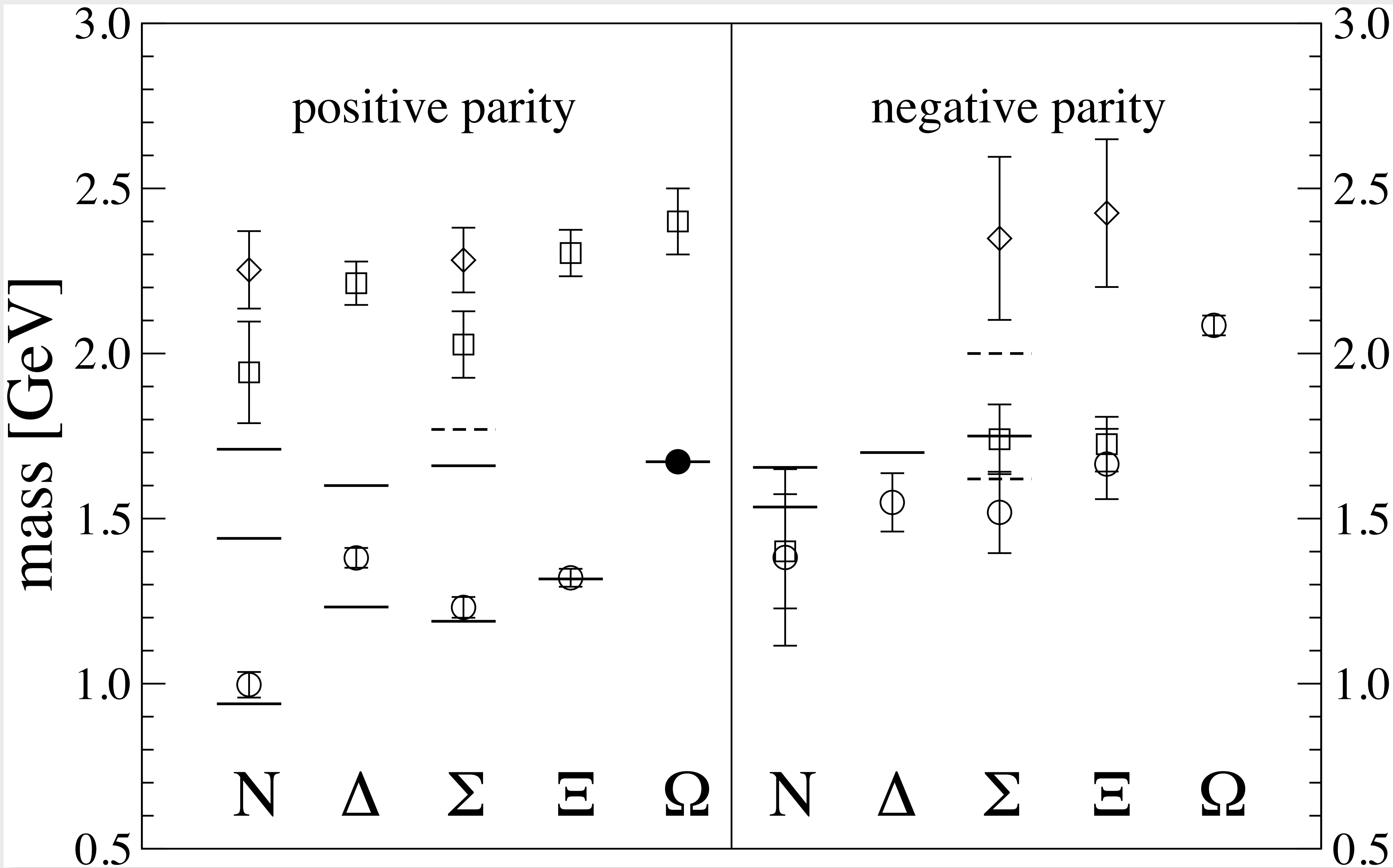
Meson summary





Engel et al. PRD 82 (2010) 034505

Baryon summary



Engel et al. PRD 82 (2010) 034505

Challenge

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We need to include these in the set of hadron interpolators!

see also: Bulava et al. PRD82(10)014507

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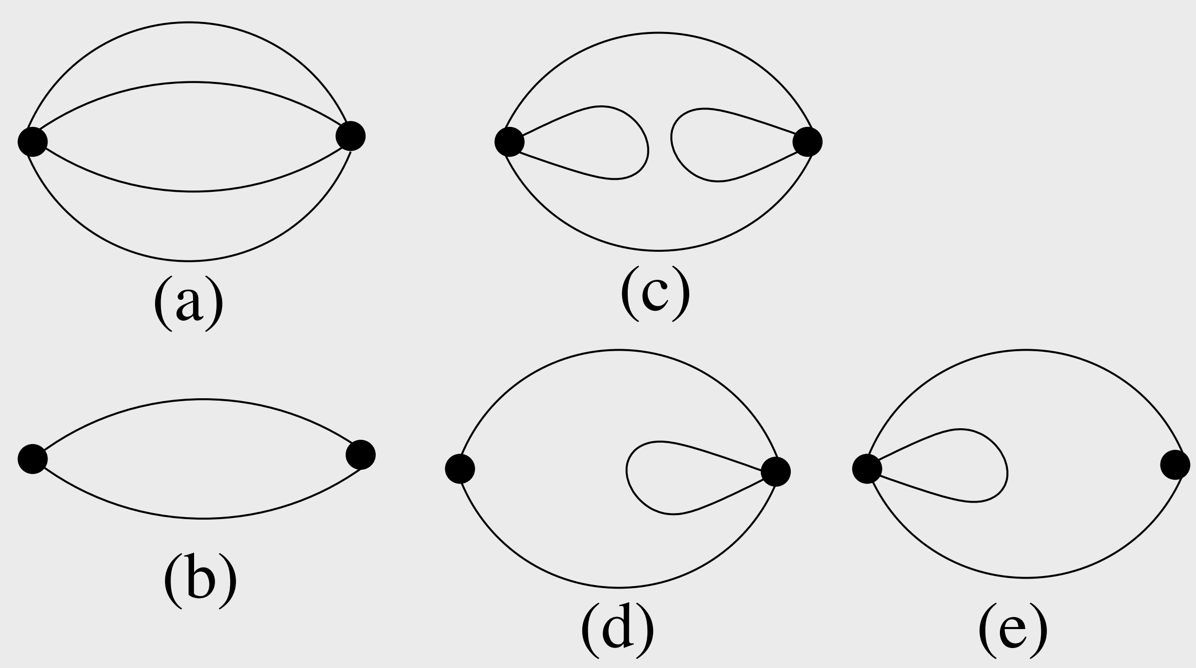
These involve (partially) disconnected contractions!



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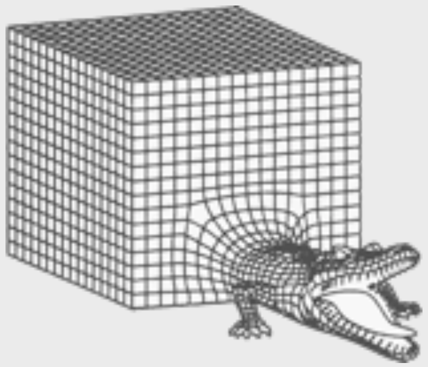

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Example 2: Rho decay

CBL,Mohler,Prelovsek,
arXiv:1105.5636

- Study $\pi\pi \to \rho \to \pi\pi$ scattering (p wave)
- N_f=2, improved Wilson fermions;
  280 configurations from A. Hasenfratz et al.
  (Thanks! See Hasenfratz et al., PRD78(08)014515,054511)
- Up to 18 interpolators
- Non-zero momentum states
- Determine p-wave phase shift

also:
Aoki et al., PoS LAT10(10)108
Feng et al., PoS LAT10(10)104
Frison et al. PoS LAT10(10)139

Interpolators

$$\mathcal{O}_1^s(t) = \sum_{\mathbf{x},i} \frac{1}{\sqrt{2}}\ \bar{u}_s(x)\ A_i\gamma_i\ \mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}\ u_s(x)\ - \{u_s \leftrightarrow d_s\} \qquad (s=n,m,w)\ ,$$

$$\mathcal{O}_2^s(t) = \sum_{\mathbf{x},i} \frac{1}{\sqrt{2}}\ \bar{u}_s(x)\ \gamma_t A_i\gamma_i\ \mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}\ u_s(x)\ - \{u_s \leftrightarrow d_s\} \qquad (s=n,m,w)\ ,$$

$$\mathcal{O}_3^s(t) = \sum_{\mathbf{x},i,j} \frac{1}{\sqrt{2}}\ \bar{u}_s(x)\overleftarrow{\nabla}_j\ A_i\gamma_i\ \mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}\ \overrightarrow{\nabla}_j u_s(x)\ - \{u_s \leftrightarrow d_s\} \qquad (s=n,m,w)\ ,$$

$$\mathcal{O}_4^s(t) = \sum_{\mathbf{x},i} \frac{1}{\sqrt{2}}\ \bar{u}_s(x)\ A_i\ \tfrac{1}{2}[\mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}\ \overrightarrow{\nabla}_i - \overleftarrow{\nabla}_i \mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}] u_s(x)\ - \{u_s \leftrightarrow d_s\} \qquad (s=n,m,w)\ ,$$

$$\mathcal{O}_5^s(t) = \sum_{\mathbf{x},i,j,k} \frac{1}{\sqrt{2}}\ \epsilon_{ijl}\ \bar{u}_s(x)\ A_i\gamma_j\gamma_5\ \tfrac{1}{2}[\mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}\overrightarrow{\nabla}_l - \overleftarrow{\nabla}_l \mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}] u_s(x) - \{u_s \leftrightarrow d_s\} \quad (s=n,m,w)\ ,$$

$$\mathcal{O}_6^{s=n}(t) = \frac{1}{\sqrt{2}}[\pi^+(\mathbf{p_1})\pi^-(\mathbf{p_2}) - \pi^-(\mathbf{p_1})\pi^+(\mathbf{p_2})]\ , \qquad \pi^\pm(\mathbf{p_i}) = \sum_{\mathbf{x}} \bar{q}_n(x)\gamma_5\tau^\pm \mathrm{e}^{\mathrm{i}\mathbf{p_i}\mathbf{x}} q_n(x)\ .$$

CBL,Mohler,Prelovsek,
arXiv:1105.5636

Interpolators

$$\mathcal{O}_1^s(t) = \sum_{\mathbf{x},i} \frac{1}{\sqrt{2}}\ \bar{u}_s(x)\ A_i\gamma_i\ \mathrm{e}^{\mathrm{i}\mathbf{P}\mathbf{x}}\ u_s(x)\ \ - \{u_s \leftrightarrow d_s\} \qquad (s=n,m,w)\ ,$$

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... include ππ operator

CBL,Mohler,Prelovsek, arXiv:1105.5636

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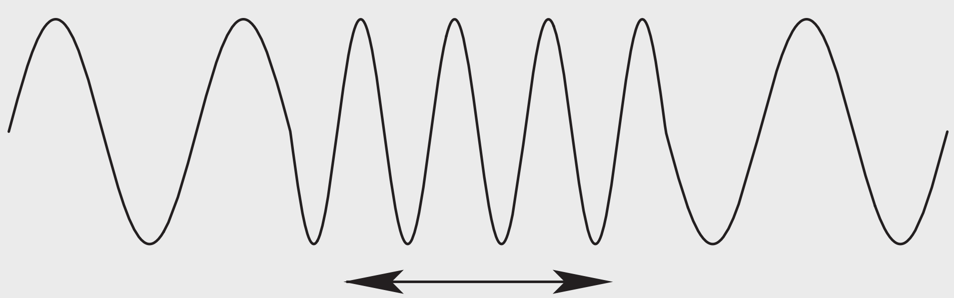
... include ππ operator

... and three quark widths (s, m, w)

CBL,Mohler,Prelovsek, arXiv:1105.5636

Energy levels and phase shift

Lüscher, CMP 105(86) 153,
NP B354 (91) 531, NP B 364 (91) 237



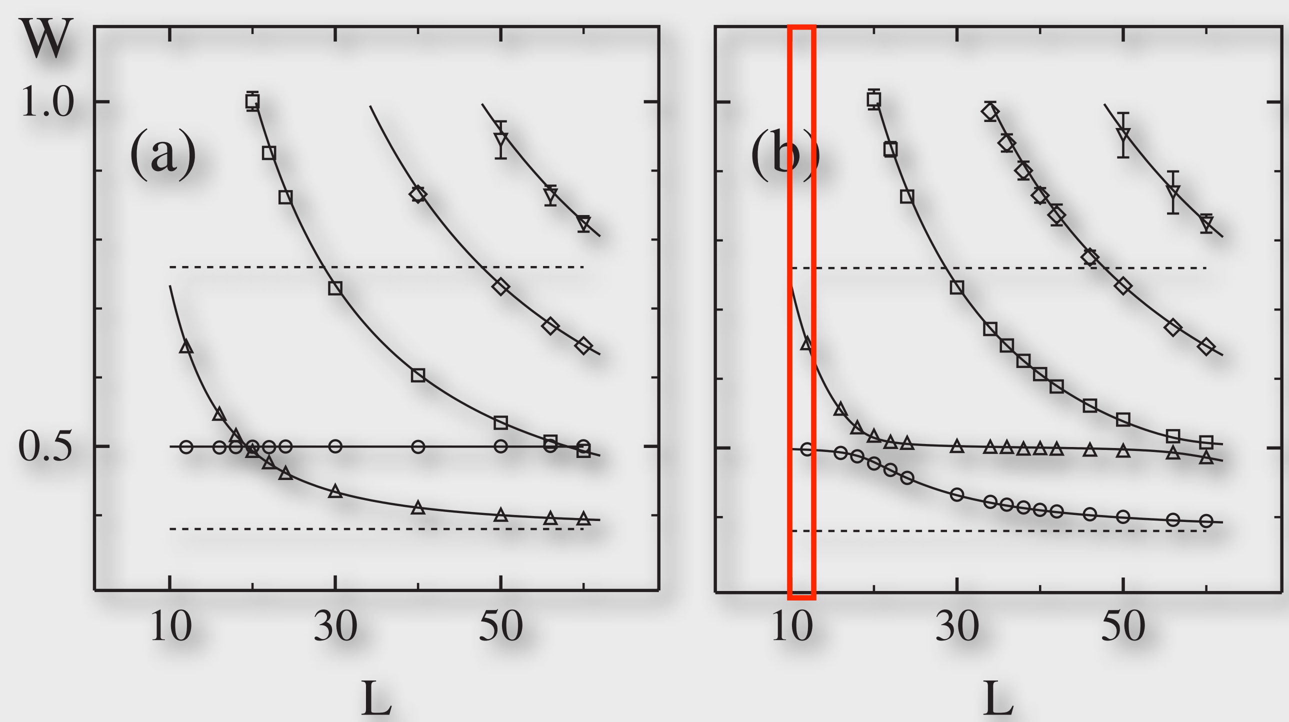
$$\mathrm{e}^{\mathrm{i}kL+2\mathrm{i}\delta(k)} = 1$$

$$k_n L + 2\delta(k_n) = 2n\pi$$

Fig. 11.2. This figure illustrates the behavior of the wave function: Outside the interaction region it is an unperturbed plane wave which picks up an extra phase shift in the interaction region (indicated by the *arrow*)

$$W_n = 2\sqrt{m^2 + k_n^2}$$

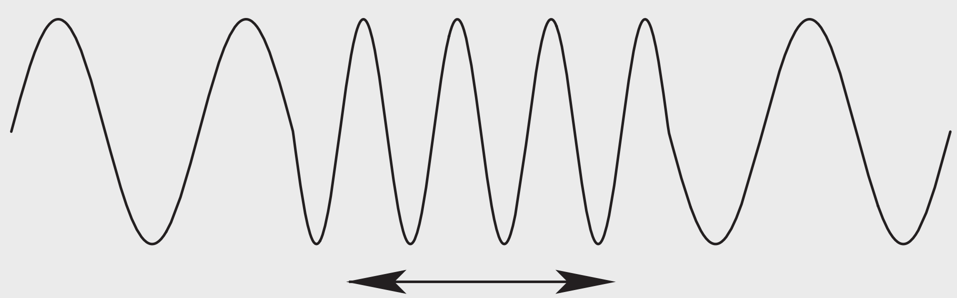
$$\tan\delta(q) = \frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1;q^2)}$$



2D example: Gattringer/CBL, NP B391 (1993) 463

Energy levels and phase shift

Lüscher, CMP 105(86) 153,
NP B354 (91) 531, NP B 364 (91) 237



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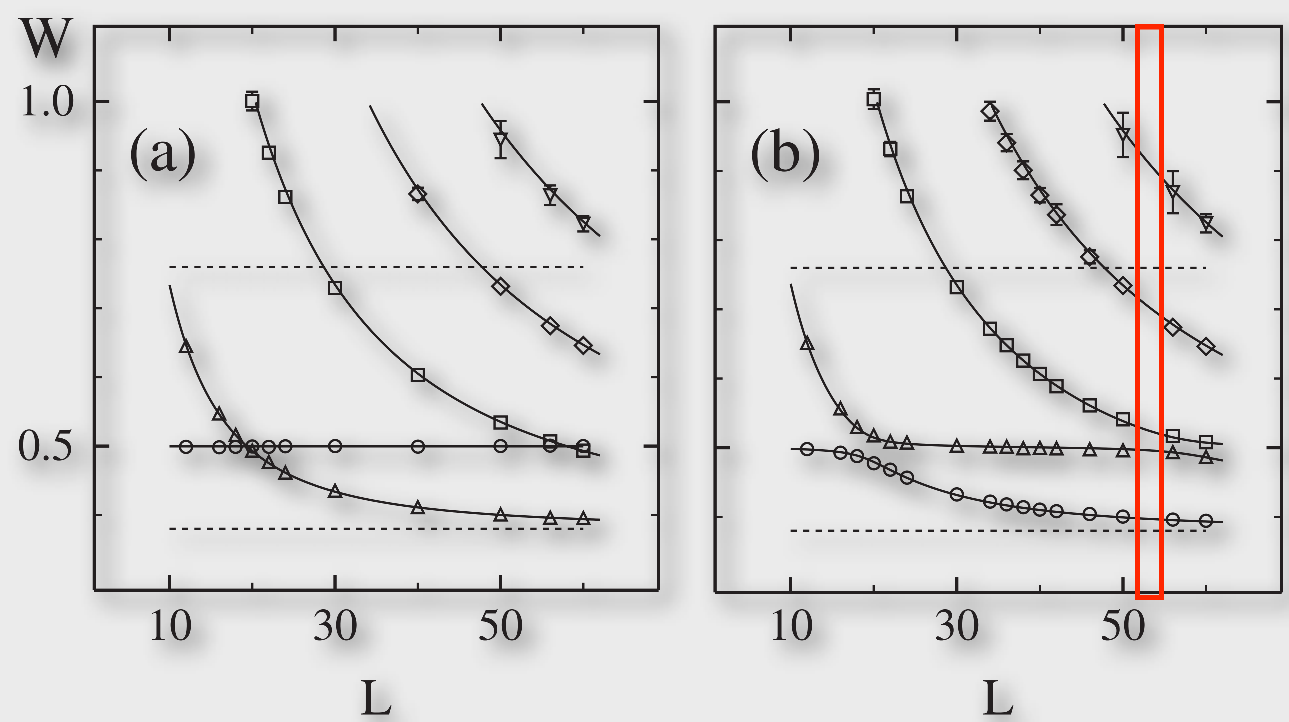


2D example: Gattringer/CBL, NP B391 (1993) 463

Energy levels and phase shift

Only 2 (3?) levels can be determined reliably for given volume!

Use different momenta ("moving frame")!

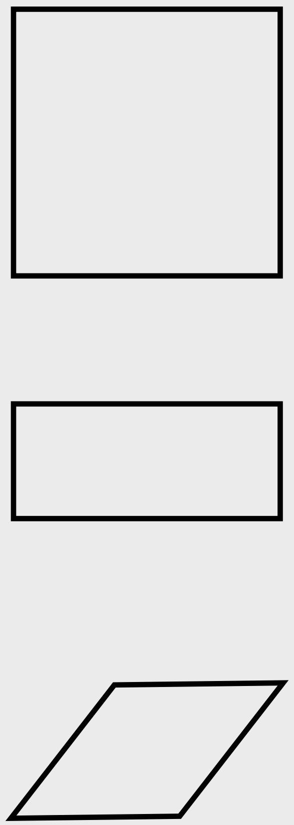
Rummukainen, Gottlieb: NP B 450(95) 397
Kim, Sharpe: NP B 727 (05) 218
Feng, Jansen, Renner: PoS LAT10 (10) 104

Rho momenta

$\vec{p} = (0,0,0) \quad (\text{units} 2\pi/L)$

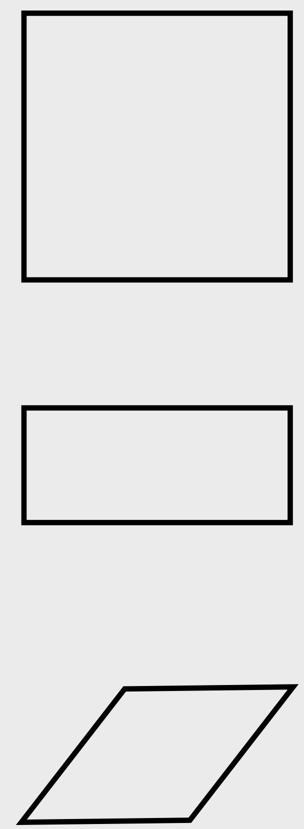
$\vec{p} = (0,0,1)$

$\vec{p} = (1,1,0)$



Relativistic distortion

Rho momenta



	Relativistic distortion	Symmetry group	Irrep for ρ
$\vec{p} = (0,0,0) \quad (\text{units} 2\pi/L)$		O_h	T_1^-
$\vec{p} = (0,0,1)$		D_{4d}	A_2^-
$\vec{p} = (1,1,0)$		D_{2d}	B_1^-

Energy levels give phase shift values

$$E,\, m_\pi \;\to\; E_{CM} \;\to\; q \;\to\; \delta(q)$$

(0,0,1) : $$\tan\delta(q) = \frac{\gamma\pi^{3/2}q^3}{q^2\,\mathcal{Z}^{\vec{d}}_{00}(1;q^2) + \sqrt{\frac{4}{5}}\;\mathcal{Z}^{\vec{d}}_{20}(1;q^2)}$$

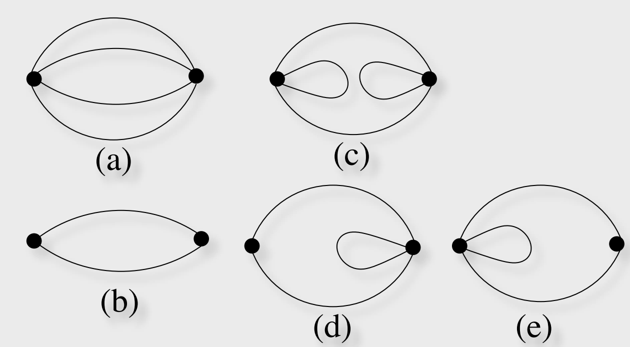
(1,1,0) : $$\tan\delta(q) = \frac{\gamma\pi^{3/2}q^3}{q^2\,\mathcal{Z}^{\vec{d}}_{00}(1;q^2) - \sqrt{\frac{1}{5}}\;\mathcal{Z}^{\vec{d}}_{20}(1;q^2) + \mathrm{i}\sqrt{\frac{3}{10}}\;\left(\mathcal{Z}^{\vec{d}}_{22}(1;q^2) - \mathcal{Z}^{\vec{d}}_{2\bar{2}}(1;q^2)\right)}$$

Recipe

$$E, m_\pi \;\rightarrow\; E_{CM} \;\rightarrow\; q \;\rightarrow\; \delta(q)$$

- Up to 6 pion interpolators, var. analysis → pion mass
- Up to 18 ρ interpolators,var. analysis →energy levels E

  - the distillation method allows to include

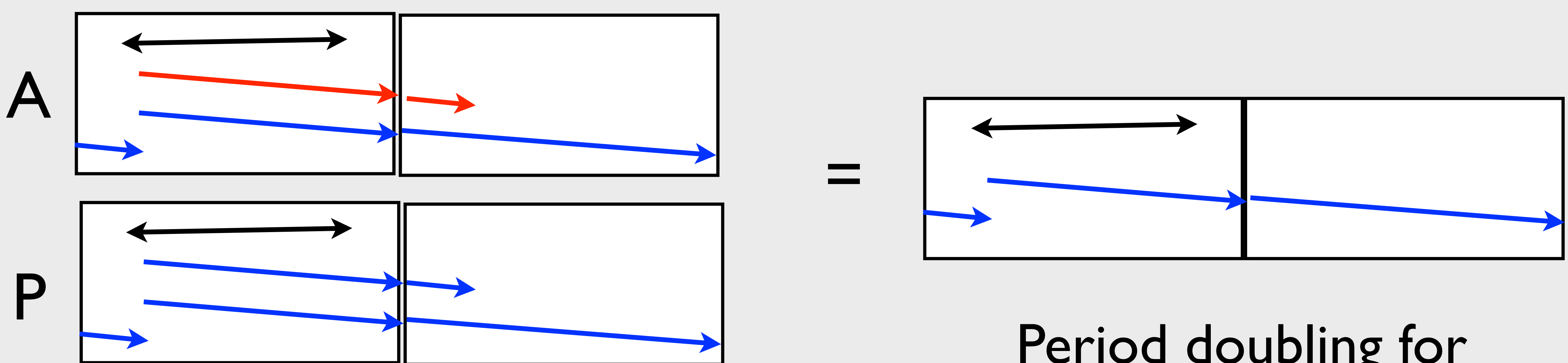


- Compute E_{CM} and q
- Compute from q the values of the phase shift
- Repeat for each momentum set → total of 6 energy values

“P+A - trick”

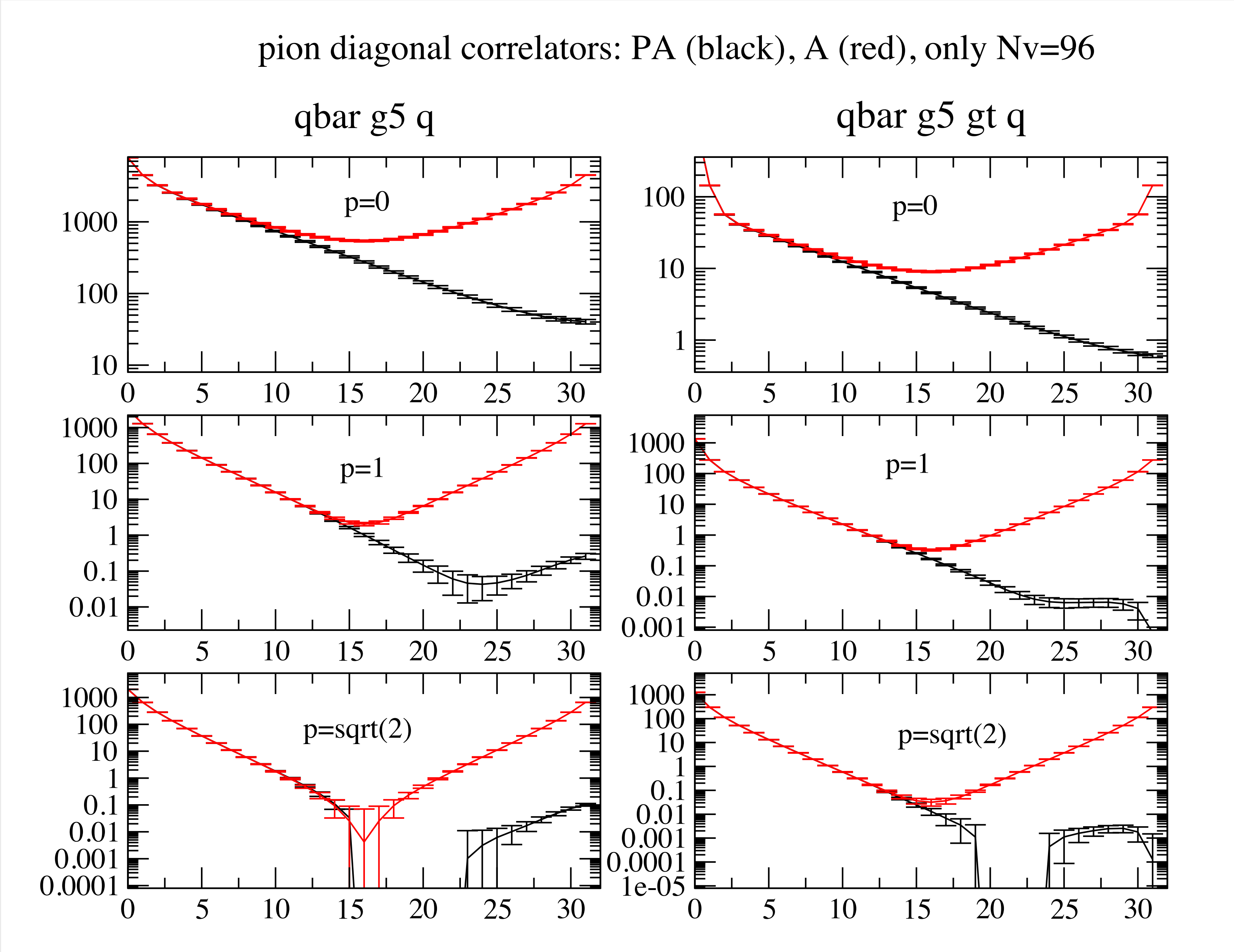
Sasaki et al., PRD65(02)074503
Detmold et al., PRD78(08)054514

$$M_{P+A}^{-1}(t_f,t_i) = \begin{cases} \frac{1}{2}[M_P^{-1}(t_f,t_i) + M_A^{-1}(t_f,t_i)] & t_f \geq t_i \\ \frac{1}{2}[M_P^{-1}(t_f,t_i) - M_A^{-1}(t_f,t_i)] & t_f < t_i \end{cases}$$



Period doubling for
valence sector

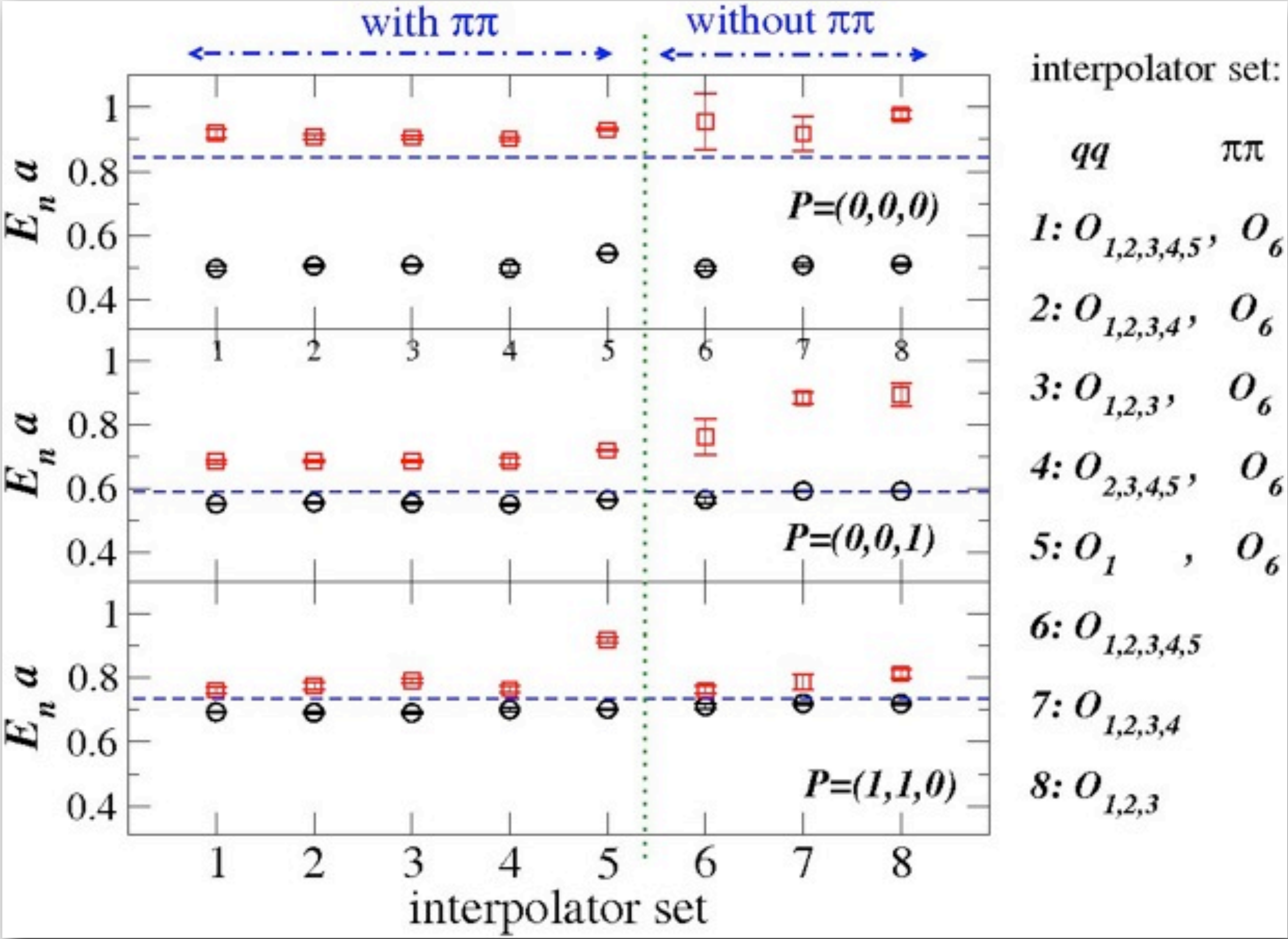
Pion diagonal correlators: A vs. P+A



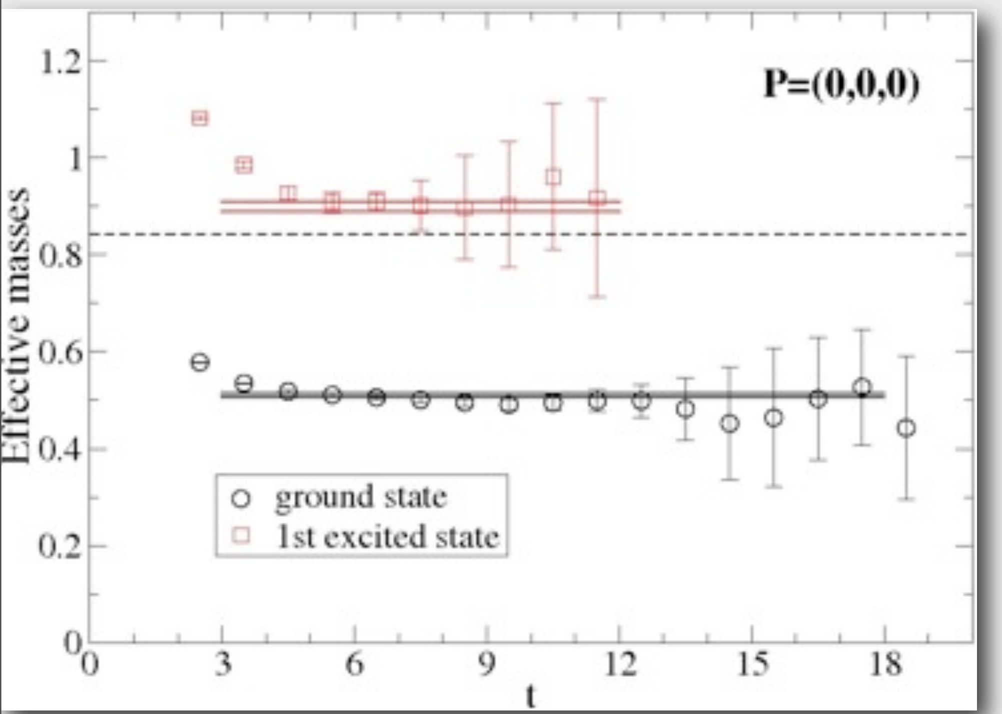
Tests - how many do we need?

Lowest two levels
(for selected
submatrices)

t_0=4
fit range 7-10



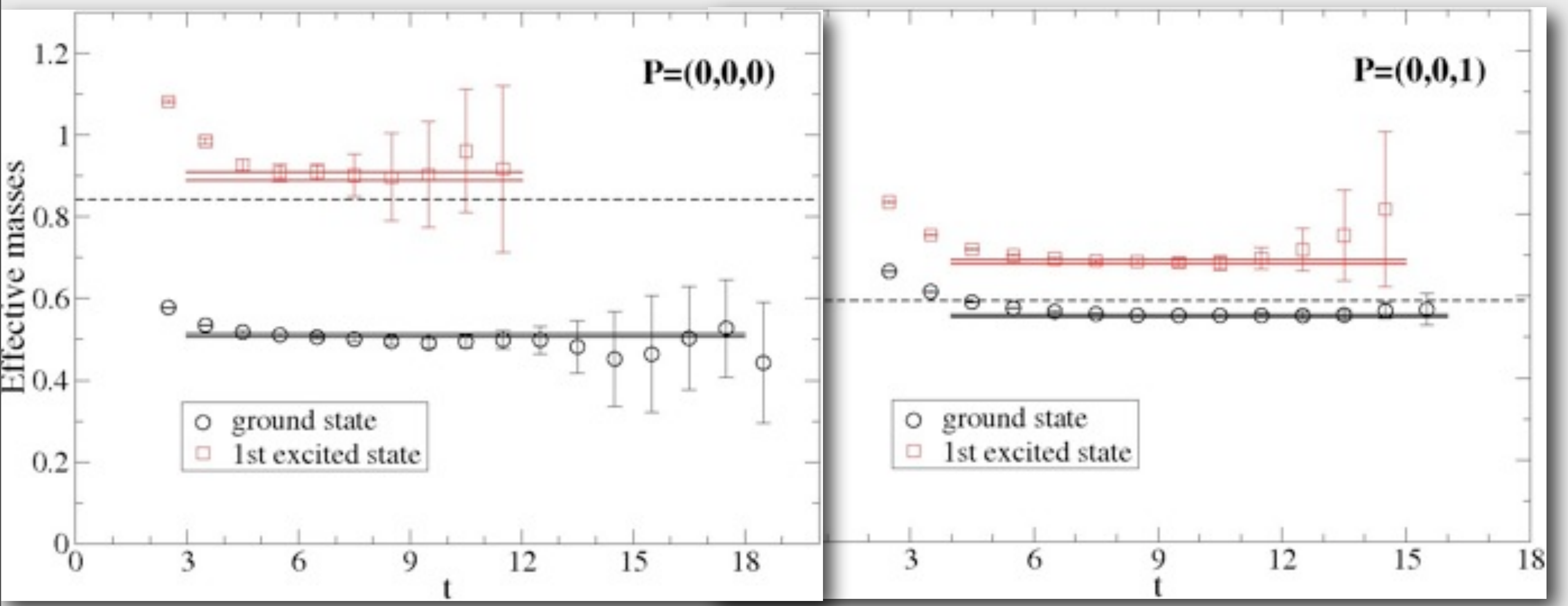
Lowest two energy levels



Bands: Fit range for λ(t) - 2 exp fits

----- noninteracting π π energy

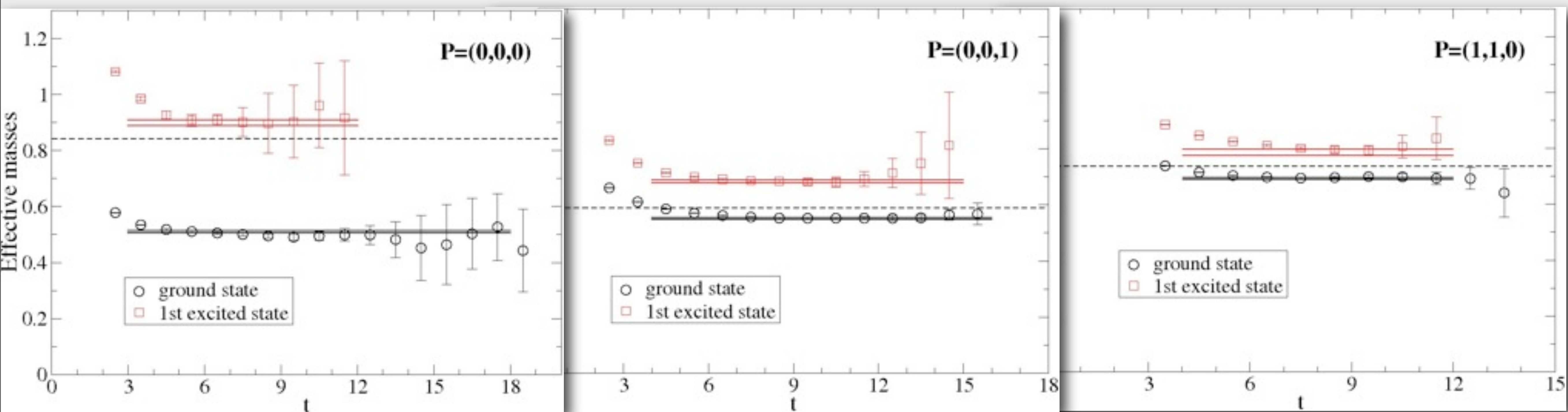
Lowest two energy levels



Bands: Fit range for λ(t) - 2 exp fits

----- noninteracting π π energy

Lowest two energy levels



Bands: Fit range for $\lambda(t)$ - 2 exp fits

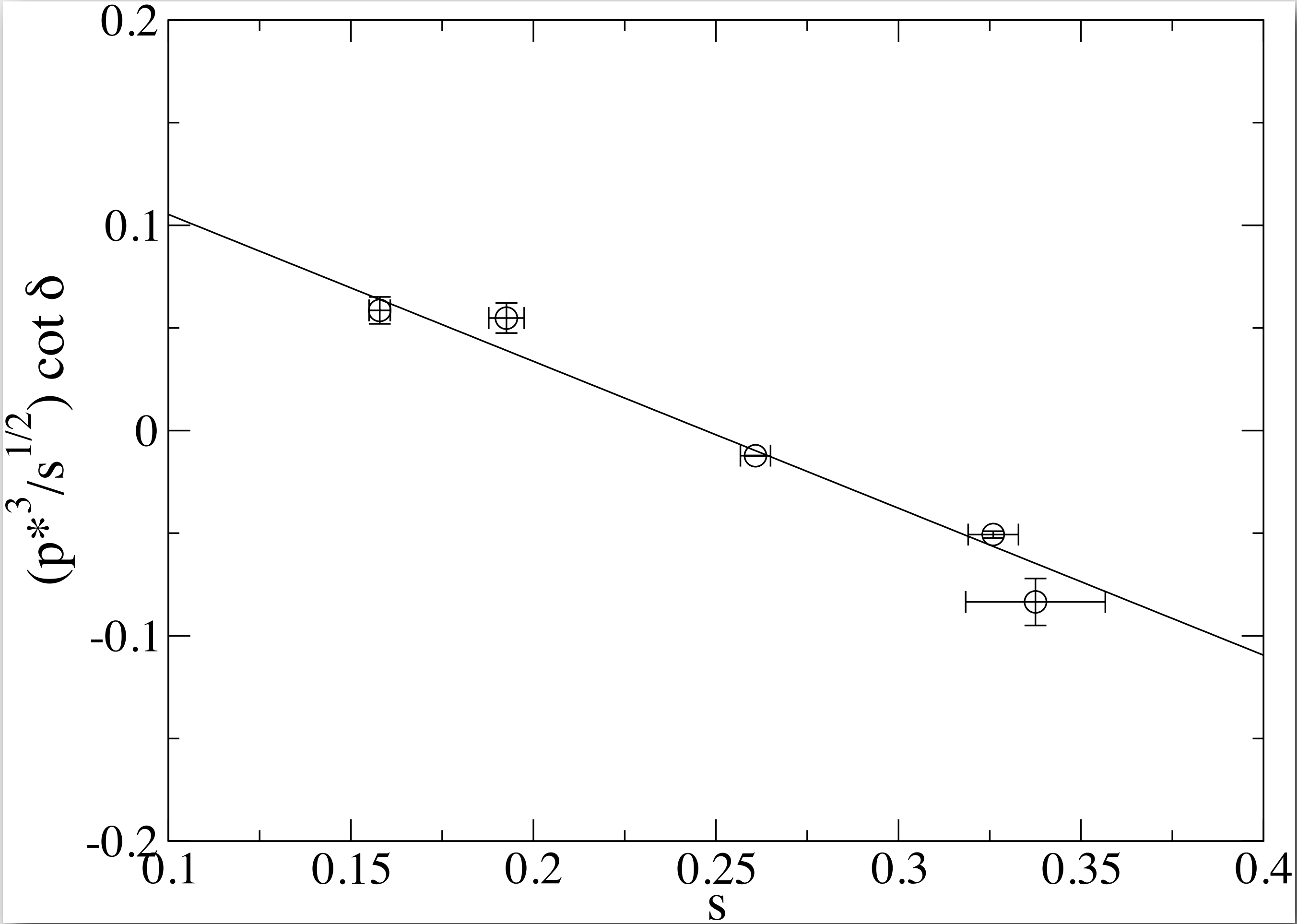
----- noninteracting π π energy

$\pi\pi \to \pi\pi$ scattering amplitude

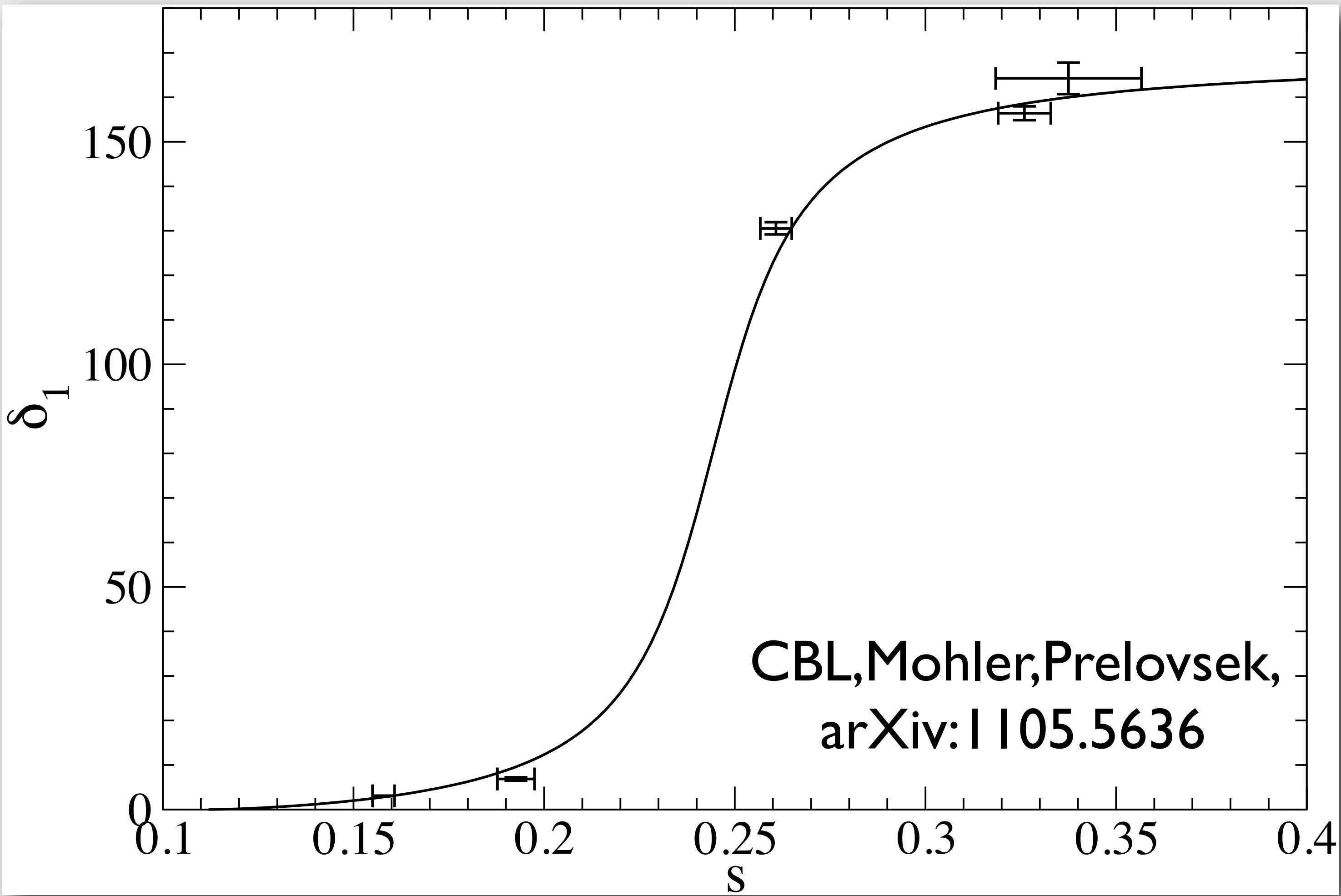
$$a_1 = \frac{-\sqrt{s}\,\Gamma(s)}{s - m_\rho^2 + \mathrm{i}\sqrt{s}\,\Gamma(s)} = \mathrm{e}^{\mathrm{i}\delta(s)} \sin\delta(s) \qquad (s = E_{CM}^2)$$

$$\sqrt{s}\,\Gamma(s)\ \cot\delta(s) = m_\rho^2 - s$$

with $$\Gamma(s) = \frac{p^3}{s}\frac{g_{\rho\pi\pi}^2}{6\pi}$$



Phase shift



$$g_{\rho\pi\pi} = 5.13(20)$$
$$m_\pi = 266(3)(3)\ \text{MeV}$$
$$m_\rho = 792(7)(8)\ \text{MeV}$$

$$g_{\rho\pi\pi,exp} = 5.96$$

Aoki et al. (PACS-CS)
PoS LAT10(10)108

$$g_{\rho\pi\pi} = 5.24(51)$$
$$m_\pi = 410\ \text{MeV}$$
$$m_\rho = 891\ \text{MeV}$$

Feng et al. (ETMC)
PoS LAT10(10)104

$$g_{\rho\pi\pi} = 6.77(67)$$
$$m_\pi = 290\ \text{MeV}$$
$$m_\rho = 980\ \text{MeV}$$

Frison et al. (BMW)
PoS LAT10(10)139

$$g_{\rho\pi\pi} = 5.5(2.9)/6.6(3.4)$$
$$m_\pi = 200/340\ \text{MeV}$$

Summary

One needs to bring together several sophisticated tools:

- Dynamical fermions
- Many hadron interpolators
- Variational analysis
- Momentum states
- Methods for disconnected graphs
- Phase shift methods

First results are being obtained:

- Excited hadrons, lowest levels
- Meson decay

There is a lot to do:

- Volume study
- Further hadronic channels (like scalar meson or meson-baryon states)
- Method improvement (more levels)
- Extension to inelastic region (e.g. Rusetsky et al.(09), Bernard et al.(10))

Thanks to my collaborators in related projects:

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