Lattice QCD at nonzero baryon number density

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and

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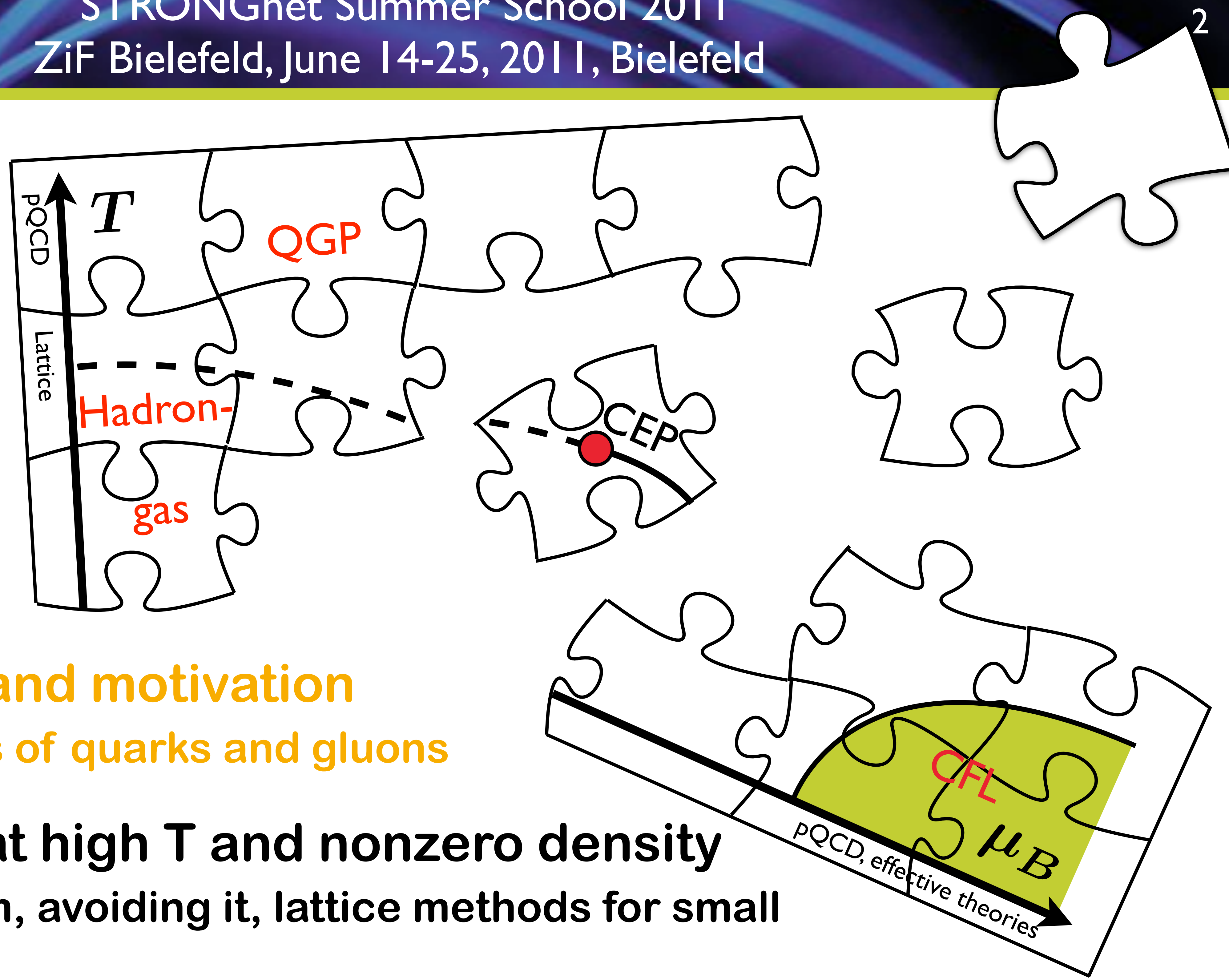


Overview:

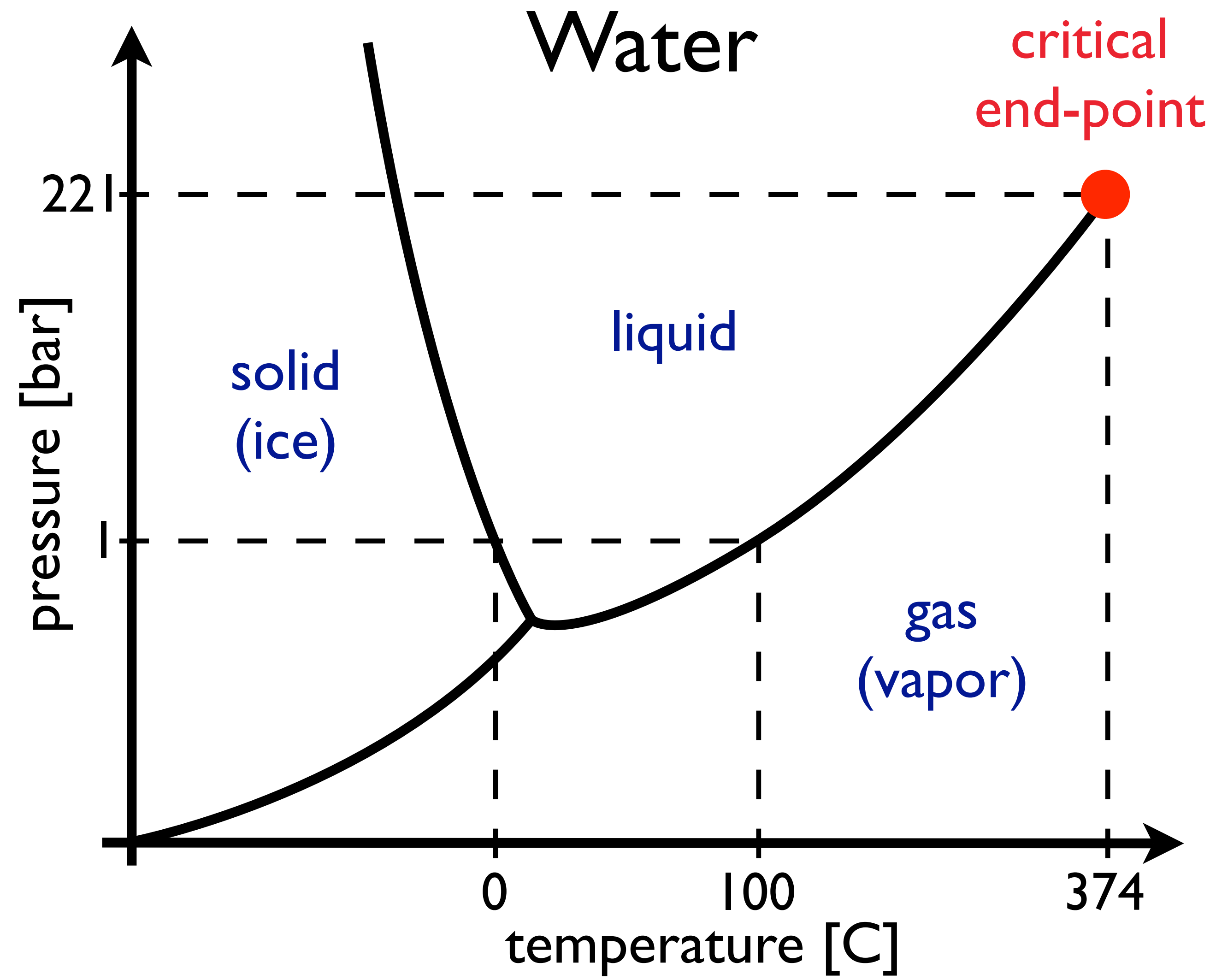
★ **Introduction and motivation**
thermodynamics of quarks and gluons

★ **Lattice QCD at high T and nonzero density**
The sign problem, avoiding it, lattice methods for small

★ **Results from the Taylor expansion method**
Hadronic fluctuations and heavy ion collisions, the critical point

★ **Summary**

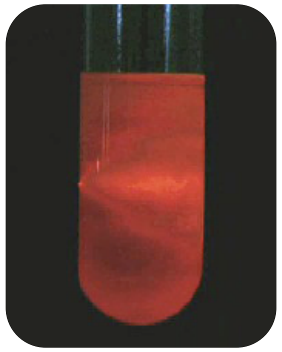
The phase diagram



⟶ determined by the **Equation of State**

⟶ exciting: **critical Phenomena**

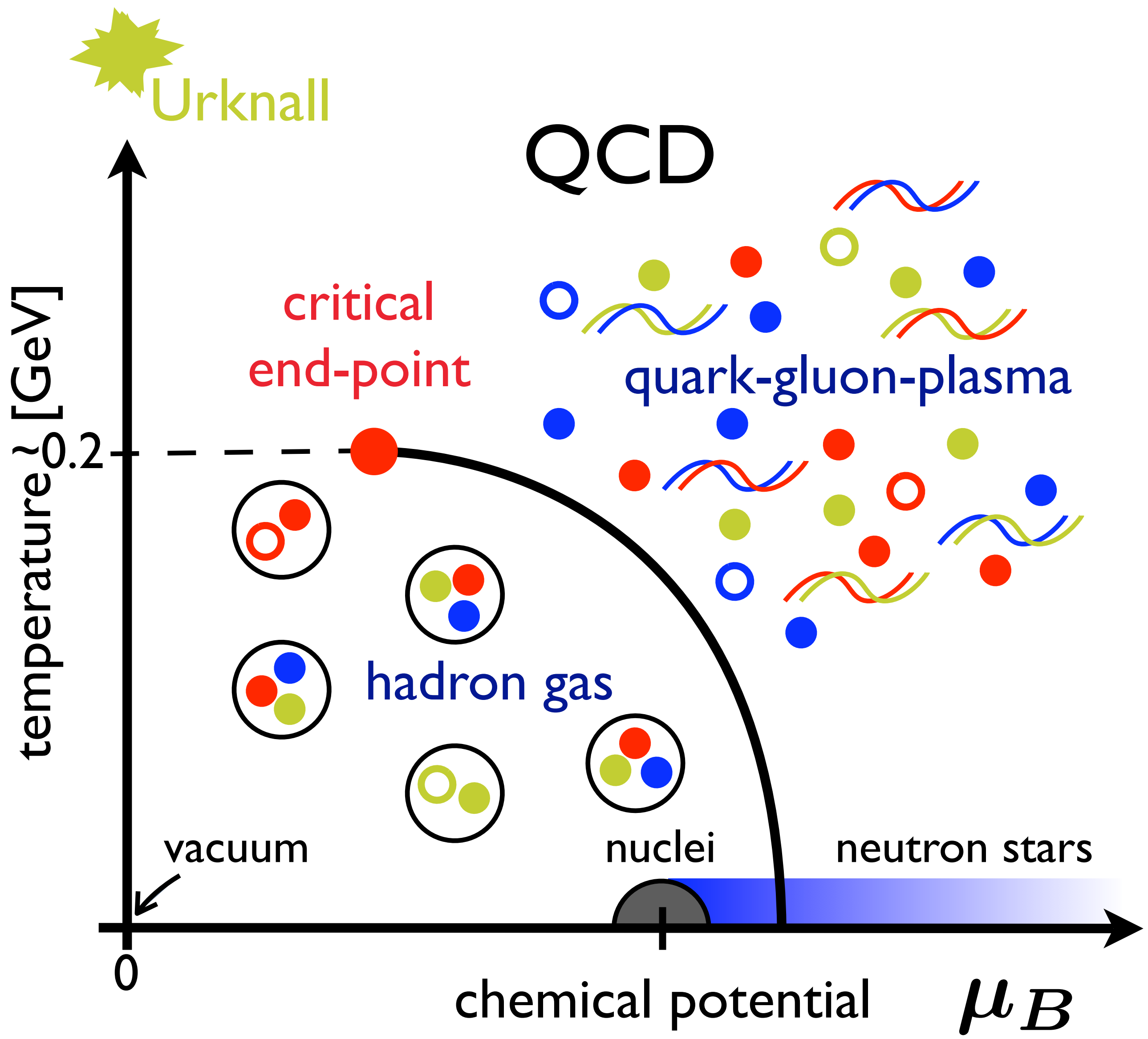
e.g. critical opalescence:



The phase diagram

Key questions

- What are the phases of strongly interacting matter and what role do they play in the cosmos ?
- What does QCD predict for the properties of strongly interacting matter ?
- What governs the transition from Quark and Gluons into Hadrons ?



The QCD phase diagram

Key questions

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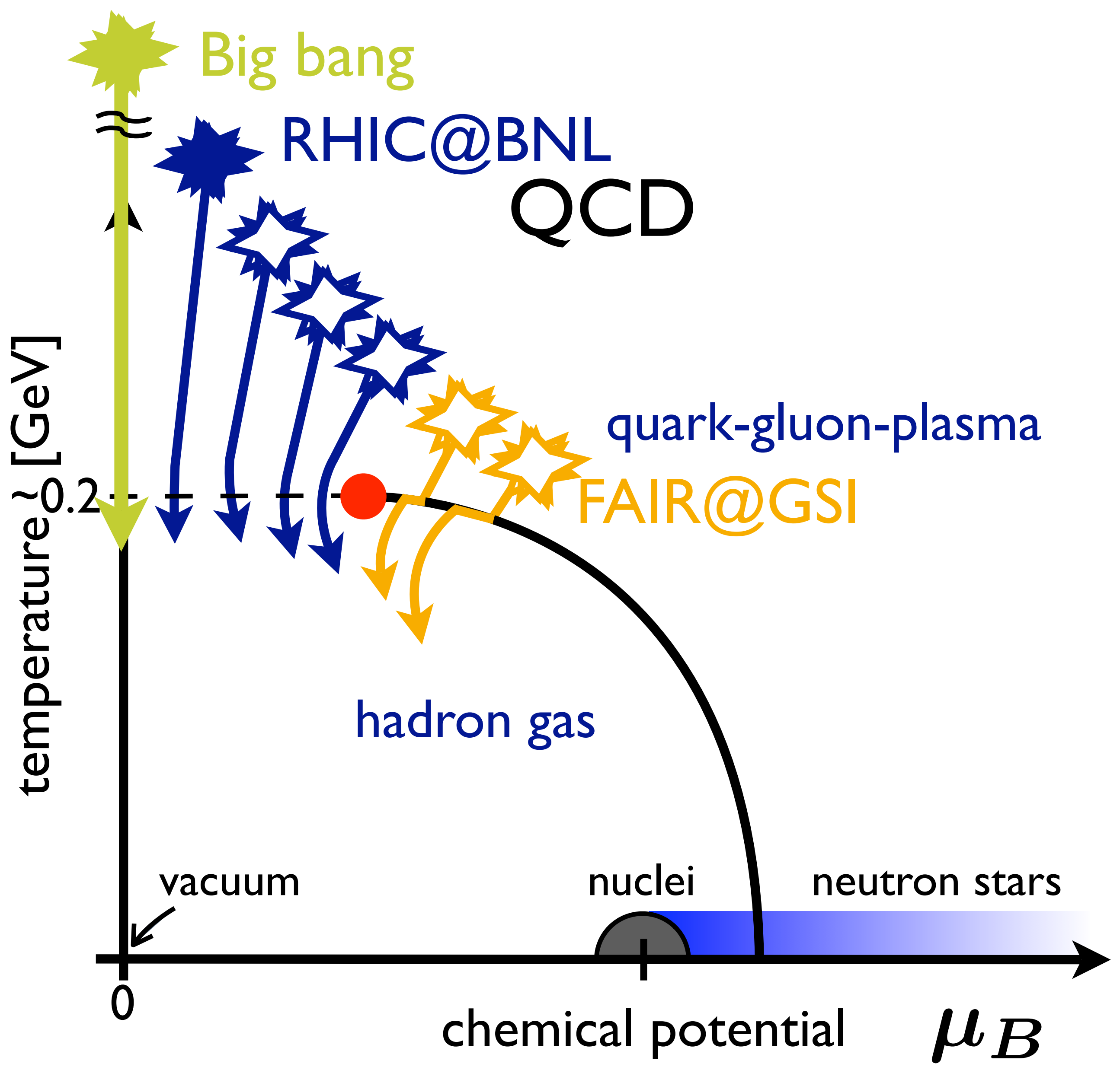
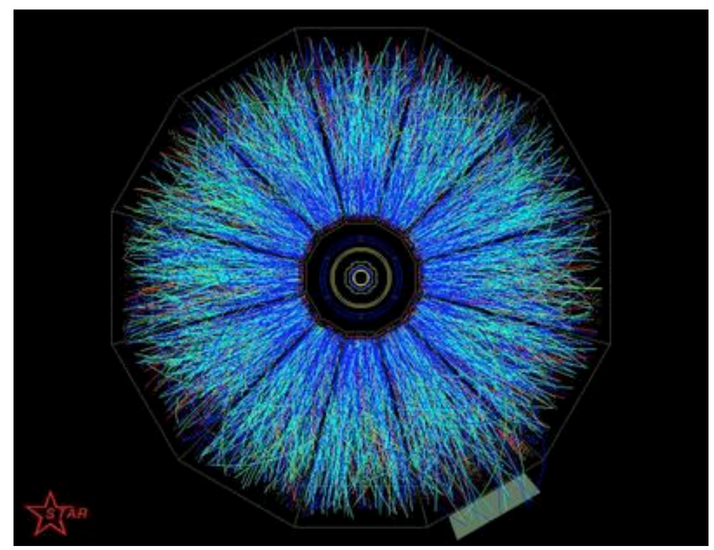


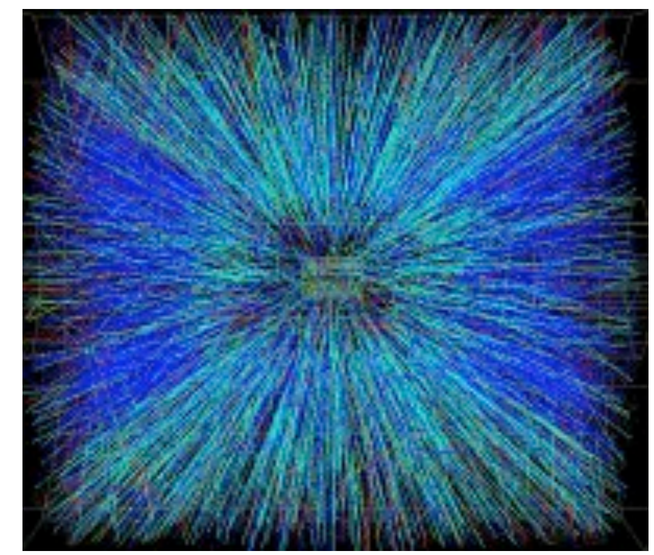
Places to find QGP ?

- In the early universe
- In the laboratory: RHIC, LHC, FAIR
- In the cores of neutron stars ?

QGP at RHIC (I)

Gold-Gold collisions at $\sqrt{s} = 130, 200\ \mathrm{GeV}/A$



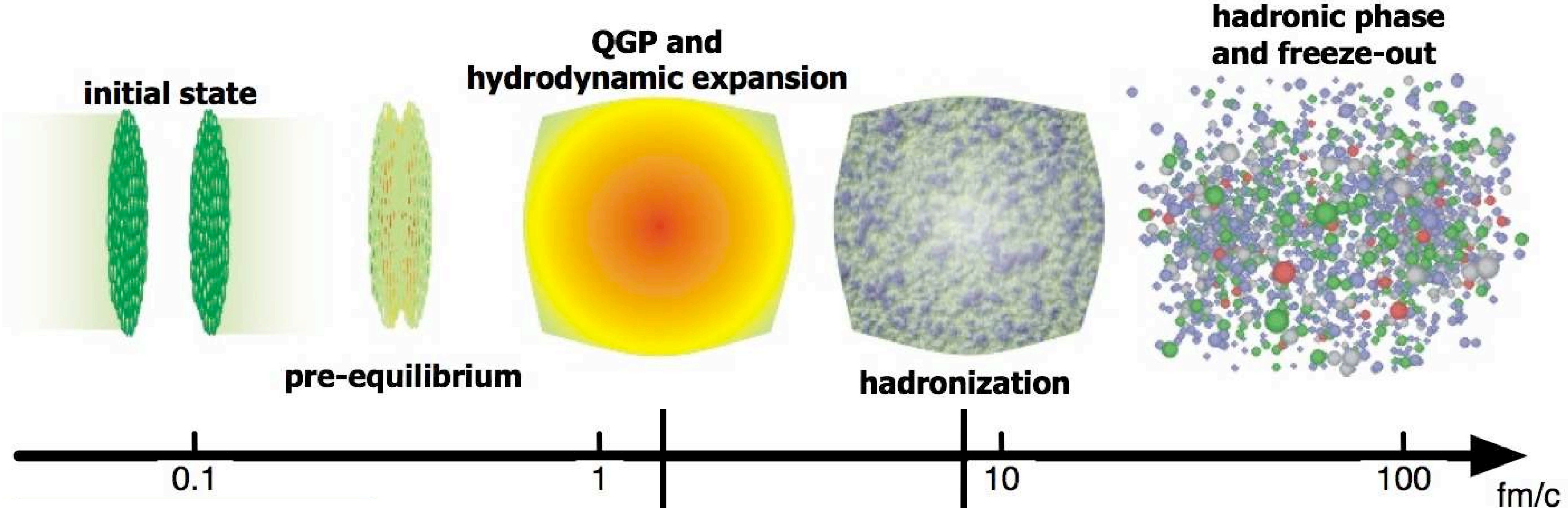


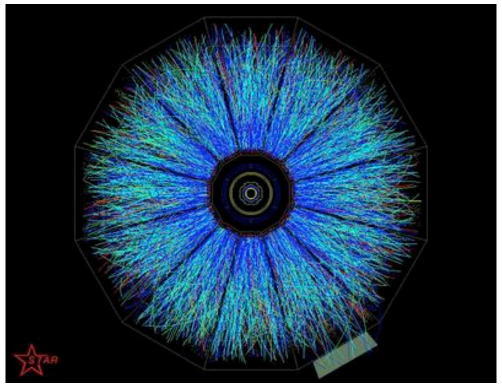
⟶ estimated temperature: $T_0 \approx (1.5 - 2) T_c$

⟶ estimated energy density: $\epsilon_0 \approx (5 - 15) \mathrm{GeV}/\mathrm{fm}^3$

QGP at RHIC (II)

(schematic picture)





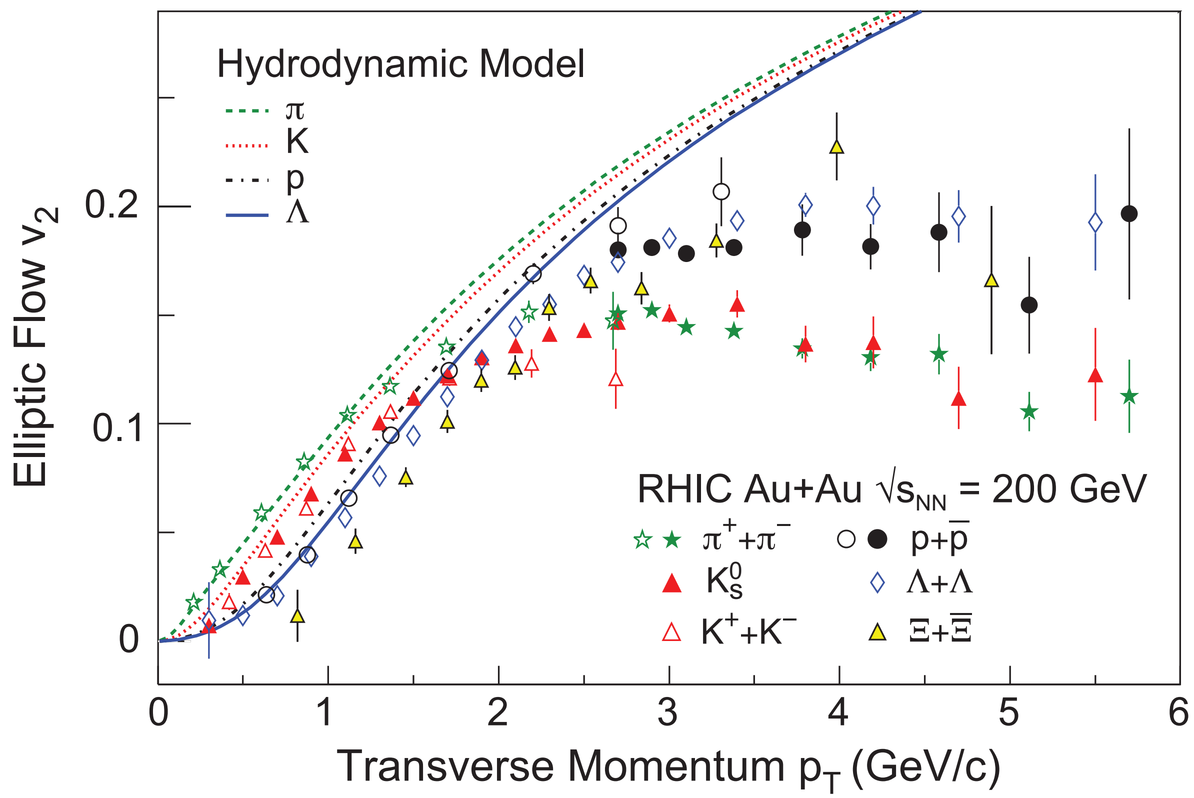
Expansion:
hydrodynamic description
needs equation of state

Hadronization:
dependent on details of the
equation of state and the nature
of the QCD transition

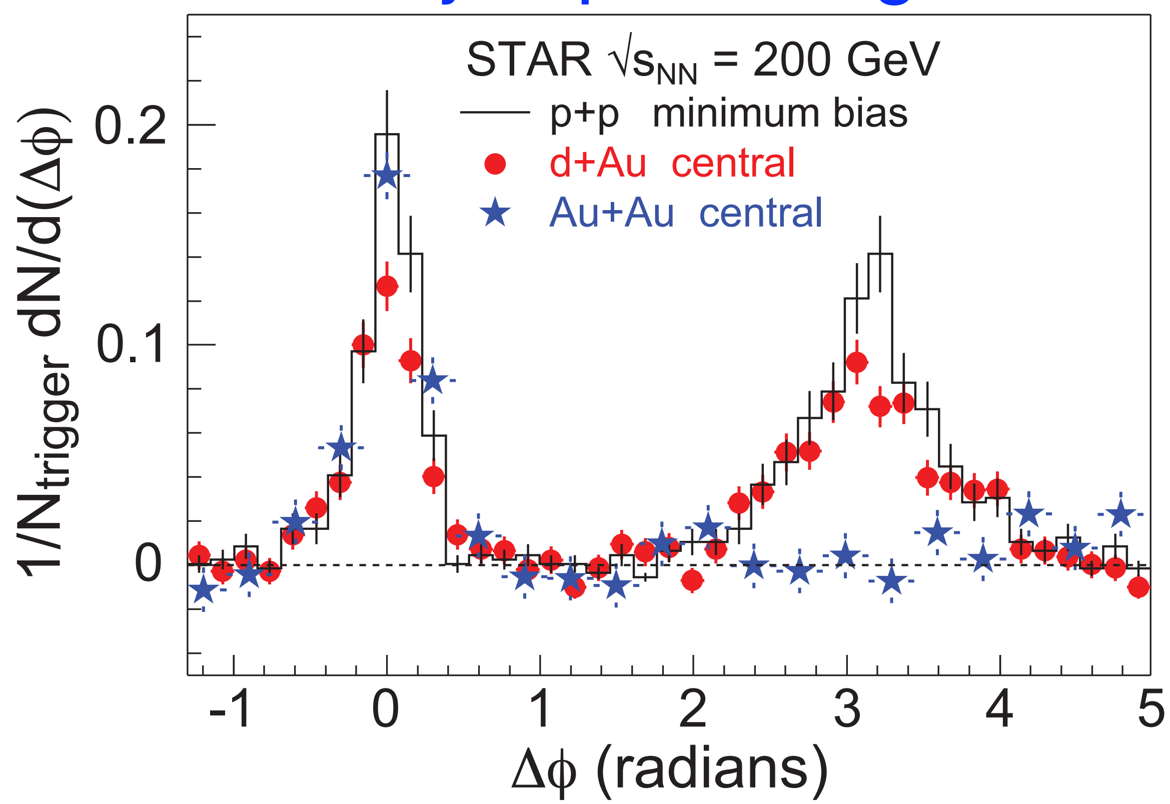
Goal:
understand the
observed tracks in the
detector

QGP at RHIC (III)

elliptic flow



jet quenching



What are the properties of the QGP at RHIC ?

- hydrodynamic models are very successful in the description of the RHIC data
- viscosity extremely small

→ perfect liquid?

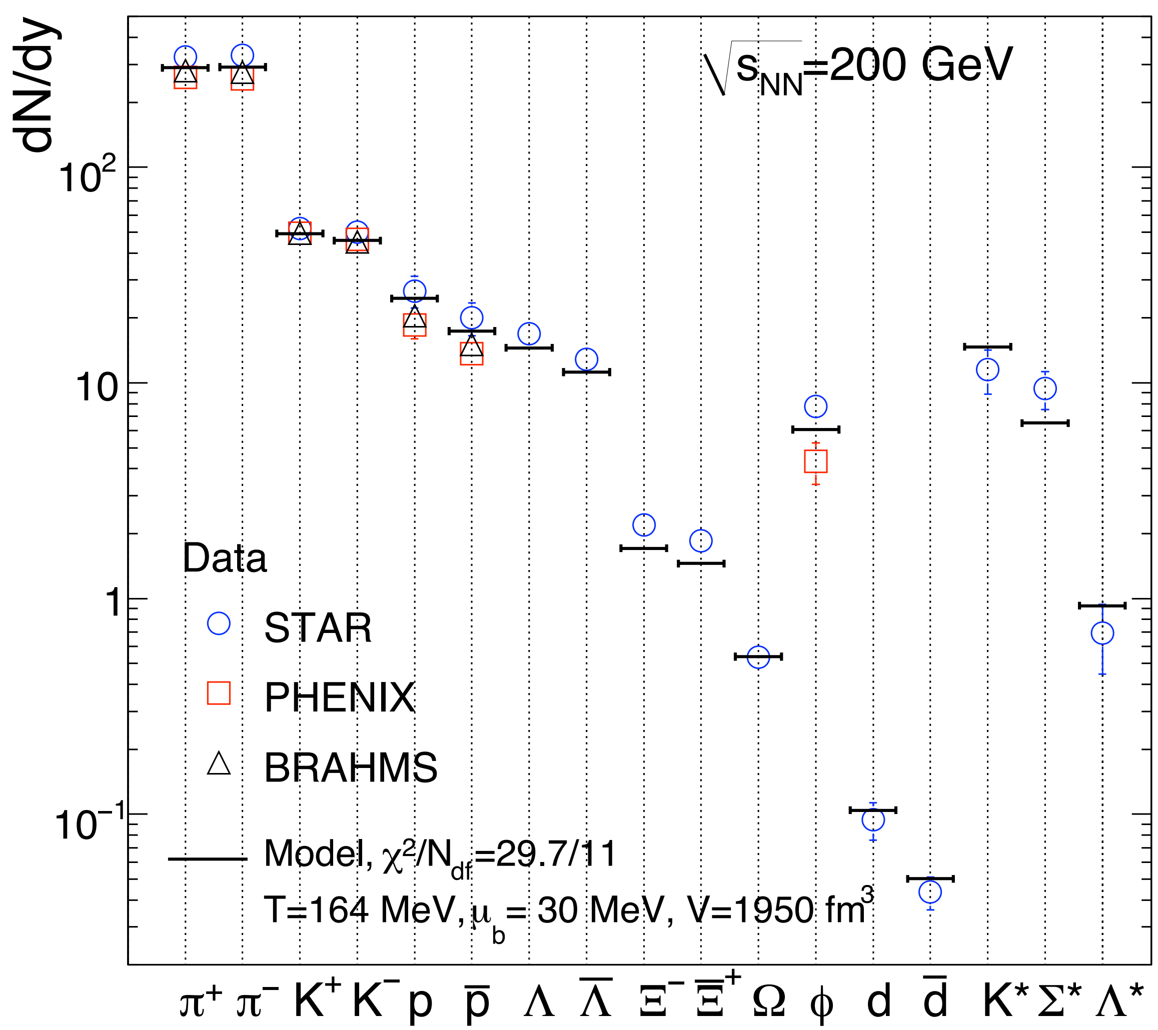
- dens medium, away-side jet is strongly/completely suppressed

HIC at RHIC (IV)

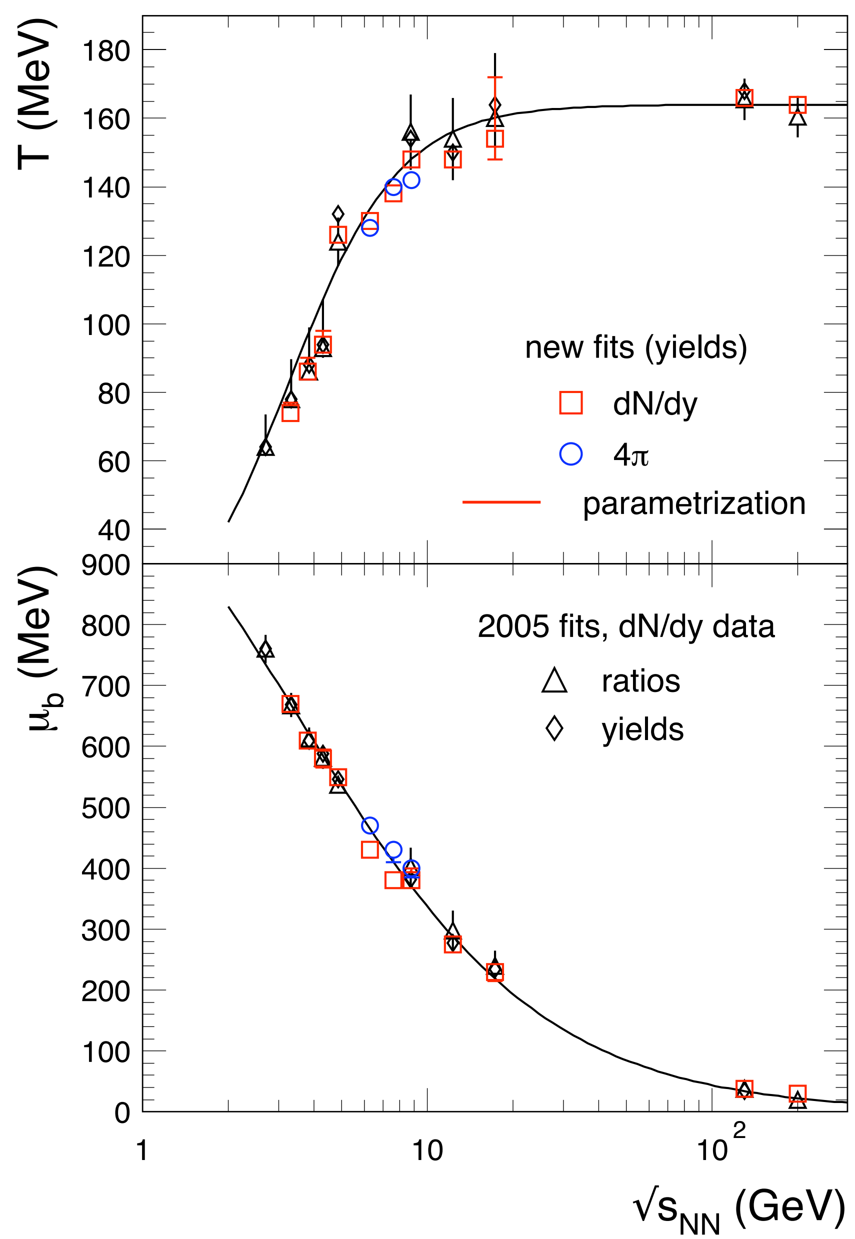
- The resonance gas is describing the observed hadron-spectrum

$$\begin{aligned}\ln Z(T,V,\mu_B,\mu_S,\mu_Q) &= \sum_{i\in hadrons} \ln Z_{m_i}(T,V,\mu_B,\mu_S,\mu_Q)\\ &\sum_{i\in mesons} \ln Z^B_{m_i}(T,V,\mu_S,\mu_Q) + \sum_{i\in baryons} \ln Z^F_{m_i}(T,V,\mu_B,\mu_S,\mu_Q)\end{aligned}$$

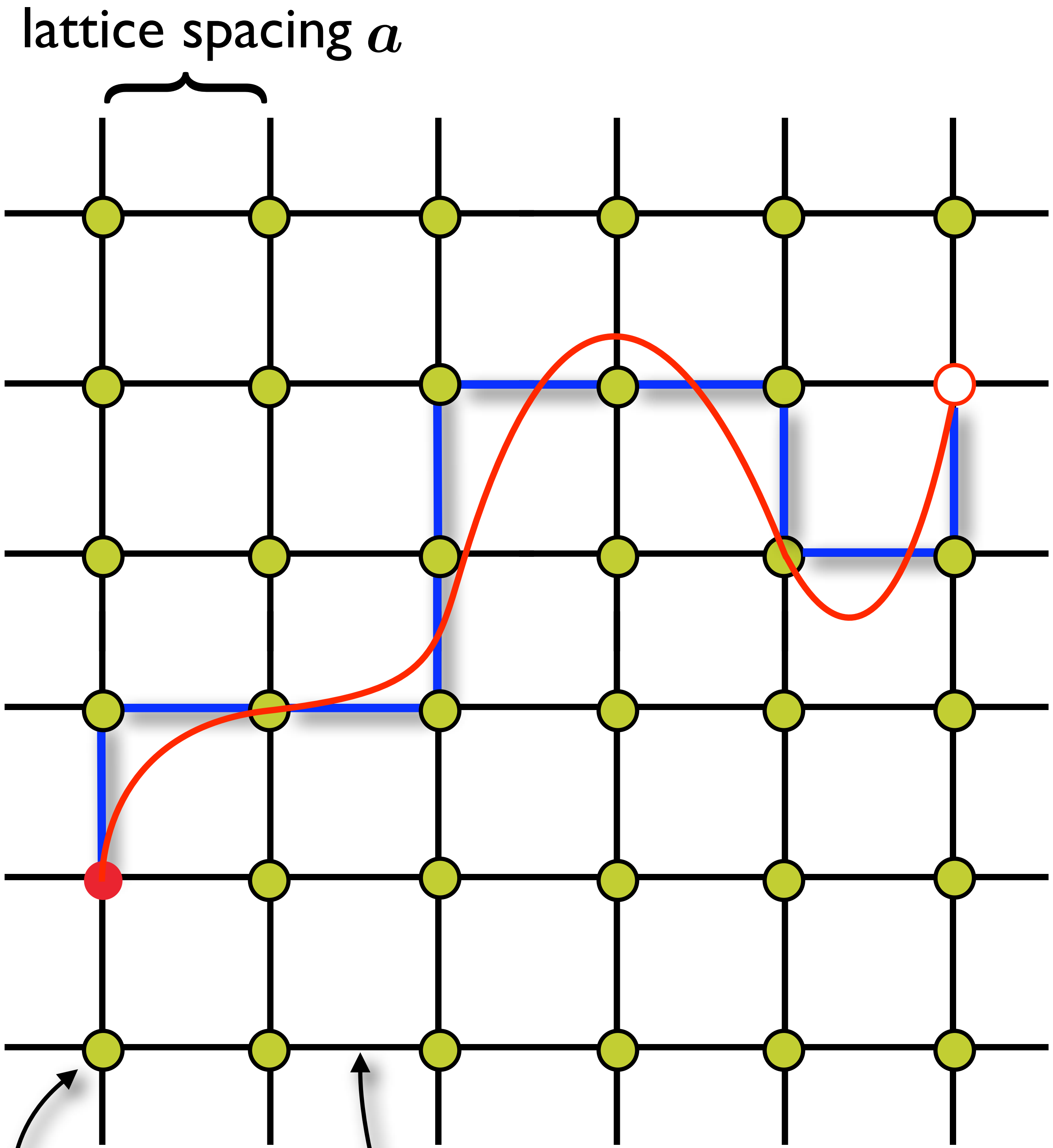
$\longrightarrow$ **produced matter is thermalized?**



Andronic, Braun-Munzinger, Stachel,
PLB 673 (2009) 142.



QCD on the lattice



discretize space time and hence all „paths“ of quarks and gluons

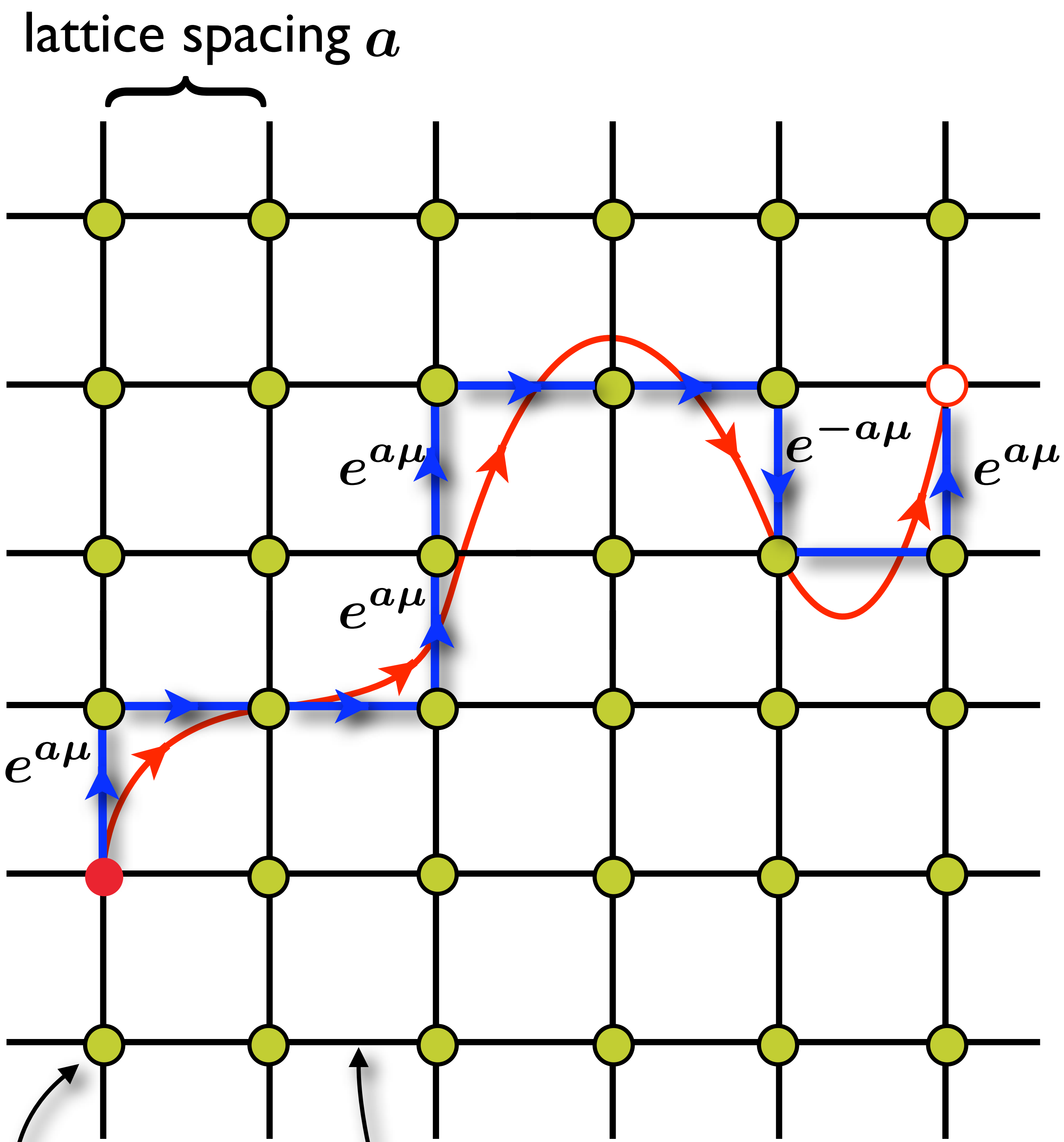
quarks

$\psi(x), \bar{\psi}(x)$

gluons

$$U_\mu(x) = P \exp\left\{ ig \int_x^{x+\hat{\mu}a} dx_\mu A_\mu(x) \right\}$$

Lattice QCD (at $T, \mu > 0$)



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$$U_\mu(x) = P \exp\left\{ ig \int_x^{x+\hat{\mu}a} dx_\mu A_\mu(x) \right\}$$

discretize space time and hence all „paths“ of quarks and gluons

at nonzero chemical potential μ:

$$A_0 \rightarrow A_0 - i\mu$$

or equivalently:

$$U_0(x) \rightarrow e^{a\mu} U_0(x)$$
$$U_0^\dagger(x) \rightarrow e^{-a\mu} U_0^\dagger(x)$$

Hasenfratz, Karsch, PLB 125 (1983) 308.

Lattice QCD (at $T, \mu > 0$)

- the QCD partition function:

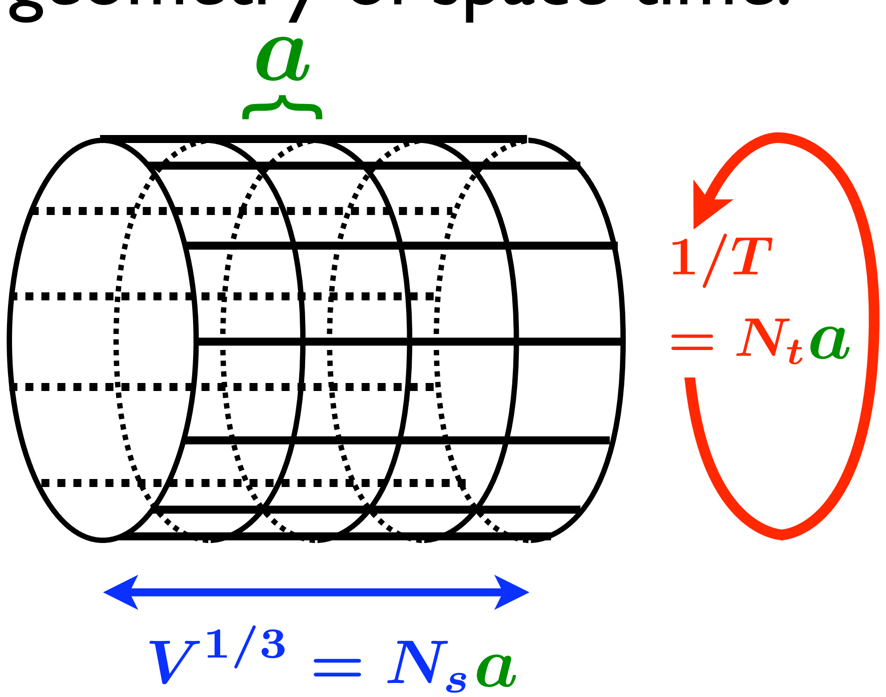
$$Z(V,T,\bar{\mu}) = \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\ \exp\{-S_E\}$$

$$S_E = \bar{\psi}_x M_{x,y} \psi_y + S_G$$

$$M_{x,y} = am\, \delta_{x,y} + \frac{1}{2}\sum_{\mu=1}^{3} \gamma_\mu \left\{U_\mu(x)\, \delta_{x+a\hat{\mu},y} - U^\dagger_\mu(y)\, \delta_{x-a\hat{\mu},y}\right\}$$

$$+\frac{1}{2}\gamma_4 \left\{e^{a\bar{\mu}}\, U_4(x)\, \delta_{x+a\hat{4},y} - e^{-a\bar{\mu}}\, U^\dagger_4(y)\, \delta_{x-a\hat{4},y}\right\}$$

- geometry of space time: $N_s^3 \times N_t$ (4d - torus)



note:

- only closed loops participate to the partition function
- only loops that wind around the torus in time direction $\mathcal{W}$-times pick up a μ-dependence:

$$\exp\{\mathcal{W}\mu/T\}$$

$\longrightarrow$ alternatively (gauge-transformation):

- choose a fixed time-slice on which all temporal links get a factor $\exp\{\pm\mu/T\}$

The sign problem

- integration over fermion fields

$$Z(V,T,\mu) = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\{S_F(A,\psi,\bar{\psi}) - \beta S_G(A)\}$$
$$= \int \mathcal{D}A\ \det[M](A,\mu)\exp\{-\beta S_G(A)\}$$

complex for $\mu > 0$

propabilistic interpretation necessary for Monte Carlo!

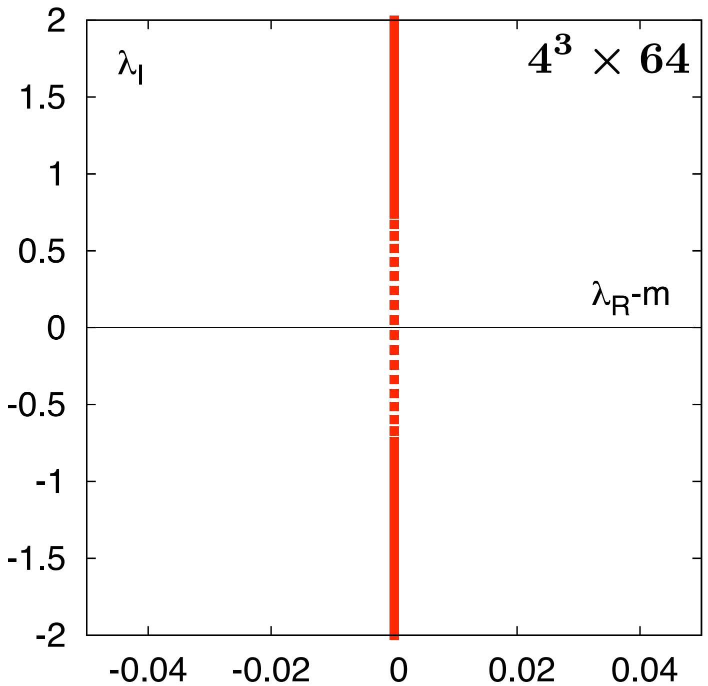
complex action can potentially be handled by the Langevin Algorithm

$\longrightarrow$ see talk by G. Aarts

The sign problem

- properties of the fermion matrix and eigen-spectrum

free case

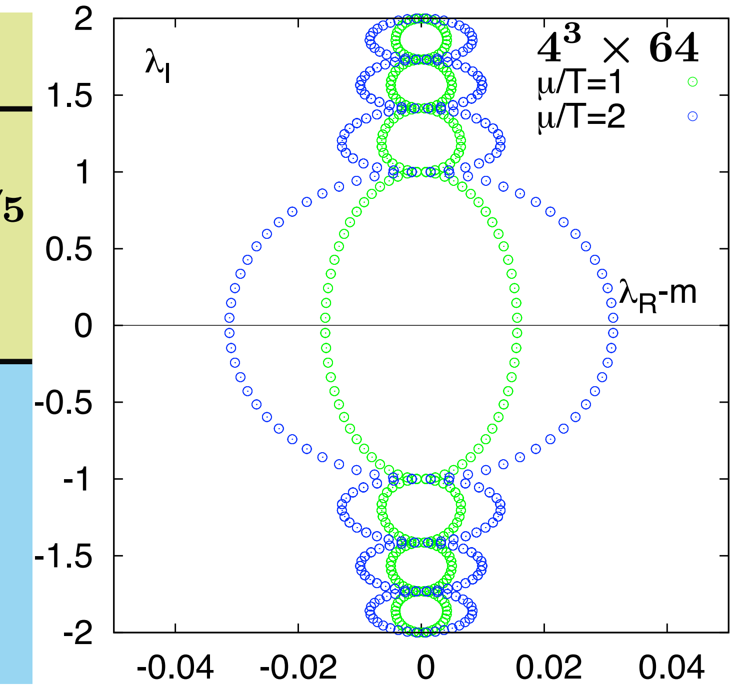


$\mu = 0$	$\mu > 0$
$M^\dagger(0) = \gamma_5 M(0) \gamma_5$ (γ_5- Hermiticity)	$M^\dagger(\mu) = \gamma_5 M(-\mu) \gamma_5$

free naive fermions

$$\lambda_p = m \pm i \sqrt{\sum_{k=1}^{3} \sin^2(p_k) + \sin^2(p_4 + i\mu)}$$

free case



$M^\dagger M$ is

- positive definite
- block diagonal in parity (even-odd) space, use even-odd preconditioning
- regulated by the mass: $\lambda_{\min} = m^2$

$M^\dagger M$ is

- not block diagonal in parity (even-odd) space
- not regulated, zero-modes possible for sufficiently large μ

The sign problem

- factorization of the fermion determinant into modulus and phase

$$\det[M] \equiv |\det[M]| \exp\{i\phi\}$$

consider the phase quenched ensemble:

$$\langle \mathcal{O} \rangle (\mu) = \frac{\langle \mathcal{O} \cos(\phi) \rangle_{|\det M(\mu)|}}{\langle \cos(\phi) \rangle_{|\det M(\mu)|}} \longrightarrow \frac{0}{0}$$

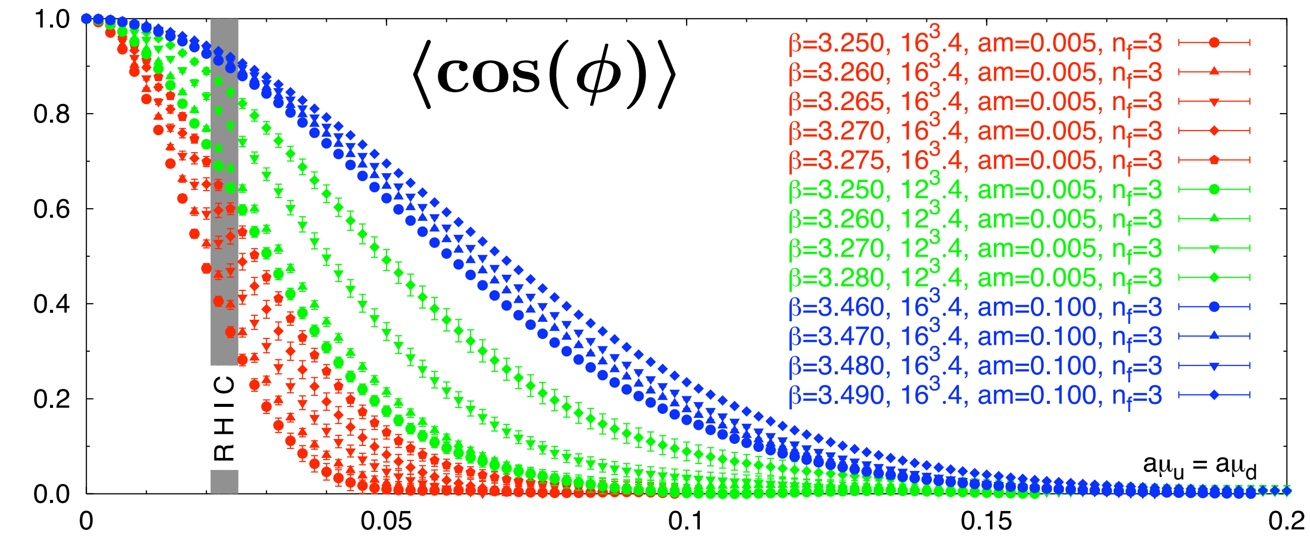
$\longrightarrow$ the signal gets lost due to the sign problem

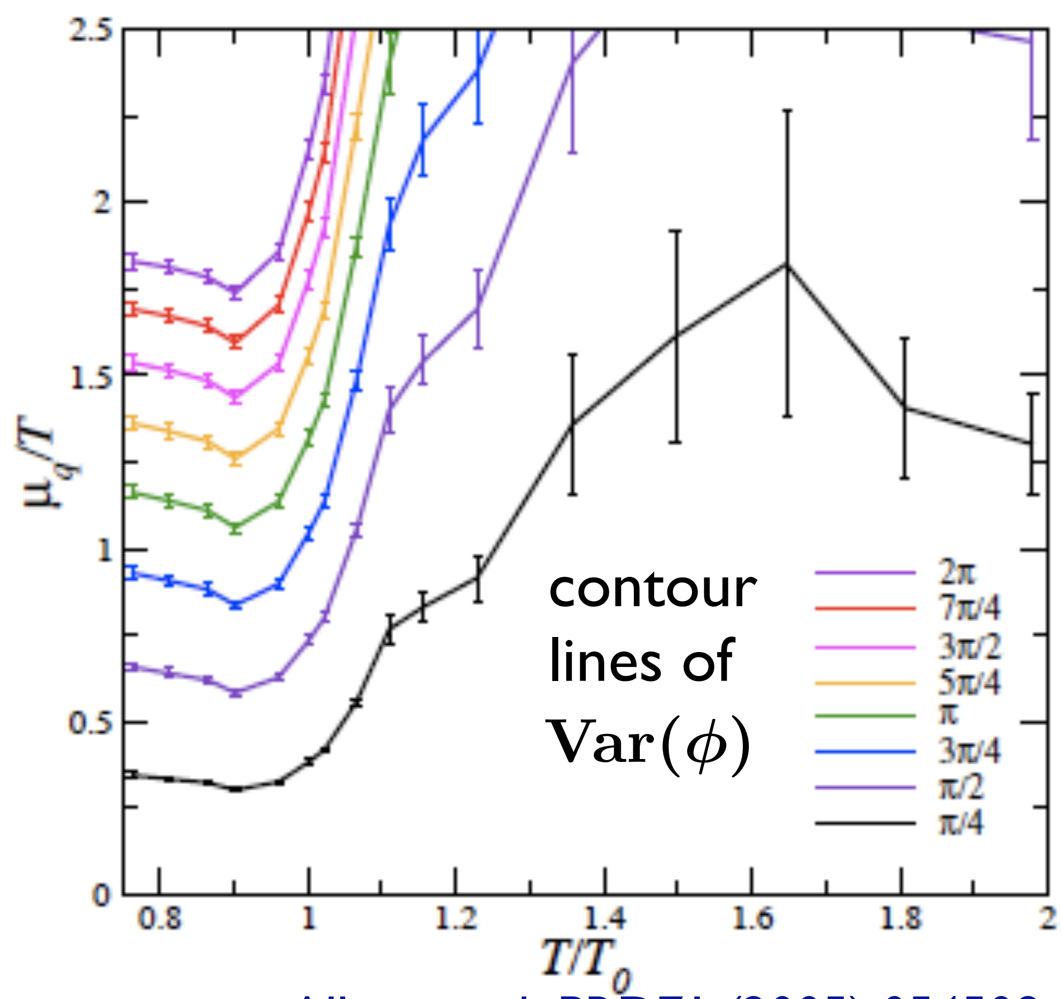
in the microscopic limit of QCD:
$\left(m_\pi^2 \ll \frac{1}{\sqrt{V}} \, , \quad \mu^2 \ll \frac{1}{\sqrt{V}} \right)$

$$\langle \cos(\phi) \rangle = \left(1 - \frac{4\mu^2}{m_\pi^2} \right)^{N_f+1}$$

Splittorff, Verbaarschot, PRL98 (2007) 031601.

$\longrightarrow$ the sign problem is not severe for $\mu < m_\pi/2$





Allton *et al.*, PRD71 (2005) 054508.

QCD-like theories

- dense two color matter: $U_\mu(x) \in SU(2)$

the 2-flavor action:

$$\begin{aligned} S_F &= \bar{\psi}_1 M(\mu)\psi_1 + \bar{\psi}_2 M(\mu)\psi_2 \\ &\quad + J\bar{\psi}_1(C\gamma_5)\tau_2\bar{\psi}_2^{tr} + \bar{J}\psi_2^{tr}(C\gamma_5)\tau_2\psi_1 \end{aligned}$$

$\longrightarrow$ diquark source terms to regulate eigenvalues and to study spontaneous symmetry breaking

symmetries:

$$\gamma_5 M(\mu)\gamma_5 = M^\dagger(-\mu)$$
$$K M(\mu) K^{-1} = M^*(\mu)$$

with $K \equiv C\gamma_5\tau_2$

$\longrightarrow$ for the latter equality we use the Pauli-Gürsey symmetry: $\tau_2 U_\mu(x)\tau_2 = U_\mu^*(x)$

$\longrightarrow$ it implies that $\det M(\mu)$ is real but not necessary positive

some lattice studies:

- Hands, Montvay, Scorzato, Skullerud, EPJC 22 (2001) 451
- Kogut, Toublan, Sinclair, PRD 68 (2003) 054507
- Hands, Kim, Skullerud, PRD 81 (2010) 091502
- Hands, Kenny, Kim, Skullerud, EPJA 47 (2011) 60

QCD-like theories

- dense two color matter: $U_\mu(x) \in SU(2)$

the 2-flavor action with change of variables

$$S_F = (\bar\psi \bar\phi) \begin{pmatrix} M(\mu) & J\gamma_5 \\ -\bar{J}\gamma_5 & M(-\mu) \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \bar\Psi \mathcal{M} \Psi$$

with $\bar\phi = -\psi_2^{tr} C \tau_2$ and $\phi = C^{-1}\tau_2 \bar\psi_2^{tr}$

symmetries:

$$\gamma_5 M(\mu) \gamma_5 = M^\dagger(-\mu)$$
$$K M(\mu) K^{-1} = M^*(\mu)$$

with $K \equiv C\gamma_5\tau_2$

consider:

$$\mathcal{M}^\dagger \mathcal{M} = \begin{pmatrix} M^\dagger(\mu)M(\mu) + |J|^2 & \\ & M^\dagger(-\mu)M(-\mu) + |\bar{J}|^2 \end{pmatrix}$$

with $\bar{J} = J^*$

⟶ block-diagonal in ψ, ϕ and regulated by $J^* J$
⟶ use ψ, ϕ-preconditioning, take square root analytically

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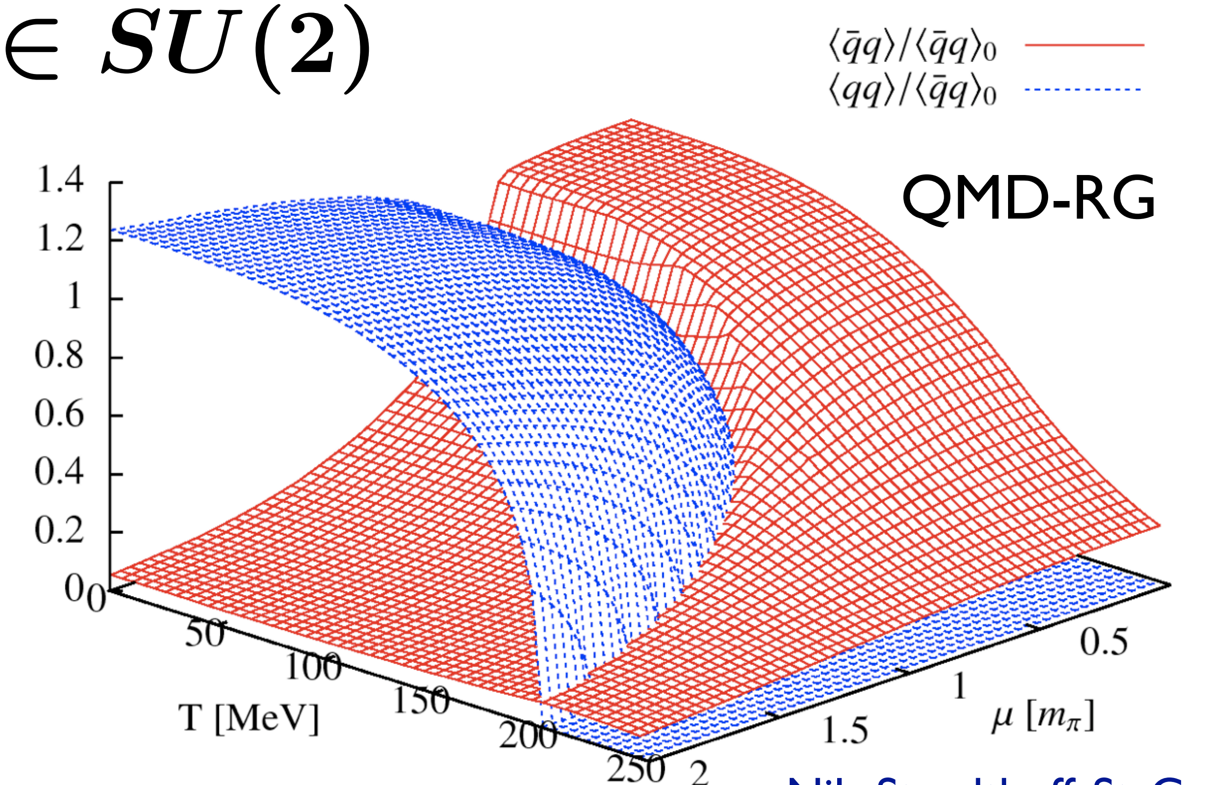
- Hands, Montvay, Scorzato, Skullerud, EPJC 22 (2001) 451
- Kogut, Toublan, Sinclair, PRD 68 (2003) 054507
- Hands, Kim, Skullerud, PRD 81 (2010) 091502
- Hands, Kenny, Kim, Skullerud, EPJA 47 (2011) 60

QCD-like theories

- dense two color matter: $U_\mu(x) \in SU(2)$

some results for the chiral and diquark condensates using a quark-meson-diquark model with proper-time RG flow

Nils Strodthoff,
St.Goar, March 16, 2011

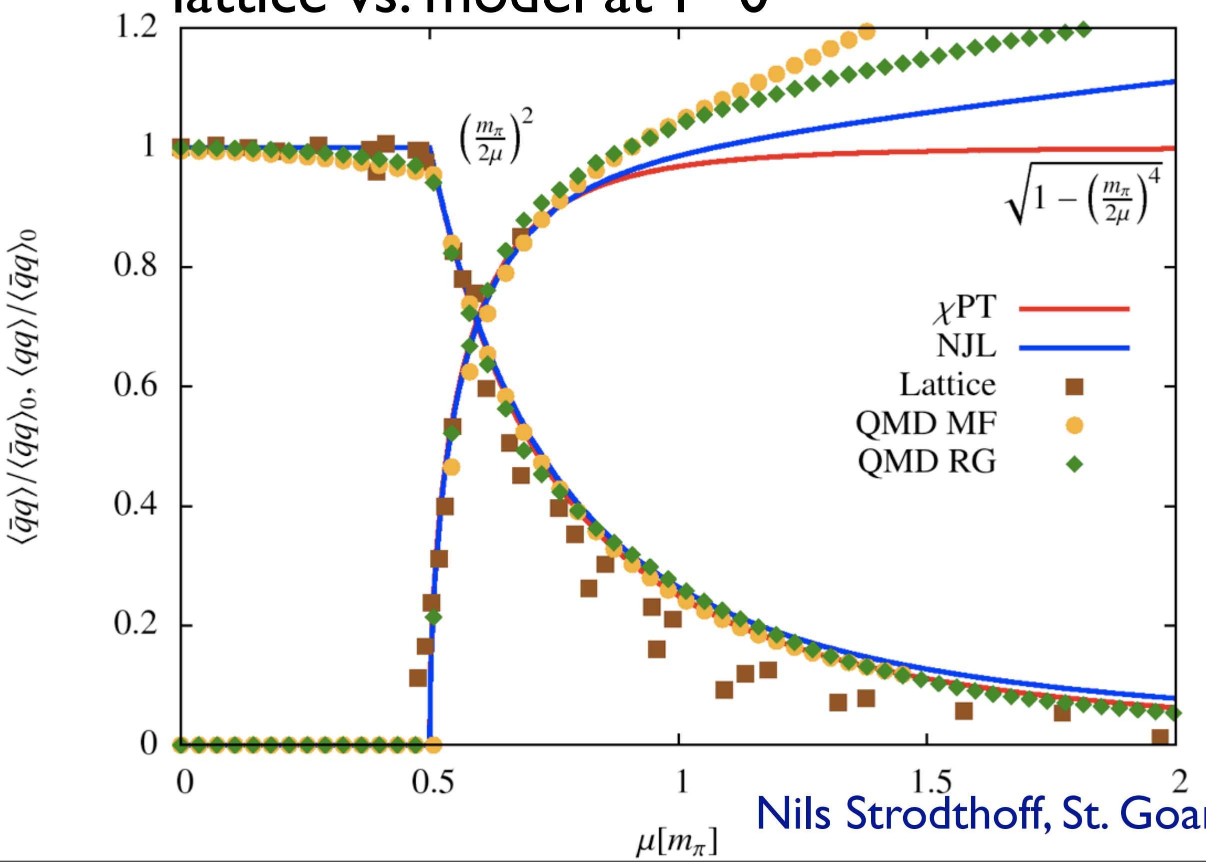


Nils Strodthoff, St. Goar 2011

short-comings / differences:

- color-neutral bound states of two quark: bosonic baryons
- enhanced symmetry (Pauli-Gürsey) $SU(N_f)_L \times SU(N_f)_R \to SU(2N_F)$
- more complex symmetry breaking pattern (5 pseudo Goldstone bosons: 3 pions + 2 diquarks)

lattice vs. model at T=0



Nils Strodthoff, St. Goar 2011

QCD with imaginary μ or iso-spin μ_I

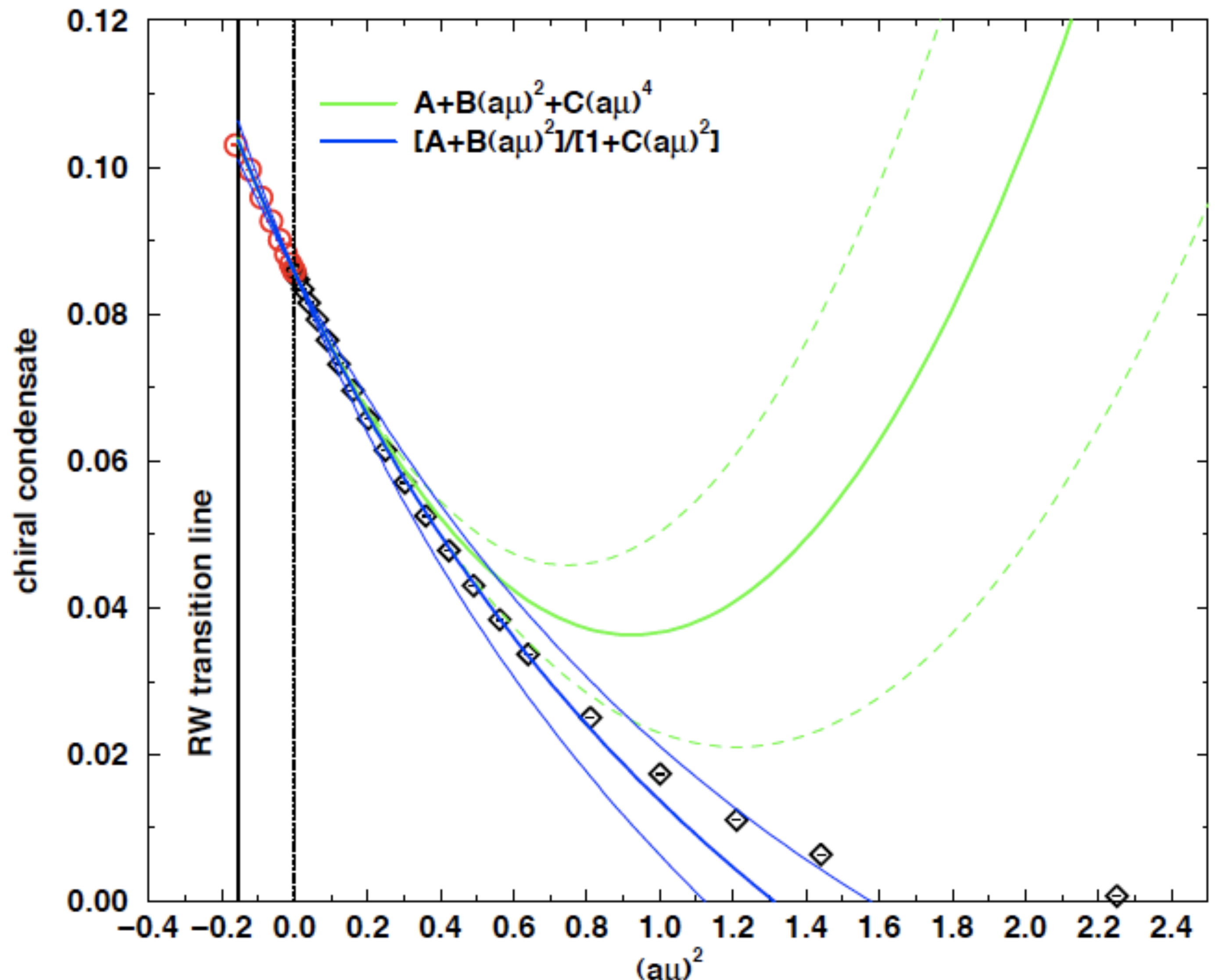
- iso-spin chemical potential: $\mu_u = -\mu_d$
  - fermion matrix acting on the iso-spin doublet has real determinant
  - introduce source term with quantum numbers of the pion condenstate
  - find variables in which $\mathcal{M}^\dagger \mathcal{M}$ is block-diagonal, use preconditioning
    Kogut, Sinclair, PRD 66 (2002) 034505

- pure imaginary chemical potential:
  - determinant is real, use standard HMC
  - partition function is periodic in $\mathrm{Im}\mu/T$ with periodicity of $2\pi T/3$
  - complex phase structure in the complex plane
    Roberge, Weiss, NPB 275 (1986) 734
  - critical behavior connected to the Roberge-Weiss transition may govern also QCD thermodynamics at $\mathrm{Re}(\mu) > 0$
    D'Elia, Massimo and Di Renzo, Francesco and Lombardo, PRD 76 (2007) 114509
    de Forcrand, Philipsen PRL 105 (2010) 152001

Extrapolation methods

- imaginary chemical potential:
  - perform HMC for $\mu^2 < 0$
  - extrapolate to $\mu^2 > 0$ by fitting data to an a appropriate Ansatz and perform analytic continuation
  - note: fitting range is limited by the periodicity of the partition function

two color QCD



Papa *et al.*, PoS Lat2006 (2006) 143

some lattice studies:

Phillipsen, Forcrand, JHEP 0811 (2008) 012;
Phillipsen, Forcrand, JHEP 0701 (2007) 077;
Phillipsen, Forcrand, NPB 673 (2003) 170;
D'Elia *et al.*, PRD 76 (2007) 114509;
D'Elia *et al.*, PRD 70 (2004) 074509 ;
D'Elia *et al.*, PRD 67(2003)014505 .

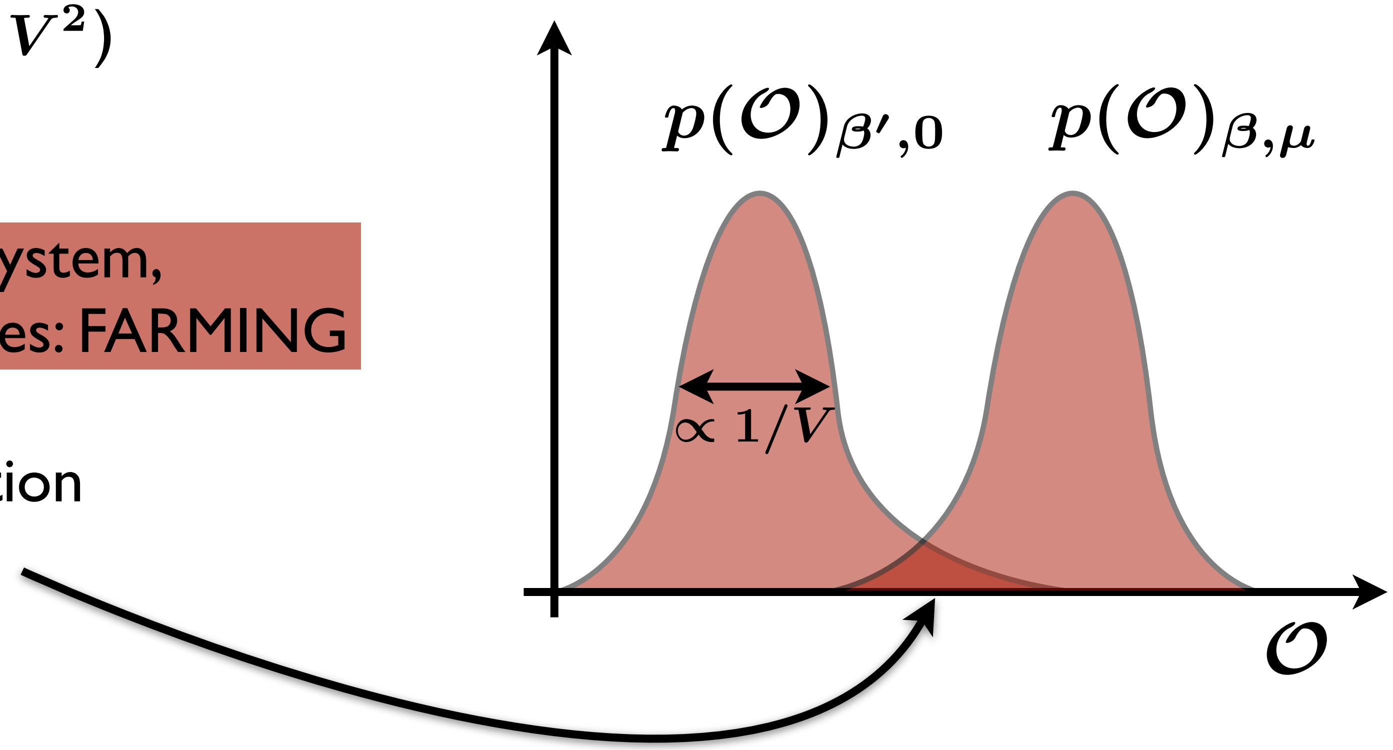
Extrapolation methods

- reweighting:

$$\langle \mathcal{O} \rangle_{\beta,\mu} = \frac{\langle \mathcal{O} R \rangle_{\beta',0}}{\langle R \rangle_{\beta',0}}, \qquad R = \frac{\det M(\mu)}{\det M(0)} \exp\{-\Delta S_G\}$$

re-weighting method by Budapest-Wuppertal group:

- exact determination of the determinant respectively all eigenvalues is required ($\propto V^2$)
  - no efficient parallel algorithms
  - small system sizes
  - ⟶ use only a few powerful nodes per system, distribute lattice parameters on nodes: FARMING
- overlap problem:
  - exponentially small tails of the distribution need to be determined very precisely
  - applicability range shrinks with volume
  - ⟶ very high statistics required



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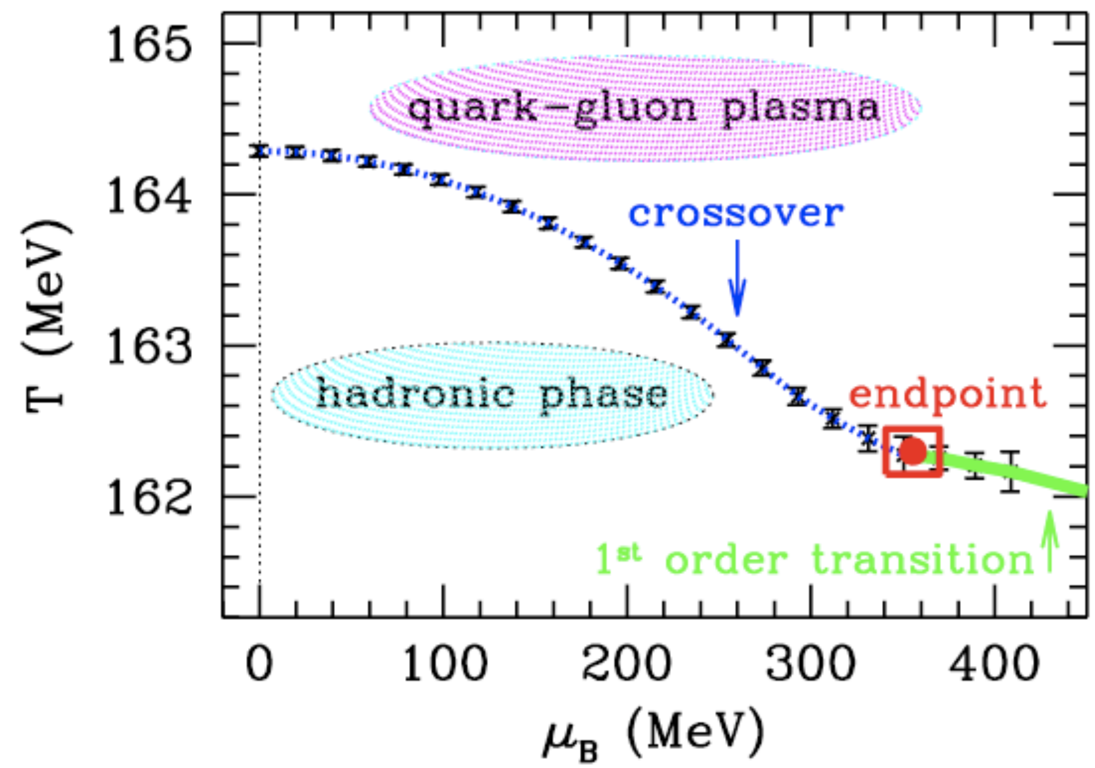


Fodor, Katz, JHEP 0404 (2004) 050;
Fodor, Katz, JHEP 0203 (2002) 014 ;
Fodor, Katz, PLB 534 (2002) 87.

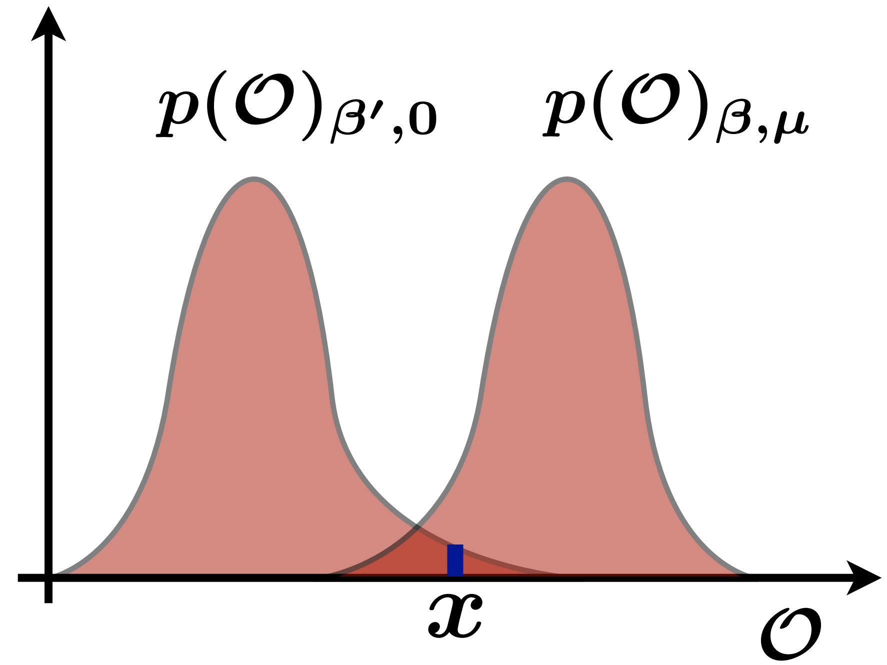
Extrapolation methods

$$\mathcal{Z}_{\mathcal{O}}(x,\mu) = \int dU\, |\det M(\mu)| \exp\{-S_G\}\, \delta(x-\mathcal{O})$$

Attempts to improve the re-weighting:

- modify the Monte-Carlo sampling in order to get a precise tail of the distribution
  - simulate at a fixed value of $\mathcal{O}$ (DOS)
  - introduce an additional weight function (Wang-Landau sampling)

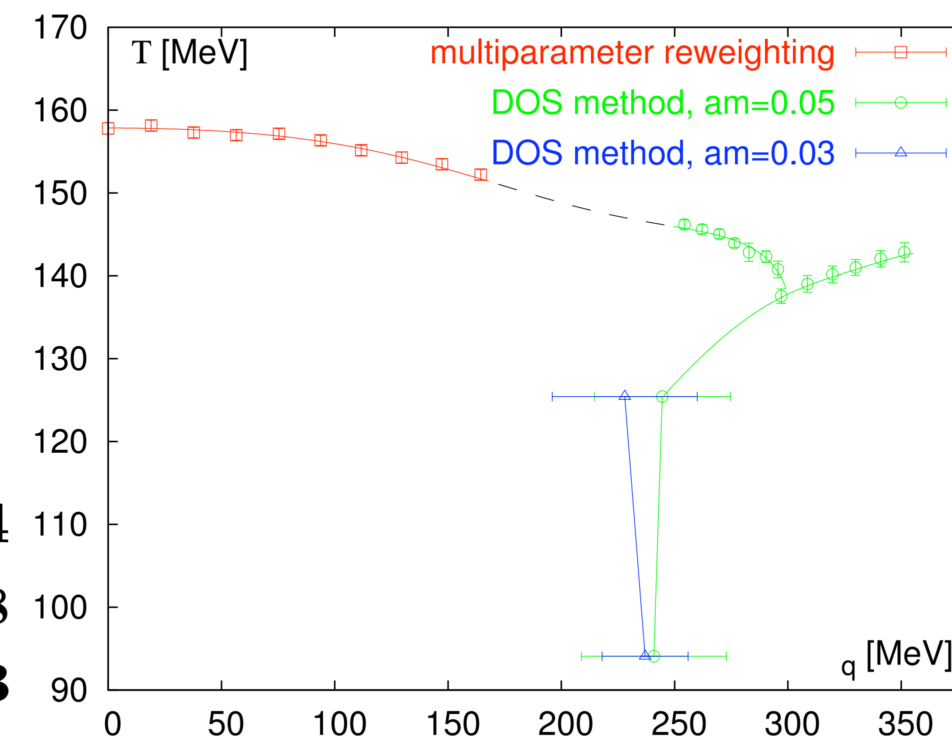
re-weight not only from the $(\mu = 0)$-ensemble but also from the phase quenched ensemble



→ much larger parameter space

→ even more FARMING

$N_f = 4$
$6^4,\ 6^3 \times 8$
$am_q = 0.05,\ 0.03$



Fodor, Katz, CS, JHEP 0703:121,2007.

Extrapolation methods

- Taylor expansion:
  - start from Taylor expansion of the pressure,

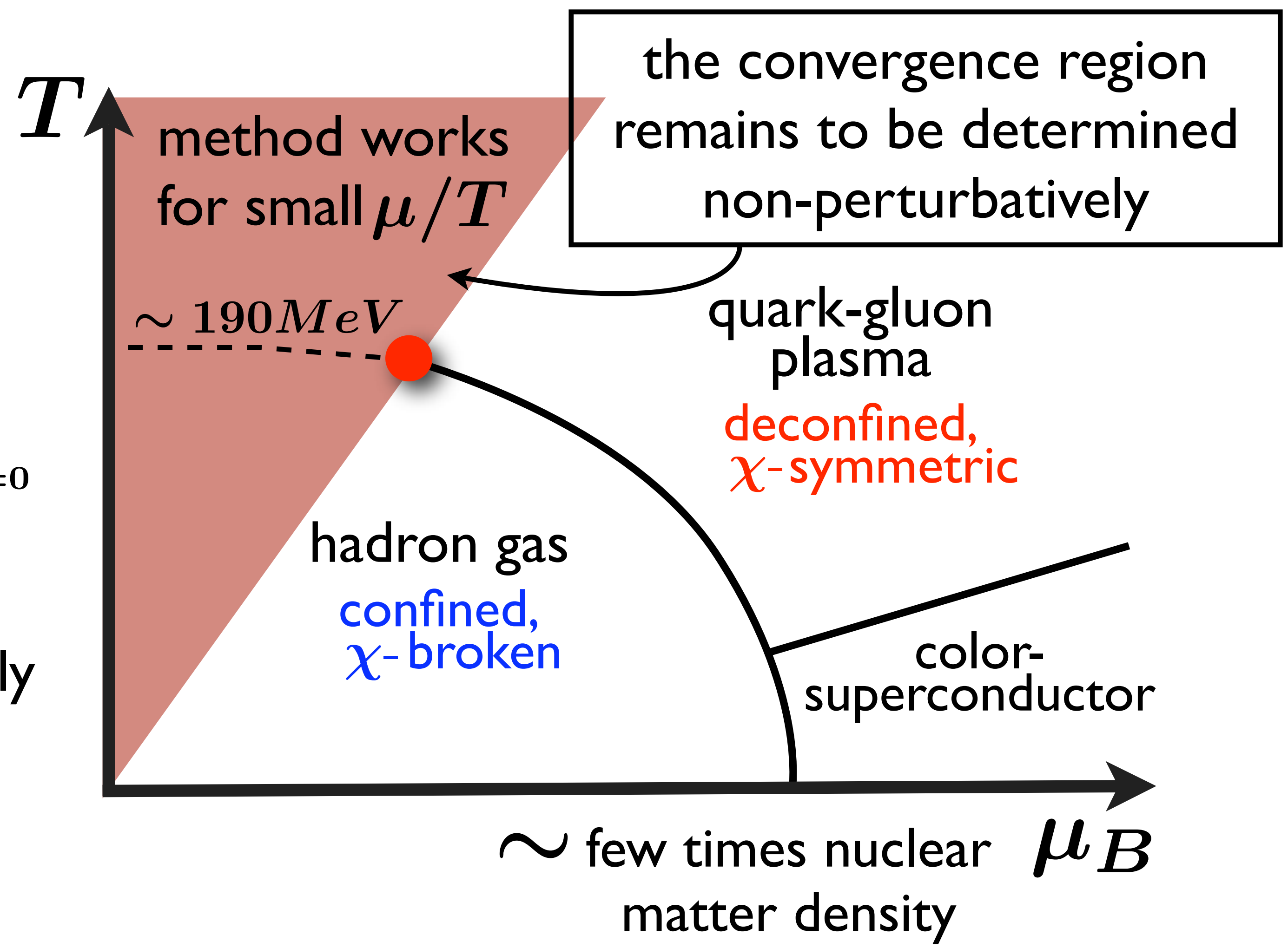
$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\mu_u,\mu_d,\mu_s) = \sum_{i,j,k} c^{u,d,s}_{i,j,k} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

  - calculate expansion coefficients for fixed temperature
  - no sign problem:
    all simulations are done at $\mu = 0$

$$c^{u,d,s}_{i,j,k} \equiv \frac{1}{i!j!k!} \frac{1}{VT^3} \cdot \left. \frac{\partial^i \partial^j \partial^k \ln Z}{\partial(\frac{\mu_u}{T})^i \partial(\frac{\mu_d}{T})^j \partial(\frac{\mu_s}{T})^k} \right|_{\mu_{u,d,s}=0}$$

  - method is straight forward:
    all terms can be generated automatically

Allton *et al.*, PRD66:074507,2002;
Allton *et al.*, PRD68:014507,2003;
Allton *et al.*, PRD71:054508,2005.



Extrapolation methods

- formulate all operators in term of space-time, color (and spin) traces:

$$\frac{\partial(\ln\det M)}{\partial\mu} = \mathcal{D}_1 = \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}\right)$$

$$\frac{\partial^2(\ln\det M)}{\partial\mu^2} = \mathcal{D}_2 = \mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) - \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right)$$

$$\frac{\partial^3(\ln\det M)}{\partial\mu^3} = \mathcal{D}_3 = \mathrm{Tr}\left(M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) - 3\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) + 2\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right)$$

$$\frac{\partial^4(\ln\det M)}{\partial\mu^4} = \mathcal{D}_4 = \mathrm{Tr}\left(M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) - 4\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) - 3\mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) + 12\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) - 6\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right)$$

- evaluate all traces by noisy estimators:

$$\mathrm{Tr}\left(\frac{\partial^{n_1}M}{\partial\mu^{n_1}}M^{-1}\frac{\partial^{n_2}M}{\partial\mu^{n_2}}\cdots M^{-1}\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\eta_k^\dagger\frac{\partial^{n_1}M}{\partial\mu^{n_1}}M^{-1}\frac{\partial^{n_2}M}{\partial\mu^{n_2}}\cdots M^{-1}\eta_k$$

with N random vectors, satisfying $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\eta_{n,i}^*\eta_{n,j} = \delta_{i,j}$

- construct expansion coefficients from $\mathcal{D}_n^u, \mathcal{D}_n^d, \mathcal{D}_n^s,$ with unbiased estimators

$$c_{2,0,0}^{u,d,s} = \frac{1}{2}\frac{N_\tau}{N_\sigma^3}\left(\left\langle\mathcal{D}_2^u\right\rangle + \left\langle\left(\mathcal{D}_1^u\right)^2\right\rangle\right)$$

Hadronic fluctuations

- Taylor expansion coefficients are the moments of hadronic fluctuations

$$2c_2^X = \frac{1}{VT^3}\left\langle N_X^2 \right\rangle \qquad 24c_4^X = \frac{1}{VT^3}\left(\left\langle N_X^4 \right\rangle - 3\left\langle N_X^2 \right\rangle^2\right)$$

$$X = B, Q, S, I, \ldots$$

Main ingredients:

- fast solver for the linear equation $Ax = b$, with A being a large and sparse matrix.
  - Iterative Krylov Subspace Methods are well suited for parallelization.

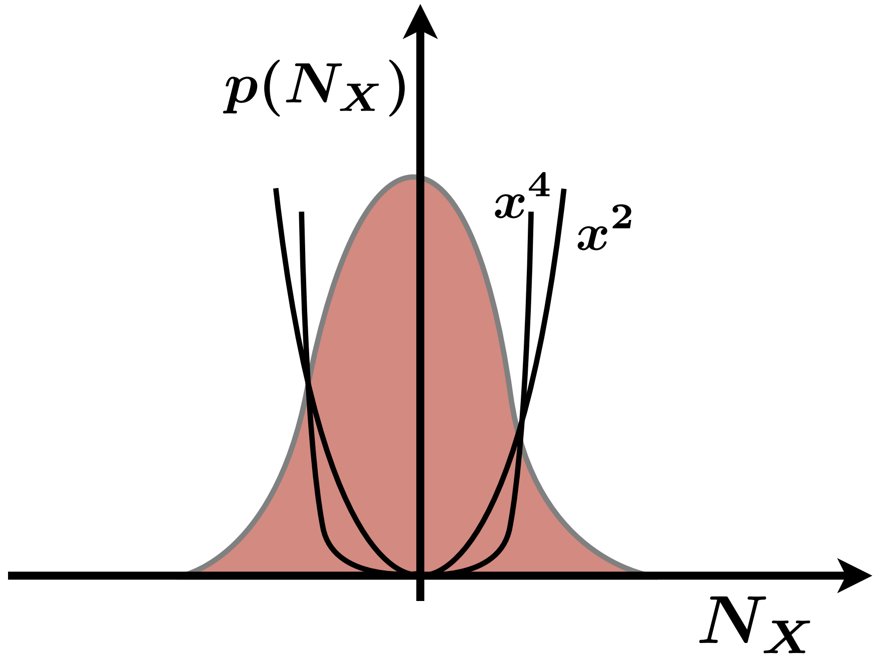
→ relatively large systems can be handled on massive parallel machines

- stochastic estimator of $\mathrm{Tr}A$
  - use noise reduction techniques

expansion coefficients with respect to μ_X are connected to the moments of the n_X-distribution

- higher order moments are getting more and more sensitive to the tail of the distribution

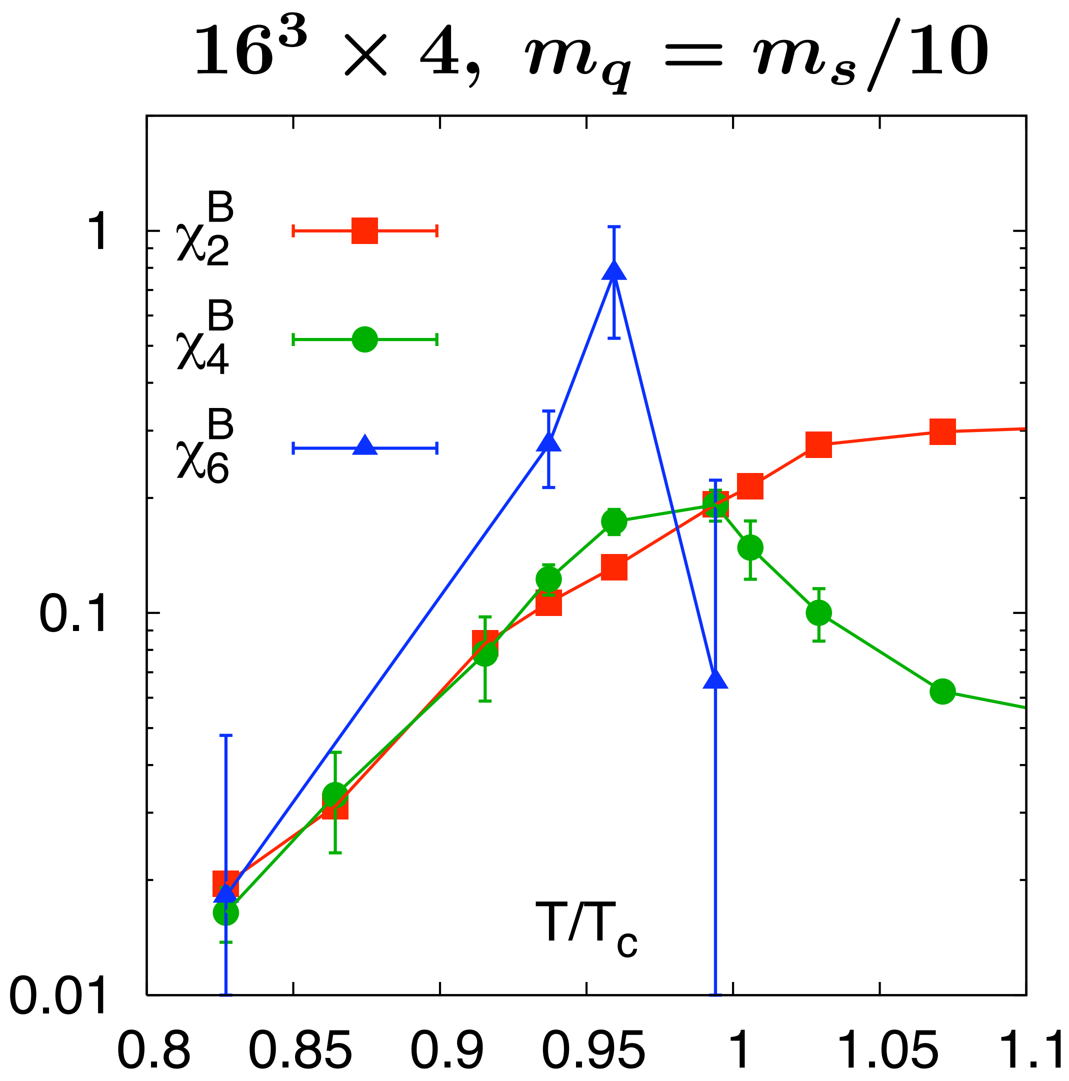
→ high statistics required



nth-moment:

$$m_n = \int dx\, x^n p(x)$$

Baryonic fluctuations



Analyzing the critical behavior:

scaling field (chiral limit):

$$t = \frac{1}{t_0}\left(\frac{T - T_c}{T_c} + \kappa \left(\frac{\mu_B}{T}\right)^2\right)$$

free energy:

$$f = A_\pm |t|^{2-\alpha} + \text{regular}$$

critical exponent:

$$-0.15 < \alpha < -0.11$$

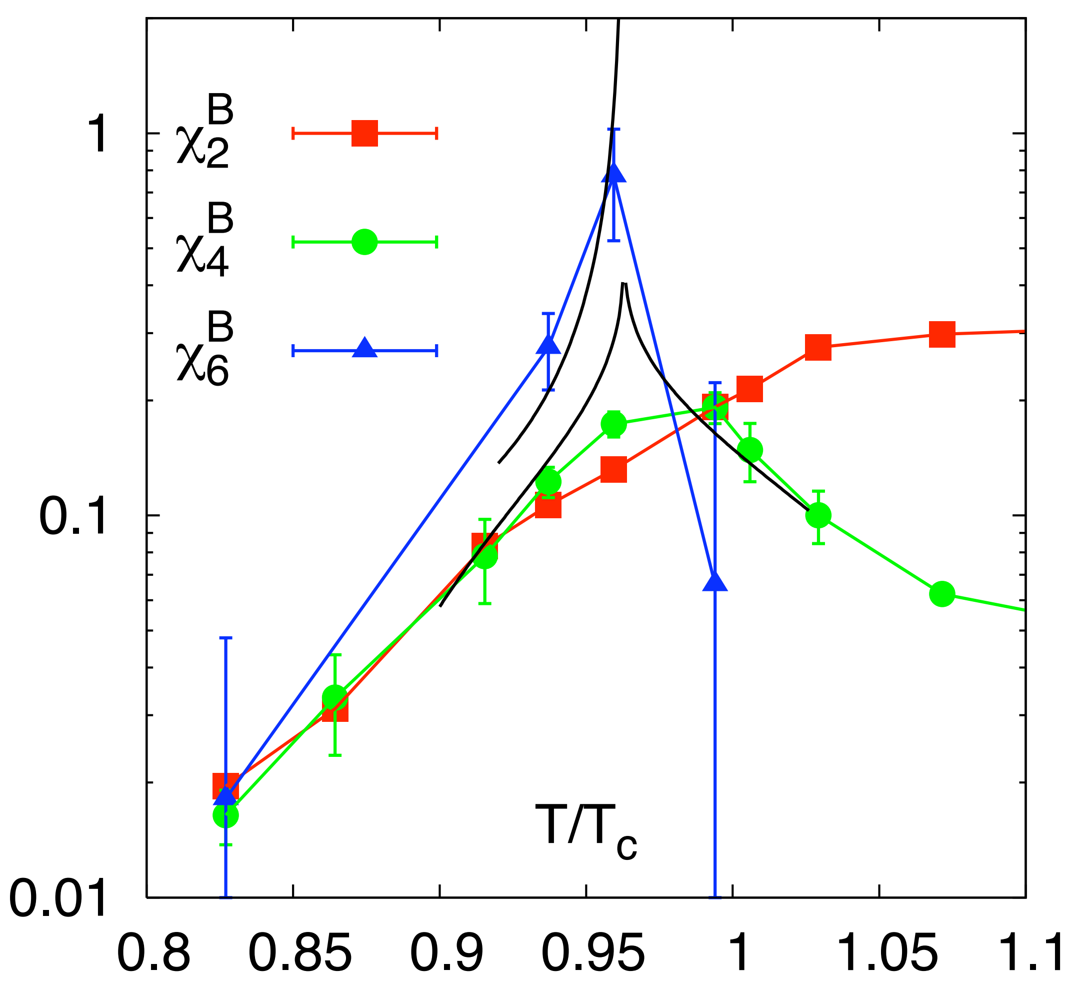
$$\chi_2^B \sim \mp 2A_\pm(2-\alpha)\kappa\,|t|^{1-\alpha} + \text{regular}$$

$$\chi_4^B \sim -12A_\pm(2-\alpha)(1-\alpha)\kappa^2\,|t|^{-\alpha} + \text{regular}$$ → kink (chiral limit)

$$\chi_6^B \sim \mp 120A_\pm(2-\alpha)(1-\alpha)(-\alpha)\kappa^3\,|t|^{-1-\alpha} + \text{regular}$$ → divergent (chiral limit)

Baryonic fluctuations

$16^3 \times 4,\ m_q = m_s/10$



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Baryonic fluctuations

hadron resonance gas

$$\ln Z(T,V,\mu_B,\mu_S,\mu_Q) = \sum_{i\in hadrons} \ln Z_{m_i}(T,V,\mu_B,\mu_S,\mu_Q)$$

$$\sum_{i\in mesons} \ln Z^B_{m_i}(T,V,\mu_S,\mu_Q) + \sum_{i\in baryons} \ln Z^F_{m_i}(T,V,\mu_B,\mu_S,\mu_Q)$$

mesons:

$$\frac{p_i}{T^4} = \frac{d_i}{\pi^2}\left(\frac{m_i}{T}\right)^2 \sum_{l=1}^{\infty}(+1)^{l+1} l^{-2} K_2(lm_i/T) \cosh(lS_i\mu_S/T + lQ_i\mu_Q/T)$$

baryons:

$$\frac{p_i}{T^4} = \frac{d_i}{\pi^2}\left(\frac{m_i}{T}\right)^2 \sum_{l=1}^{\infty}(-1)^{l+1} l^{-2} K_2(lm_i/T) \cosh(lB_i\mu_B/T + lS_i\mu_S/T + lQ_i\mu_Q/T)$$

Boltzmann approximation

ratios are independent of spectrum and volume

$\longrightarrow$ possibly large parts of cut-off effects cancel

3 ratios:

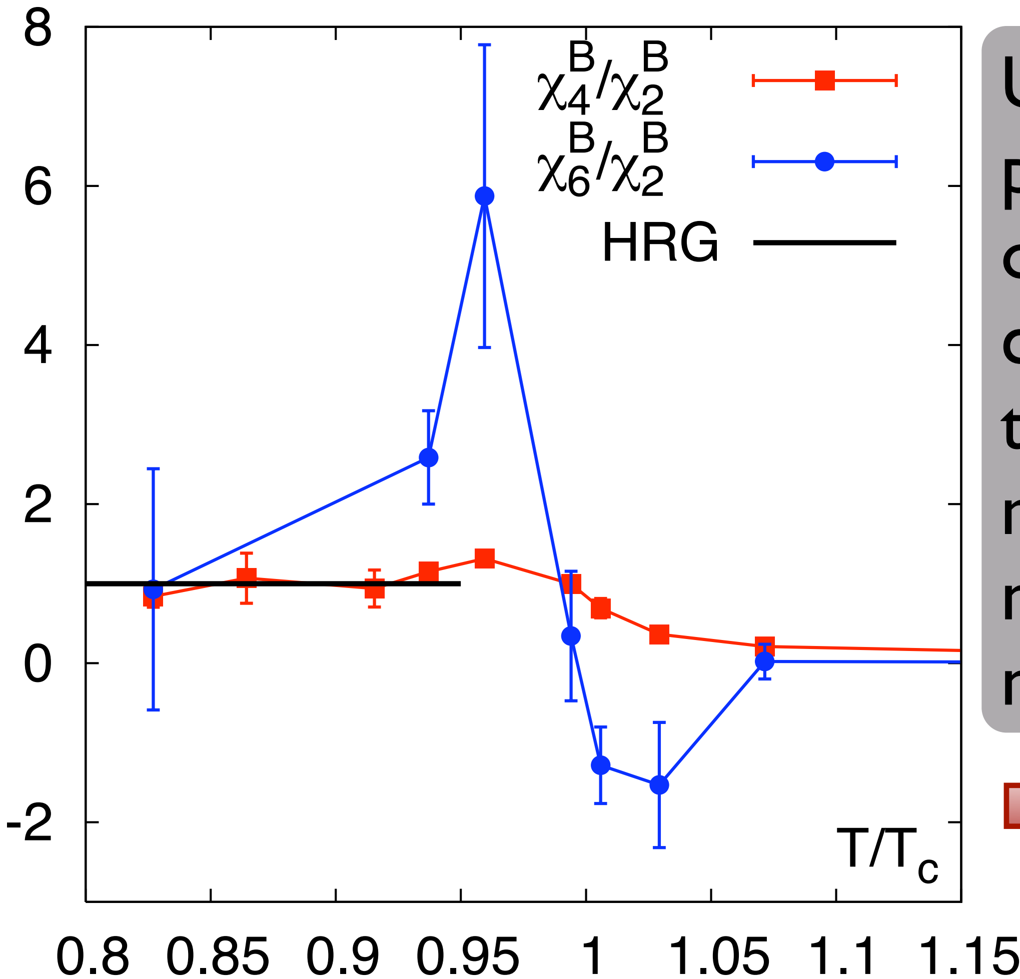
$$\frac{\chi_4^B}{\chi_2^B} = \kappa\sigma^2 = \frac{B^4}{B^2} = 1$$

$$\frac{\chi_3^B}{\chi_2^B} = S\sigma = \frac{B^3}{B^2}\tanh(\mu_B/T)$$

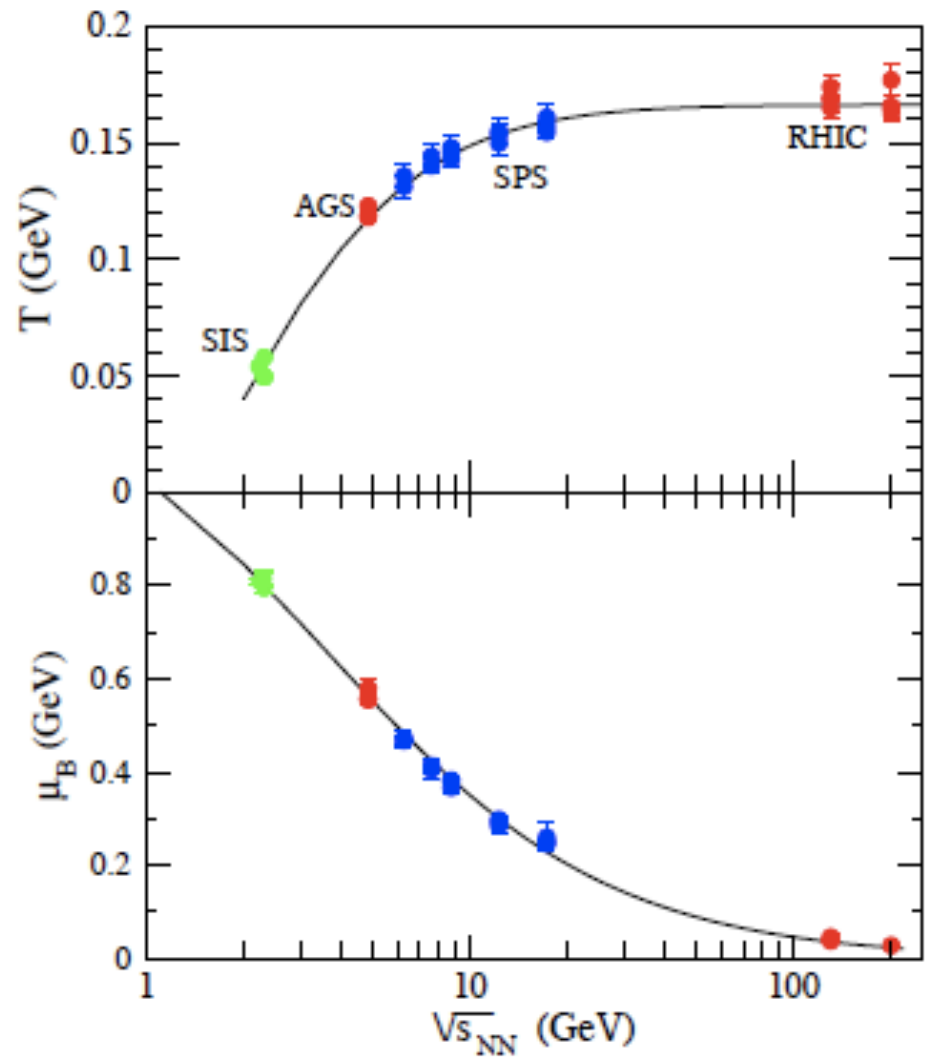
$$\frac{\chi_2^B}{\chi_1^B} = \sigma^2/N_B = \frac{B^2}{B^1}\coth(\mu_B/T)$$

Baryonic fluctuations

- sixth order fluctuations



Use parametrization of freeze-out curve to connect to STAR measurements of net-proton number



[CS, arXiv:1007.5164]

$$T(\mu_B) = 0.166\ \text{GeV} - 0.139\ \text{GeV}^{-1}\mu_B^2 - 0.053\ \text{GeV}^{-3}\mu_B^4$$

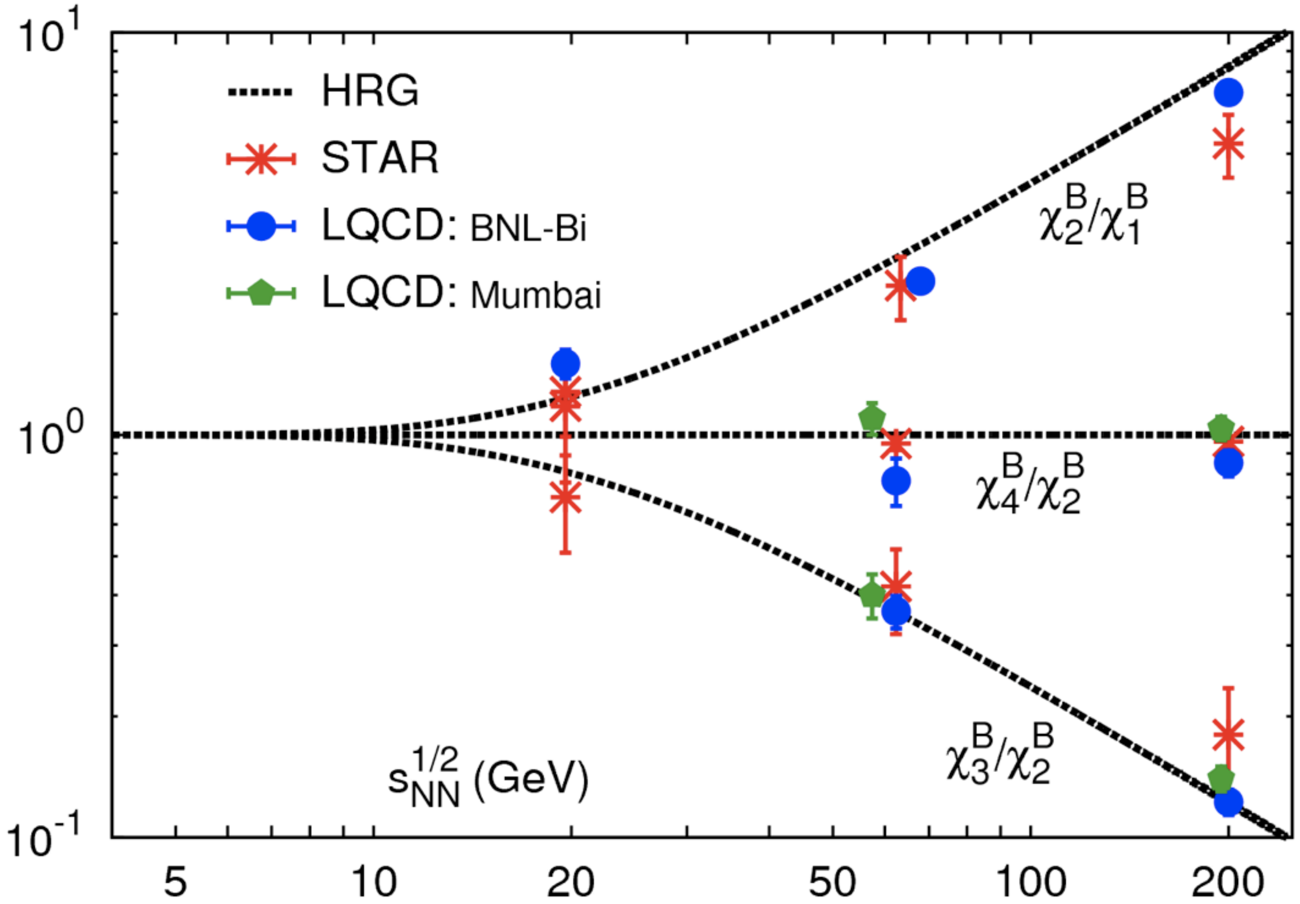
$$\mu_B(\sqrt{s}) = \frac{1.308\ \text{GeV}}{1 + 0.273\ GeV^{-1}\sqrt{s}}$$

- sensitive to relevant quantum numbers in the medium
- divergent at the critical point

[Cleymans et al., Phys. Rev. C 63 (2006) 034905]

Baryonic fluctuations

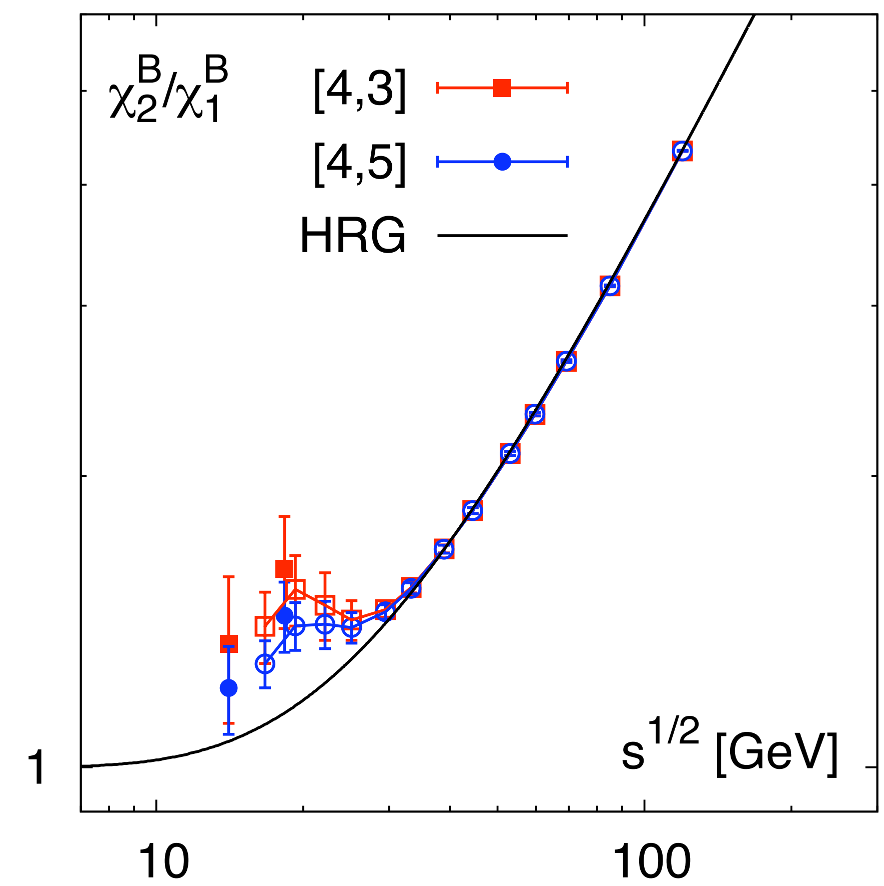
Lattice vs. Experiment:



Mukherjee, QM 2011

[HRG: Karsch, Redlich, PLB 695 (2011)]

[STAR data: Aggarwal et al, PRL (2010) 022302]



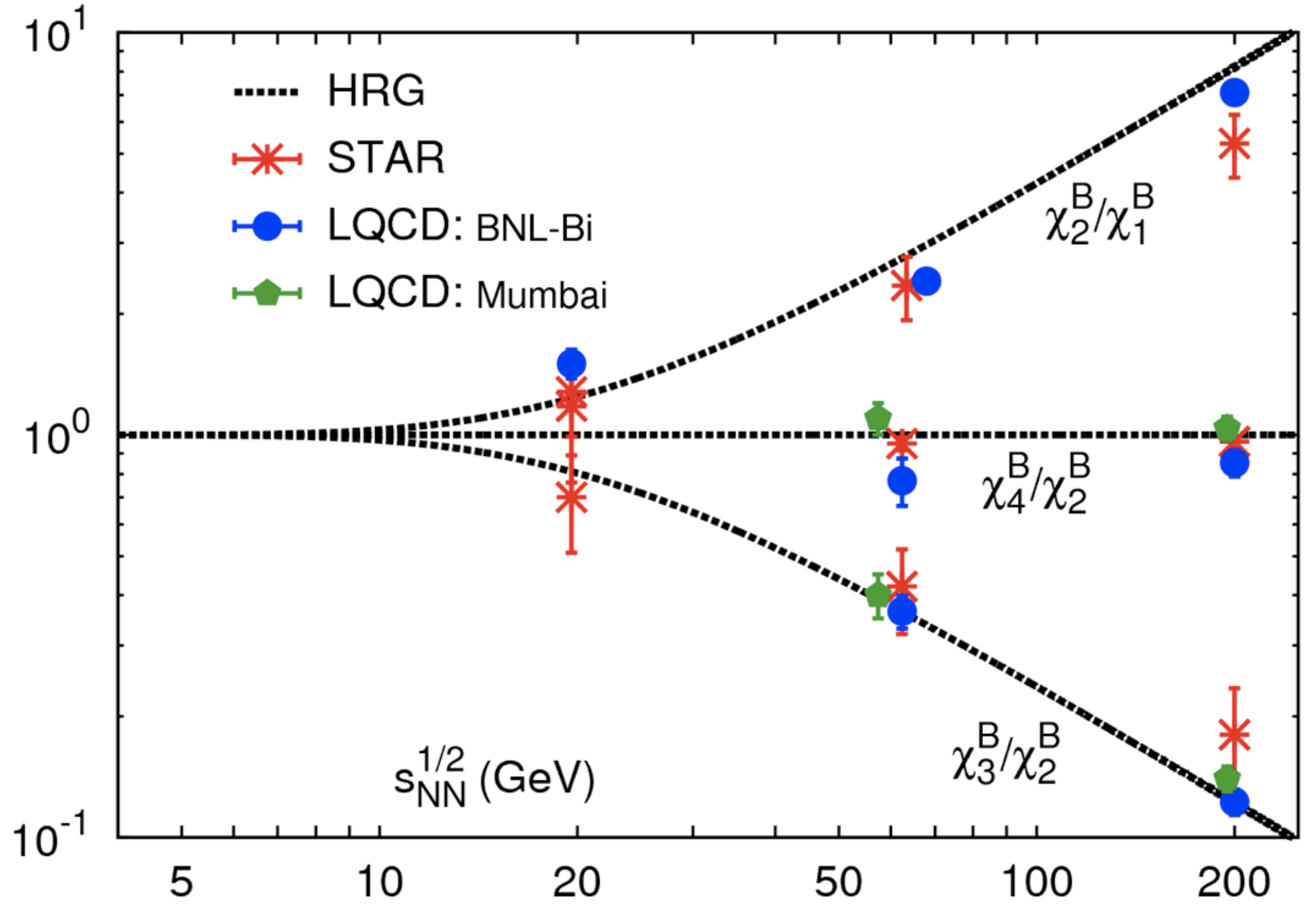
CS, *Theor. Phys. Suppl.186, 563 (2010)*

- net-proton number fluctuations can be described by the HRG
  solid lines: $\mu_Q \neq 0, \mu_S \neq 0$
  dashed lines: $\mu_Q = 0, \mu_S = 0$

- fluctuations increase for small $\sqrt{s}$
- sensitive to truncation of the series due to close radius of convergence

Baryonic fluctuations

Lattice vs. Experiment:



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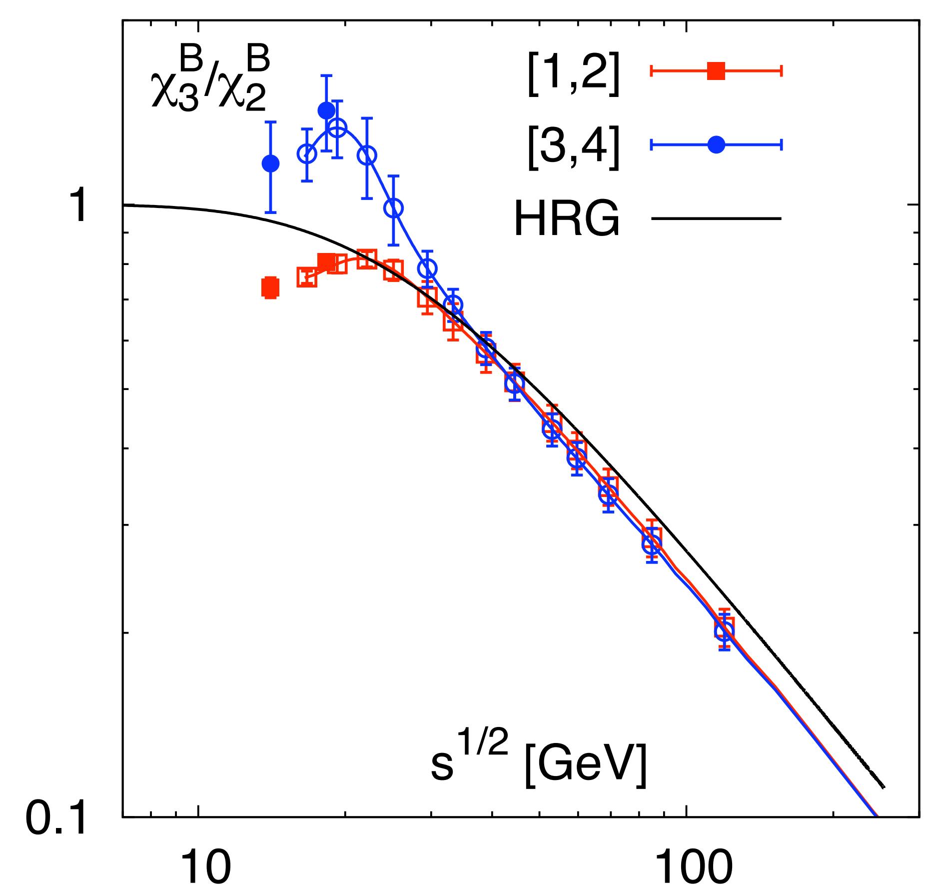

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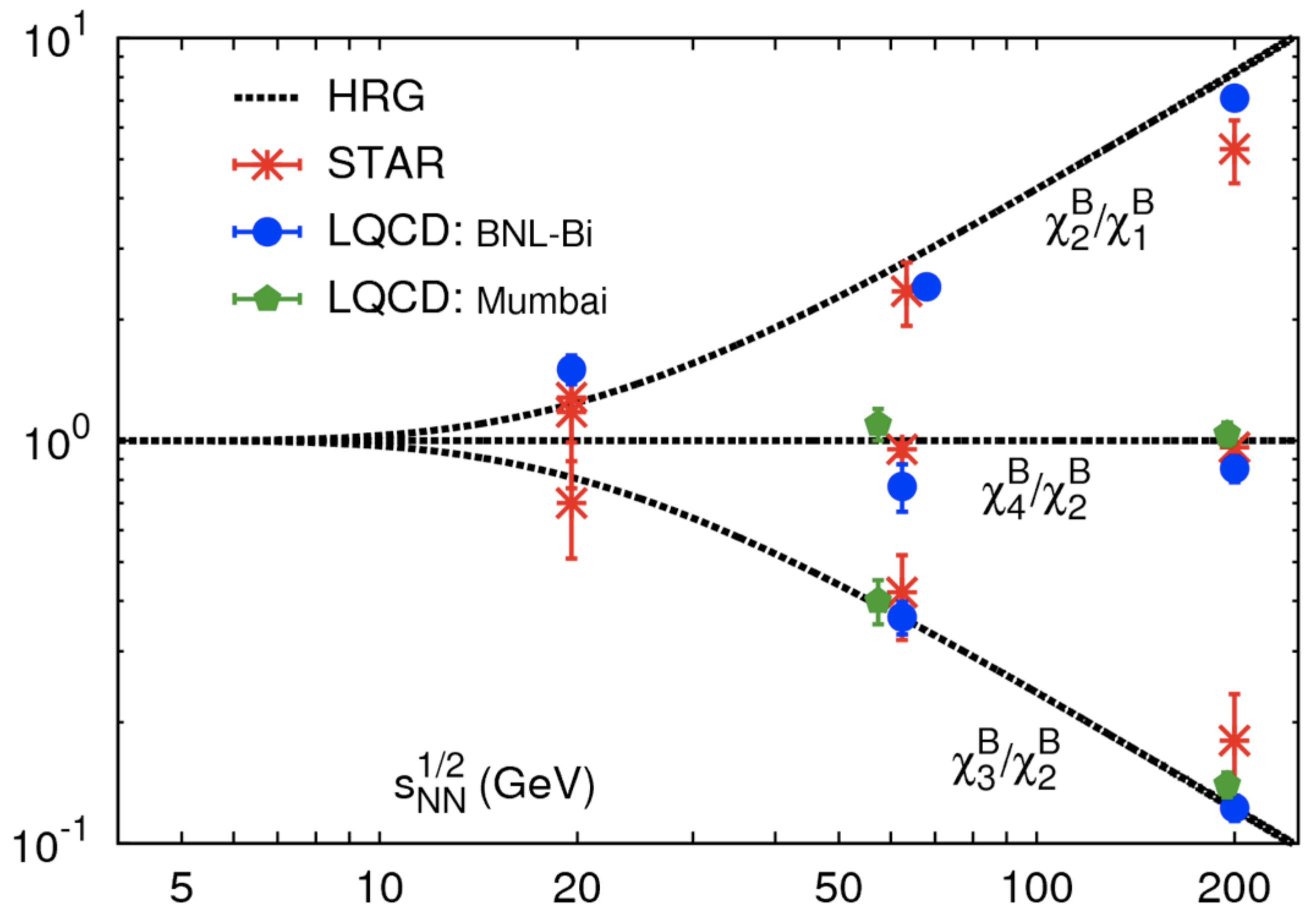
Baryonic fluctuations

Lattice vs. Experiment:



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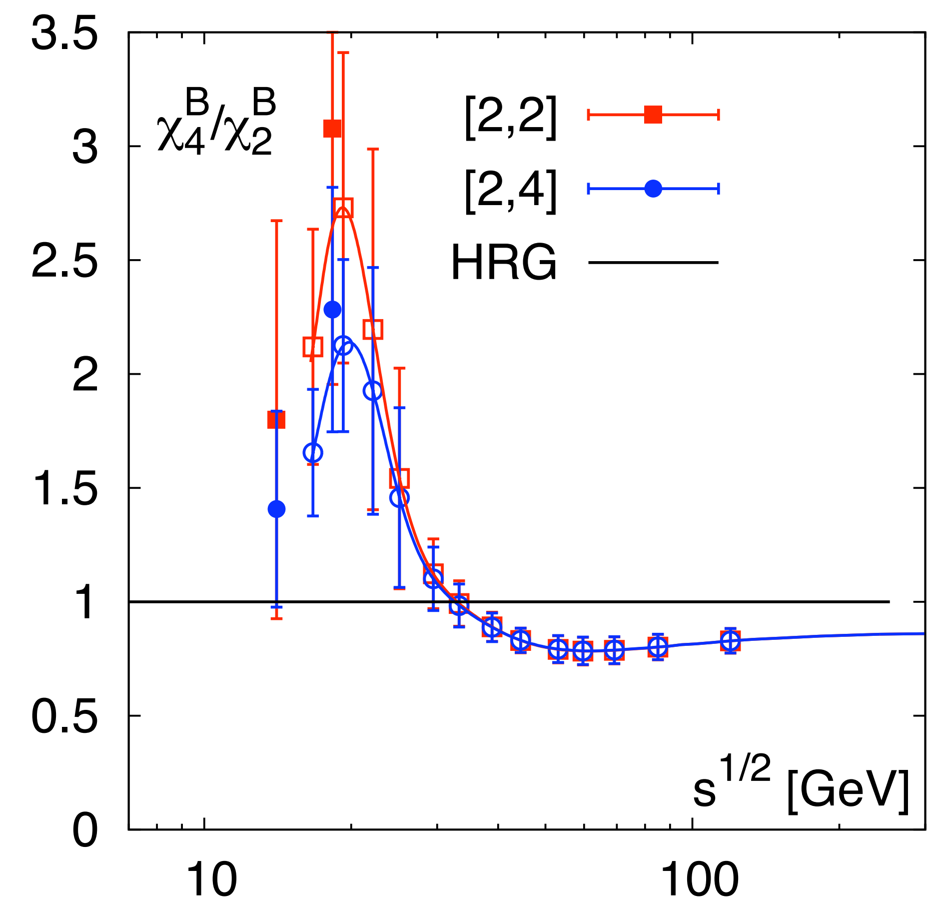

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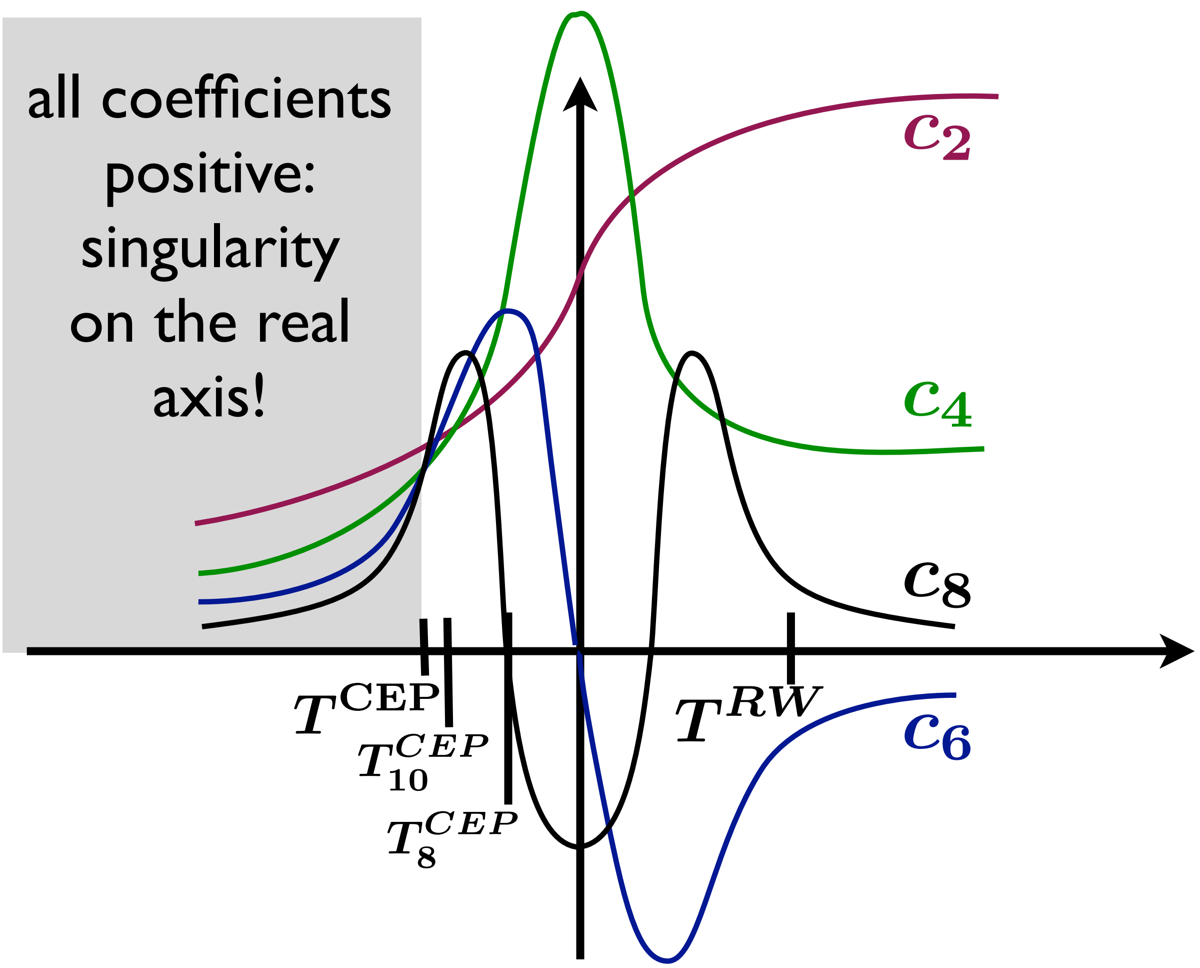
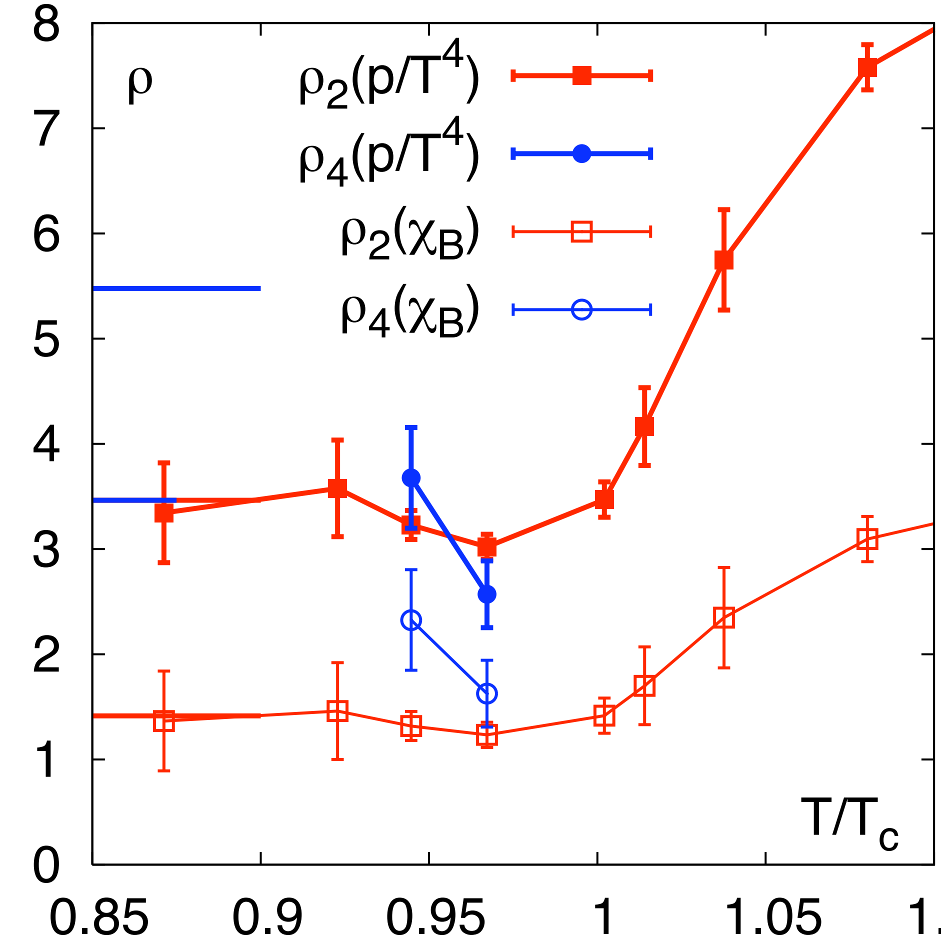
<u>method for locating of the CEP:</u>

- determine largest temperature where all coefficients are positive → $T^{\rm CEP}$
- determine the radius of convergence at this temperature → $\mu^{\rm CEP}$

$$p = c_0 + c_2\,(\mu_B/T)^2 + c_4\,(\mu_B/T)^4 + \cdots$$

$$\chi_B = 2c_2 + 12c_4\,(\mu_B/T)^2 + 30c_6\,(\mu_B/T)^4 + \cdots$$





CS, Theor. Phys. Suppl. 186, 563 (2010)

first non-trivial estimate of $T^{\rm CEP}$ by c_8
second non-trivial estimate of $T^{\rm CEP}$ by c_{10}

$$\rho_n(p) = \sqrt{c_n/c_{n+2}}$$

$$\rho = \lim_{n\to\infty} \rho_n$$

The critical endpoint

method for locating of the CEP:

- determine largest temperature where all coefficients are positive → T^{CEP}
- determine the radius of convergence at this temperature → μ^{CEP}

$$p = c_0 + c_2\left(\mu_B/T\right)^2 + c_4\left(\mu_B/T\right)^4 + \cdots$$

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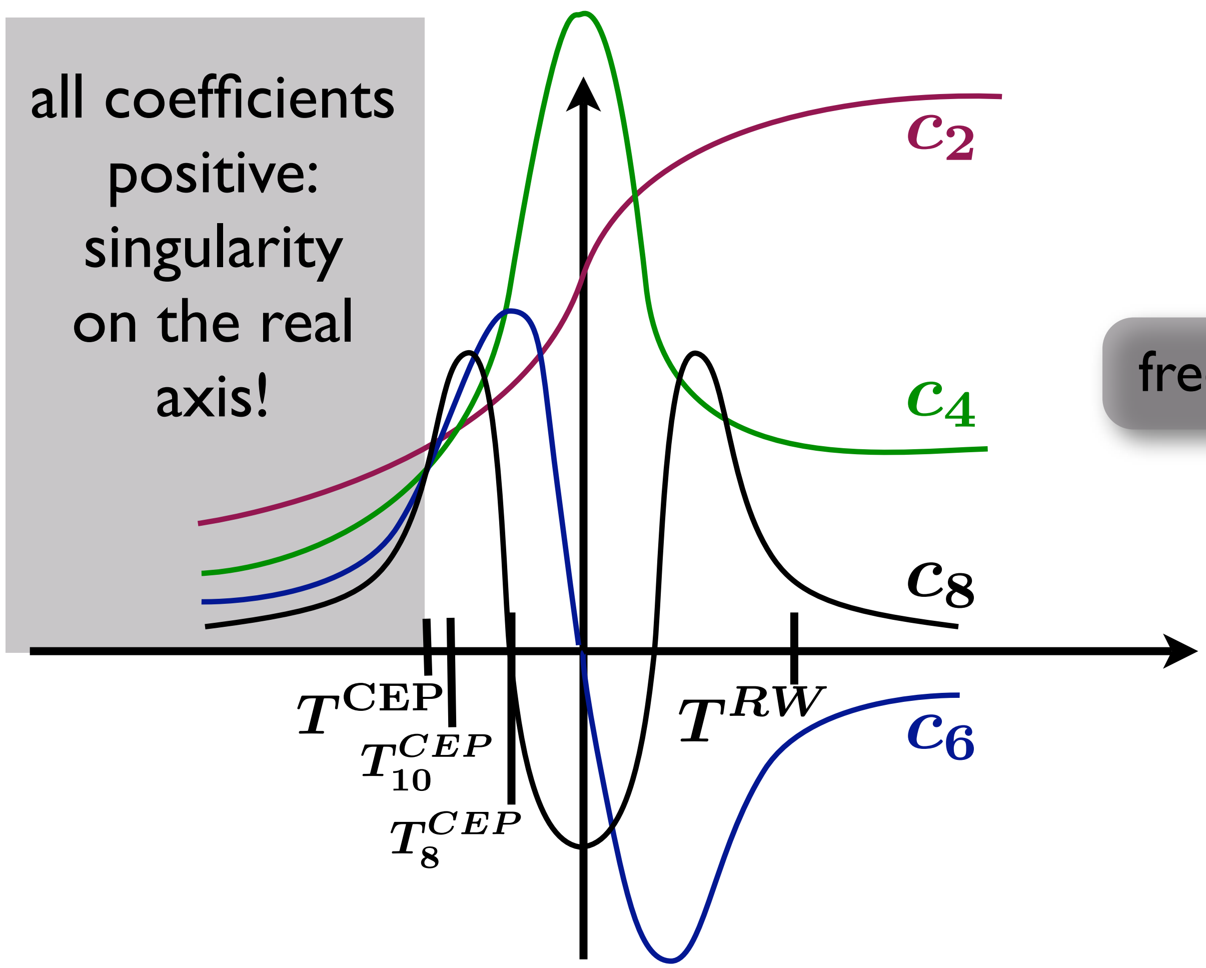

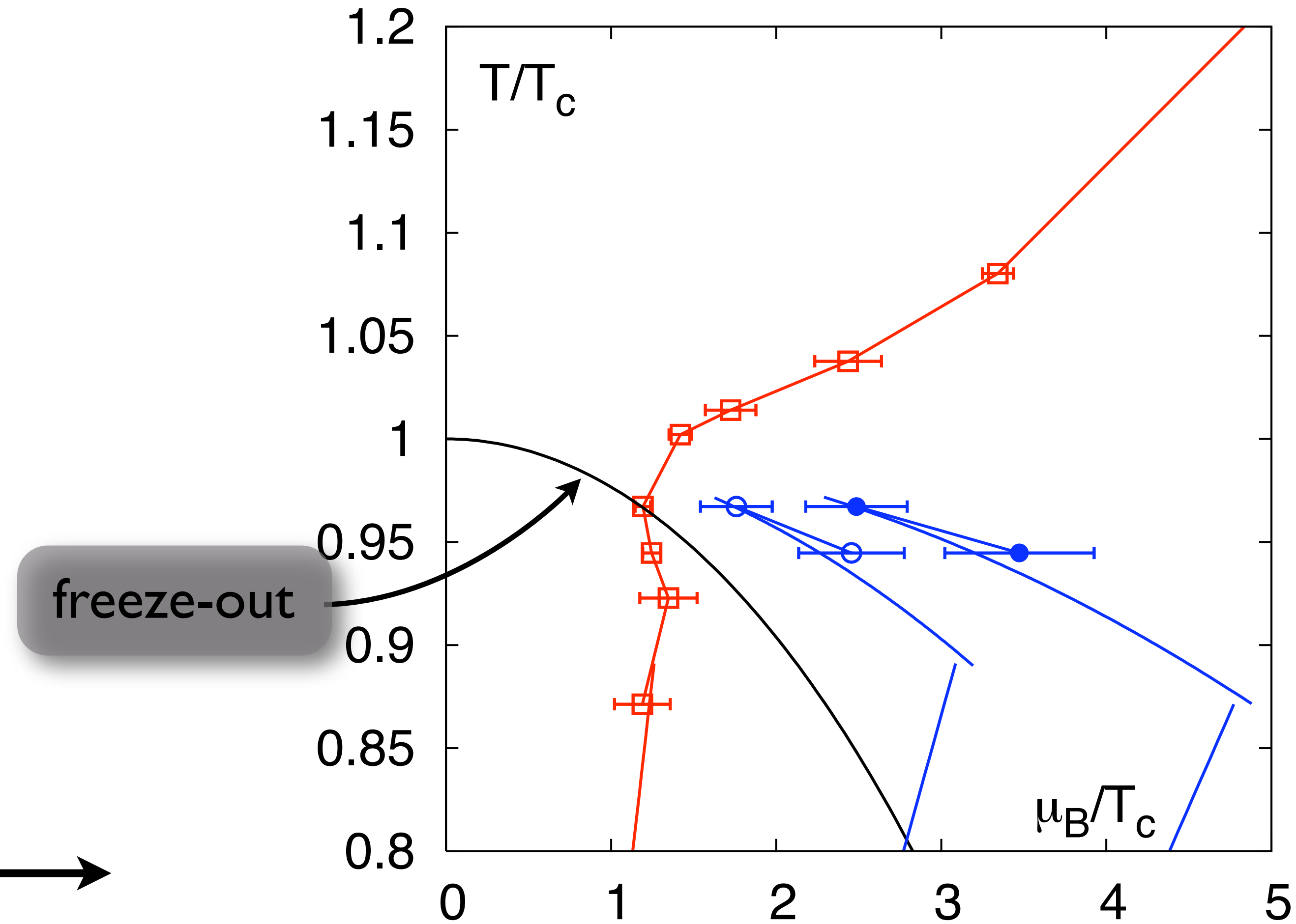

CS, Theor. Phys. Suppl. 186, 563 (2010)

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The critical endpoint

method for locating of the CEP:

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- determine the radius of convergence at this temperature $\rightarrow \mu^{\mathrm{CEP}}$

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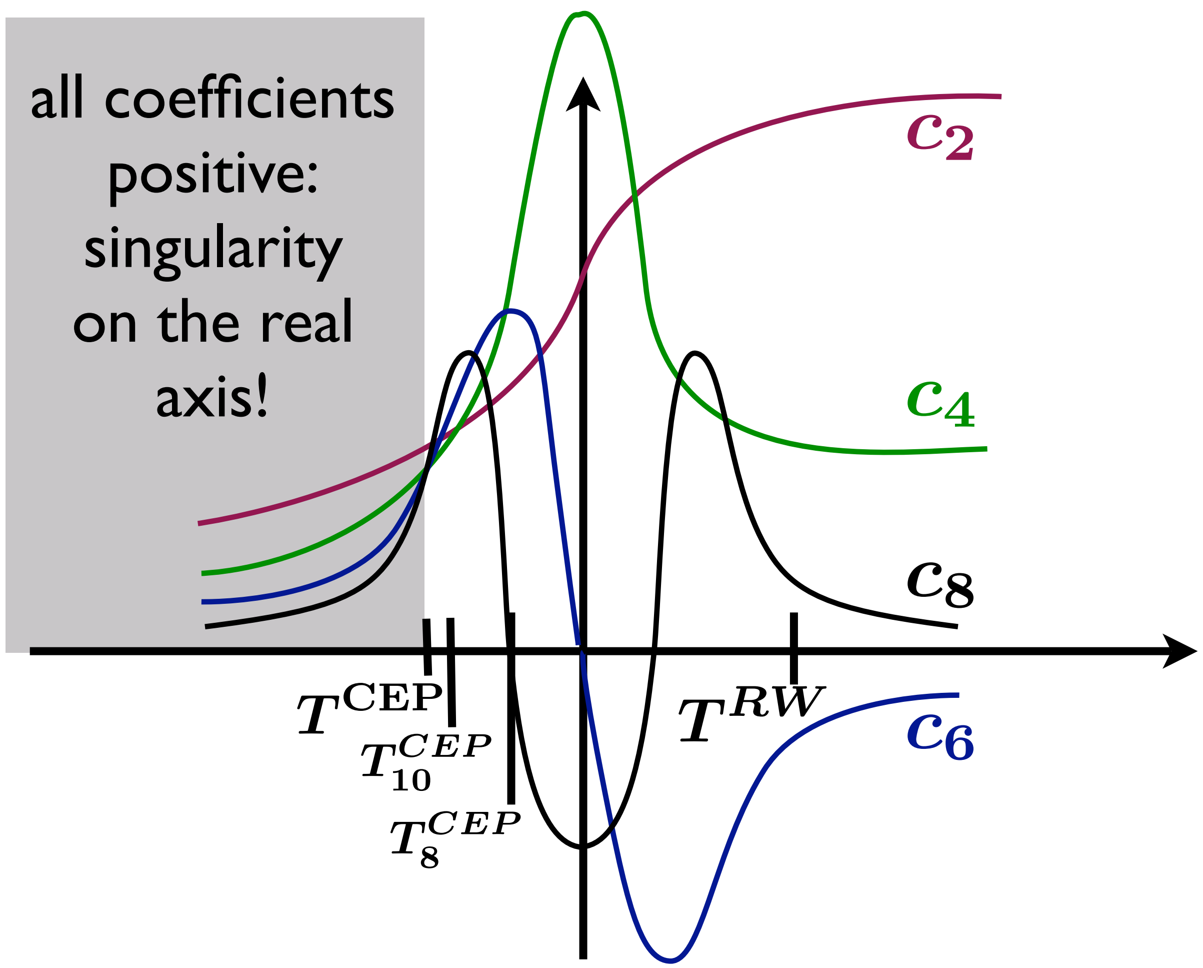

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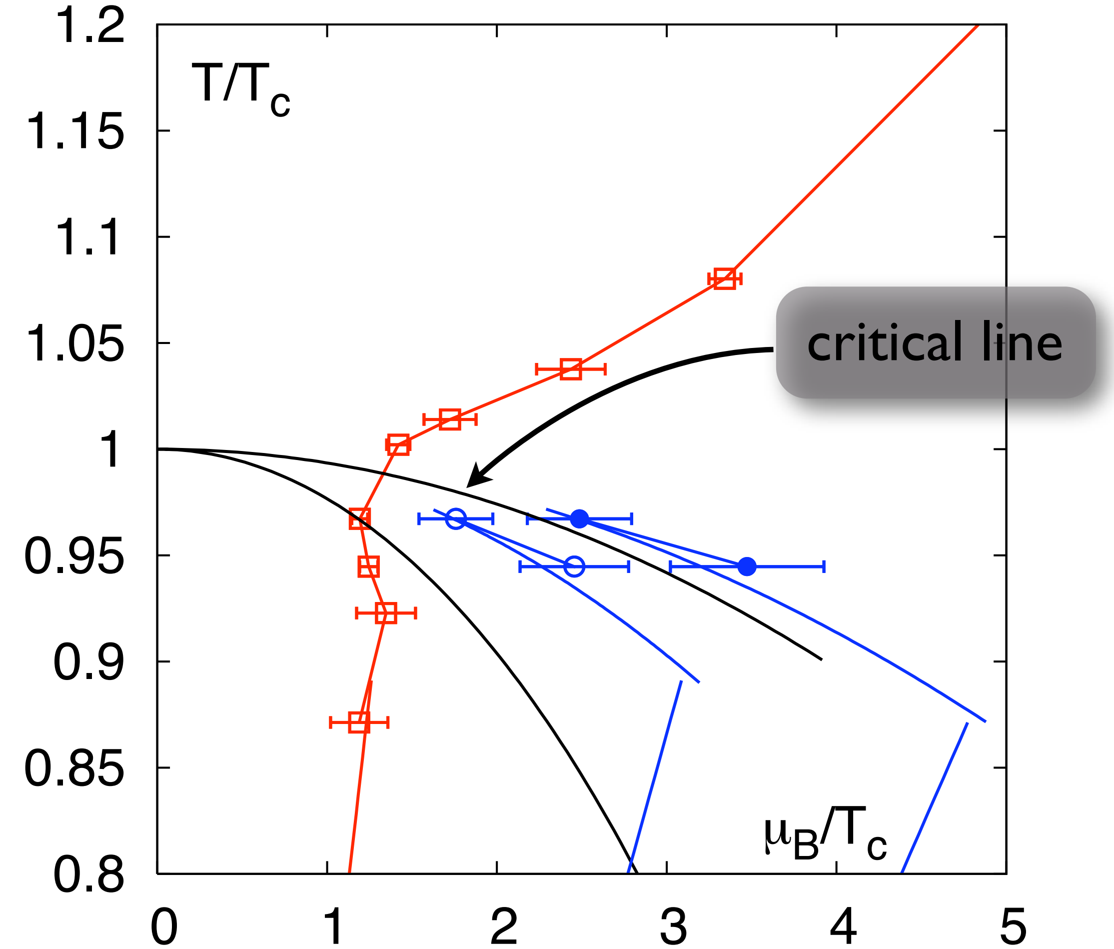


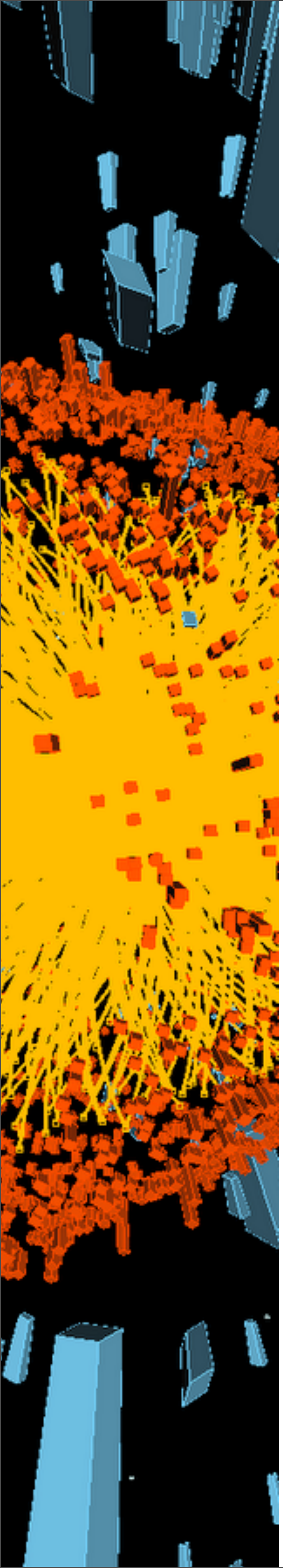
CS, Theor. Phys. Suppl.186, 563 (2010)

- radius of convergence is consistent with critical line in the chiral limit

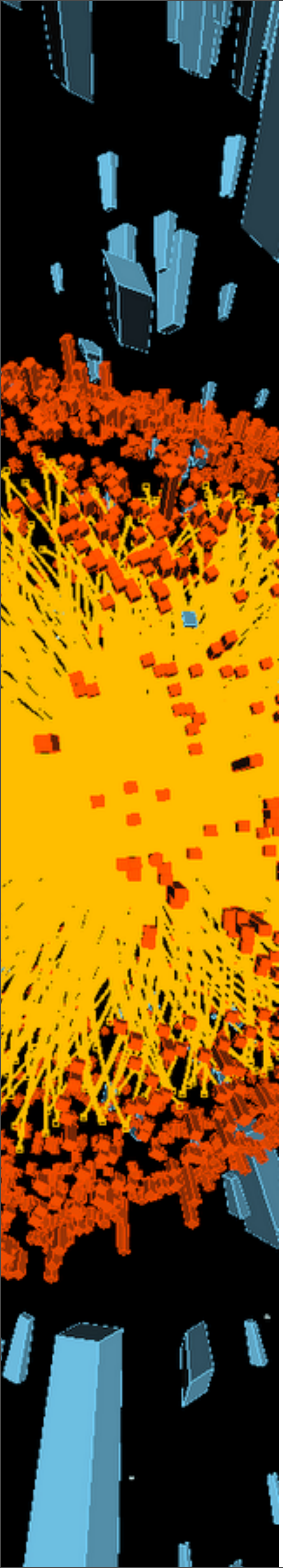
O. Kaczmarek, et al., PRD 83 (2011) 014504

Summary

- Some QCD-like theories such as two color QCD, as well as QCD with pure imaginary chemical potential or isospin chemical potential can be simulated without a sign problem.
- A bunch of extrapolation techniques exist that can be used to obtain results for small chemical potentials (μ/T).
- The expansion coefficients of the pressure are connected the hadronic fluctuations, which can be compared to experimental data from heavy ion collisions.
- The Taylor expansion naturally provides a method to locate the critical point of QCD by an analysis of the radius of convergence.



Back Up



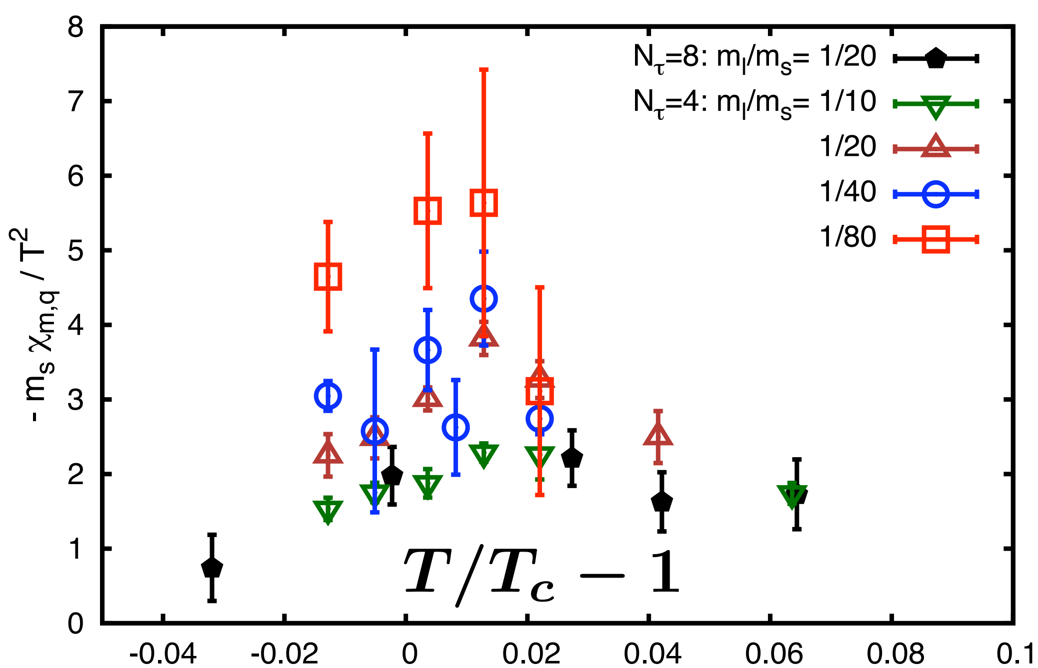
phase boundary for small chemical potentials

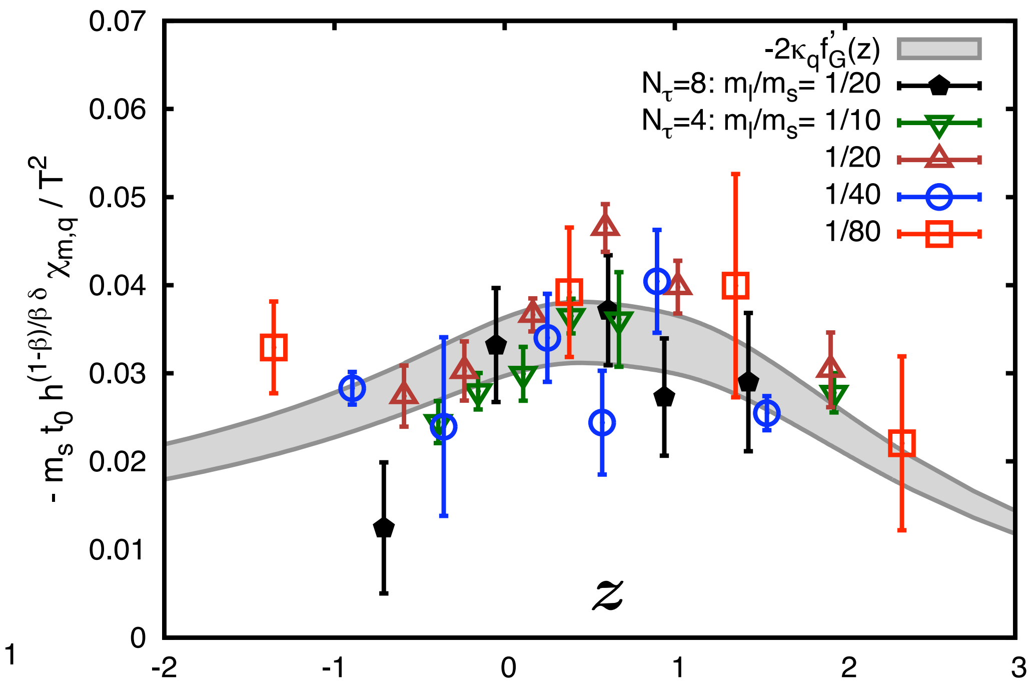
Following the critical line:

- Three parameters (T_c, t_0, h_0) have been fixed by the magnetic equation of state $M = h^{1/\delta} f_G(z)$
- Determine κ_q by a scaling analysis of the mixed susceptibility

$$\chi_m = \frac{\partial^2 M}{(\partial \mu/T)^2} = \frac{2\kappa_q}{t_0 T_c} h^{(\beta-1)/\beta\delta} f'_G(z) \propto \chi_t$$

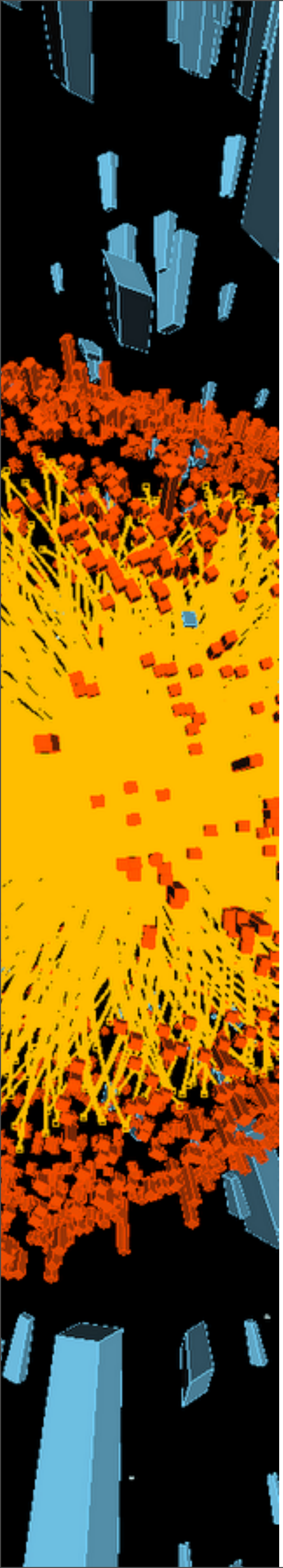
$\Rightarrow$ **one fit parameter:** κ_q





$\Rightarrow$ **obtain from p4-action,** $N_\tau = 8, 4: \kappa_q = 0.059(6)$

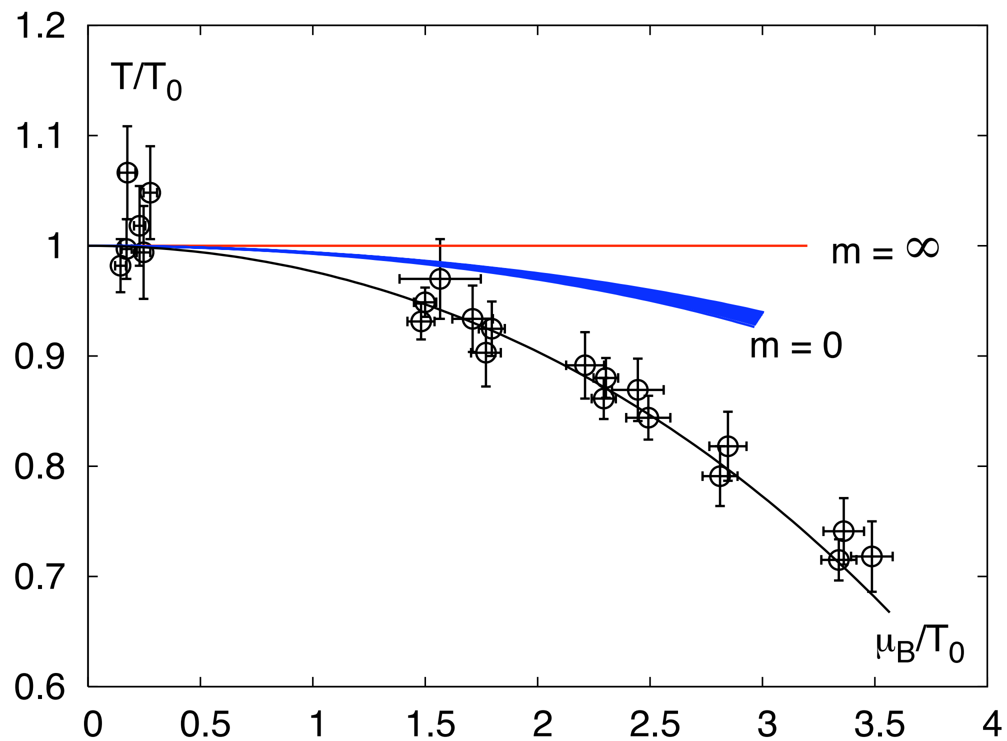
Kaczmarek *et al*, PRD 83 (2011) 014504



phase boundary for small chemical potentials

Comparison with the freeze-out line:

- Statistical models are very successful in describing particle abundances observed in heavy ion collision; use a parametrization of the freeze-out curve



statistical model:

$$\frac{T_c}{T} = 1 - 0.023 \left(\frac{\mu_B}{T}\right)^2 - d \left(\frac{\mu_B}{T}\right)^4$$

Cleymans, *et al.*, PRC 73 (2006) 034905

lattice:

$$\frac{T_c}{T} = 1 - 0.0066(7) \left(\frac{\mu_B}{T}\right)^2$$

Kaczmarek *et al*, accepted by PRD, arXiv:1011.3130 [hep-lat]

$\Rightarrow$ **curvature of the freeze-out curve seems to be larger**

- **open issues**: continuum limit, strangeness conservation, nonzero electric charge