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Importance sampling techniques in finance

S. Shcheredin

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Bielefeld University

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- KPMG's Credit Portfolio Management (CPM) team provides advisory services across the full range of portfolio management activities. These range from bespoke credit portfolio risk measurement and modelling to portfolio management strategy (including unit set-up and organisational structure review). Our aim is to provide clients with the infrastructure, tools and insight to enable more effective portfolio management. We do this by leveraging off a wealth of industry experience and dealing with these issues first hand.

- Current market conditions typified by economic uncertainty, scarcity of capital and increased risk at the counterparty and asset level, along with high hedging costs, mean a thorough understanding of aggregate portfolio risk and how to manage risk and return should be critical items on an institutions agenda. The CPM team has a great deal of practical experience implementing frameworks and models in the areas of new deal assessment, risk aggregation and Economic Capital implementation, concentration measurement and portfolio optimisation.

- CPM team members have also produced some of the most cutting edge and well-respected quantitative research in the market over the past ten years. These innovative ideas have been adopted widely across the industry and have added to our understanding and experience of management of portfolio risk, capital and pricing.

- KPMG has a truly global coverage, operating in 146 countries with more than 140,000 professionals working around the world. Our global network allows us to draw upon a vast range of expertise with detailed local market knowledge to address the specific needs of our clients. This includes 1,600 financial risk management specialists serving the world's largest financial institutions. For more information about KPMG visit **www.kpmg.co.uk/cpm**, for more information about our CPM advisory services please email, **cpm@kpmg.co.uk**.

Contents

Motivation

- The recent credit crunch resulted in one of the biggest write-downs in the history of the banking industry
- Many institutions have ceased to exist

[As of: 23 June 2009]			
	Write-down/Loss ($bn)		
Firm	**June 2009**	**3Q 2008**	**1Q 2008**
Wachovia	101.9	52.0	9.5
Citi	101.8	68.4	43.4
Bank of America Corp.	56.6	28.2	16.0
Merrill Lynch & Co.	55.9	55.9	35.0
UBS AG	53.1	48.7	38.3
WaMu	45.3	45.4	9.0
HSBC Holdings Plc	42.2	26.7	11.1
JP Morgan Chase & Co.	41.2	23.8	11.7
RBS	31.6	13.1	3.1
HBOS Plc	29.1	10.2	5.8

Source:Bloomberg.

- Regulators and banks are now trying to come up with more prudent risk practices to make the financial system more resilient
- The capital markets are very complicated and call for more sound modelling approaches to understand and quantify the risk inherent in bank's portfolios

Motivation (cont.)

The financial modelling uses techniques from:

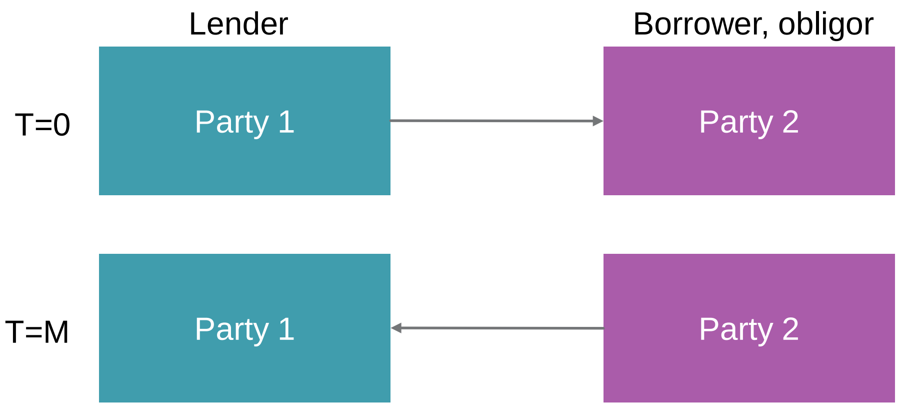
- Probability theory and stochastic processes
- Game theory
- Combinatorics
- Mathematical statistics
- Linear algebra, functions of complex variables analysis
- Numerical techniques
- Physics
  - Gibbs ensembles
  - Asymptotic expansions
  - Saddlepoint approximation
  - (Markov chain) Monte Carlo methods, importance sampling
  - Phase transitions
  - Chaos theory
  - Renormalisation group theory

Computing grids and video cards are also used by leading financial institutions to run intensive Monte Carlo simulations which are used for

- Trading support
- Exposure (£[…] amount that can be lost by the bank) calculation
- Deal pricing and approval support
- Risk assessment and reporting
- Portfolio optimisation

Introduction

- Historically a fundamental role of banks has been to lend money
- A loan is a bilateral agreement between two parties. The first party (lender) agrees to provide money for a certain period of time and the second party (borrower, obligor or also counterparty) receives the loan and is obligated to repay it at the maturity
- It is conceivable that the borrower may run into financial problems over time and will not be able to meet the obligations at maturity. This is called a default event and the lender is thus exposed to credit risk on its lending book



- The situation can be more complicated if a bank has a derivative contract with a counterparty
- A derivative contract is a more complicated agreement than a loan
- At inception the value of the derivative contract is normally zero (i.e. no one owes money to anyone)
- However as time progresses, since the derivative contract is dependent on various market drivers (interest rates, stock prices, commodity prices, foreign exchange rates etc.) the value could become either positive (the counterparty owes money to the bank) or negative (the bank owes money to counterparty)

References:

'Options, Futures and Other Derivatives' by John Hull

Web links:

wilmott.com

Defaultrisk.com

Introduction (cont.)

The uncertainties or risks e.g. default risk are modelled as stochastic processes.

Probability space is a triple (Ω, F, P) where Ω is a set of elementary events, F is the sigma algebra defined on Ω and P is the probability measure defined on F.

Sigma algebra F is defined as:

(i) $\Omega, \emptyset \in F$

(ii) if $x \in F$ then $\Omega \backslash x \in F$

(iii) if $x_1, x_2, \ldots, x_l \in F$ then $x_1 \cup x_2 \cup \ldots \cup x_l \in F$

A particular sigma algebra generated by open intervals on R is known as the Borel sigma algebra B

P is a map $F \to [0,1]$:

(i) $P(\Omega) = 1$

(ii) $P(x_1 \cup x_2) = P(x_1) + P(x_2)$ if $x_1 \cap x_2 = \emptyset, x_1, x_2 \in F$

Random variable y is defined as a map $y: F \to R$: $y^{-1}(b) \in F, b \in B$

Expectation value or mean value E is defined $E(y) = \int_{\Omega} y(\omega) dP(\omega)$

Source: Wikipedia pages

Conditional probability is defined as $P(x_1 \mid x_2) = \dfrac{P(x_1 \cap x_2)}{P(x_2)}$

If y_1 and y_2 are two random variables then the correlation is defined as $\rho = \dfrac{E(y_1 y_2) - E(y_1)E(y_2)}{\sqrt{E(y_1^2) - (E(y_1))^2}\sqrt{E(y_2^2) - (E(y_2))^2}}$

A stochastic process is defined as a map $y_t : F \otimes [0,T] \to R : y_t^{-1}(b) \in F_t$

Let $x, y \in F$. An indicator random variable $1_x(y)$ is 1 if $y = x$ and 0 otherwise.

If $P(x) = p$ then $E(1_x) = p \cdot 1 + (1-p) \cdot 0 = p$

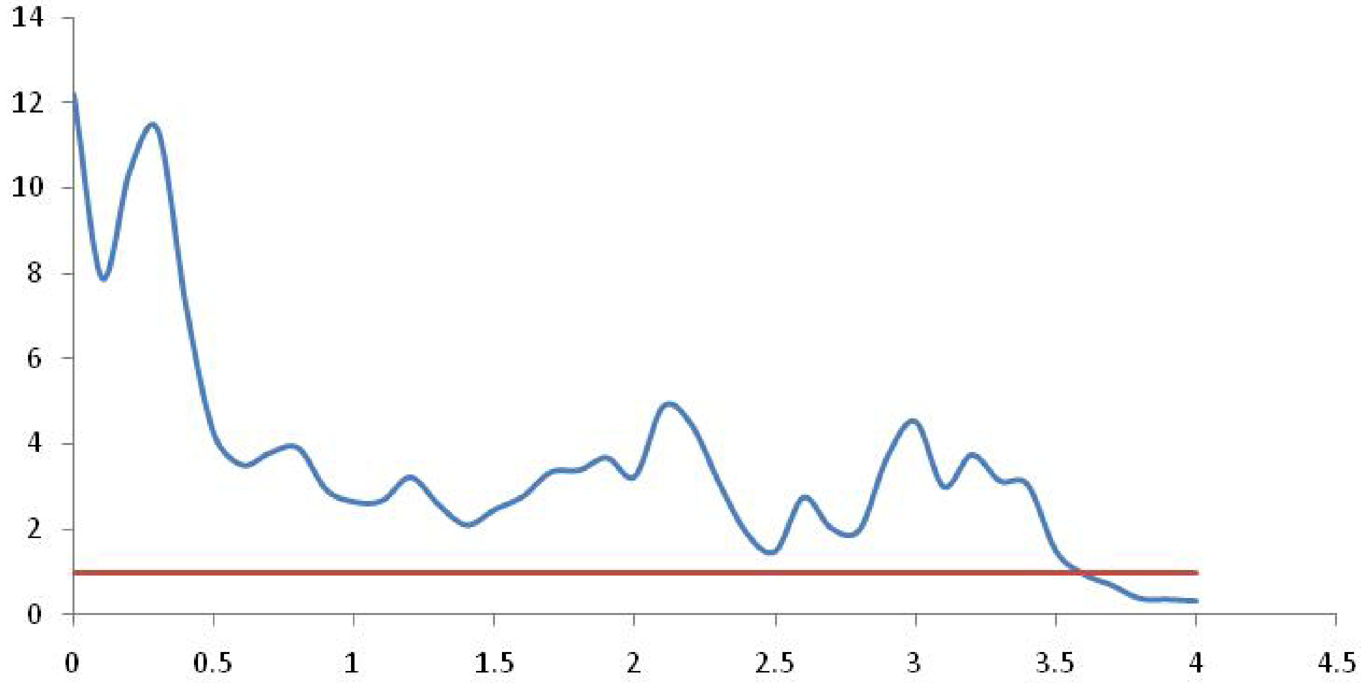
Credit portfolio modelling

- A portfolio is a set of obligors (the so called obligor view)
- The simplest modelling approach to quantify the credit risk inherent in the loan portfolio is as follows
- Let's assume that a loan portfolio consists of N loans to N obligors. Every obligor is characterised by its default probability p_i i.e. the likelihood of experiencing a default over a certain time horizon. It is estimated e.g. by a rating agency or a dedicated team in the bank
- Lets further assume that if a default event occurs then the potential loss is estimated to be a_i i.e. the default is a binary event
- We assume that the default risk (uncertainty) can be captured by
  - A random variable called the systematic risk factor which is an uncertainty common to the whole portfolio. The systematic risk factor is also referred to as the 'state of economy'
  - N random variables (uncertainties) called idiosyncratic factors (or specific risk factors) defined for each obligor in the portfolio. They model the specific default risk for each obligor

Credit portfolio modelling (cont.)

In the so-called structural models the default can be modelled as follows

- Assume that there is a stochastic process associated with each obligor and the default is defined when the stochastic process falls below certain barrier over a certain period of time
- The modelling of any default dependency between obligors in the portfolio can be addressed in this framework as well, through the correlation of the processes associated with each obligor



Vasicek model

In the Vasicek model the following additional assumptions are made

Assume that there is a stochastic process ξ_i associated with each obligor *i* and the default is defined when the stochastic process falls below certain barrier b_i over a certain period of time *T*

$$\xi_i < b_i, i = 1,.., N$$

The first assumption is that ξ_i is distributed as N(0,1) at time *T*

The second assumption is that the correlated default events are modelled through correlation of ξ_i i.e.

$$\xi_i = \beta_i Z + \sqrt{1-\beta_i^2}\,\eta_i, i = 1,..N$$

where *Z* is a random variable distributed as N(0,1) and is common for all obligors. It is referred to as the state of the economy. η_i is a set of independent random variables distributed as N(0,1). They are referred to as individual (idiosyncratic) obligor risks. A set of β_i^2 defines the percentage of the variance of the process ξ_i explained by the state of the economy *Z*

The default correlation is defined as:

$$\rho_{ij} = \frac{E(1_{\xi_i < b_i} 1_{\xi_j < b_j}) - p_i p_j}{\sqrt{p_i(1-p_i)}\sqrt{p_j(1-p_j)}}$$

Risk measures

Now we are ready to put everything together to compute the aggregated portfolio loss random variable

$$Y = \sum_{k=1}^{N} a_k Y_k, \qquad Y_k = 1_{\xi_k < b_k}$$

In finance to quantify the risk it is necessary to compute:

- probability density function of Y
- tail probability i.e. prob($Y>y$)
- quantiles of the probability density function
- conditional expectation values E(Y_k | $Y = L$)
- conditional expectation value of Y above certain thresholds i.e. E(Y| $Y > L$)

Some of the calculations may be required to be performed frequently as other models may require them as inputs. For instance if a portfolio has to be optimised with respect to default risk then the optimisation algorithm may require recalculation of the measures for a changed portfolio until convergence is established

For a general case there is no analytical exact solution for any of those problems

Therefore Monte Carlo methods are applied

Linkage with physics

The risk measures can be associated with observables in physics

Some formulae look particularly similar

Recall that the portfolio random loss variable is:

$$Y = \sum_{k=1}^{N} a_k Y_k, \qquad Y_k = 1_{\xi_k < b_k} \qquad \xi_i = \beta_i Z + \sqrt{1-\beta_i^2}\,\eta_i, i = 1,\ldots, N$$

Then e.g. the tail probability can be written as:

$$\mathrm{Prob}(Y > y) = E(1_{Y>y}) = \int dZ \left[\prod_{k=1}^{N} d\eta_k\right] 1_{Y|Z,\eta>y} \frac{1}{\sqrt{2\pi}} e^{-0.5Z^2} \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-0.5\eta_k^2}$$

N is approximately 3000 – 10000

This is similar to the computation of observables in statistical physics i.e.

$$\langle O \rangle = \frac{1}{Z} \int \left[\prod_{\mu=1}^{4} \prod_{k=1}^{K} dU_{k\mu}\right] O \det D[U_{k\mu}] e^{-S[U_{k\mu}]}$$

Monte Carlo methods

First we establish the simplest brute force Monte Carlo algorithm

The prerequisite step is to compute the default barrier $b_{i.}$

$$p_i = E(1_{\xi_k < b_k}) = \text{prob}(\xi_k < b_k)$$
$$b_k = N^{-1}(p_i)$$

Then the Monte Carlo algorithm proceeds as follows

1. Draw a sample of the state of the economy *Z* from a N(0,1)
2. Given the state of the economy *Z* compute the conditional probability of default for every obligor in the portfolio

$$p_{iZ} = \text{prob}(\xi_i < b_i \mid Z) = \text{prob}(\beta_i Z + \sqrt{1-\beta_i^2}\eta_i < b_i \mid Z) =$$
$$\text{prob}\left(\eta_i < \frac{b_i - \beta_i Z}{\sqrt{1-\beta_i^2}}\right) = N\left(\frac{b_i - \beta_i Z}{\sqrt{1-\beta_i^2}}\right)$$

   In the Vasicek model the default events are independent if the state of the economy is fixed

3. Compute a sample of *Y* by simulating a default of every obligor with a uniform distribution. The obligor has defaulted if $u < p_{iz}$

Importance sampling

For risk management purposes it is important to understand the tail risk of the probability density function of *Y*

Problem: reduction of the Monte Carlo variance for the tail probability

$$T(y) = P(Y > y) = E(1_{Y>y}) \approx \frac{1}{L}\sum_{n=1}^{L} 1_{Y_n > y}$$

$$\text{var} = E(1_{Y>y}) - \left(E(1_{Y>y})\right)^2$$

Hence variance reduction techniques

A method proposed by Avranitis et al, Finger, Kalkbrener et al shifts the distribution of the economy state risk factor *Z* to favour the 'bad' states as they are more responsible for the tail

Indeed

$$E(1_{Y>y}) = E(E(1_{Y|Z>y} \mid Z))$$

Then

$$E(1_{Y>y}) = \int dZ \frac{1}{\sqrt{2\pi}} e^{-0.5(Z-\mu)^2} E(1_{Y|Z>y} \mid Z) \frac{e^{-0.5Z^2}}{e^{-0.5(Z-\mu)^2}} = \int dZ \frac{1}{\sqrt{2\pi}} e^{-0.5(Z-\mu)^2} E\left(1_{Y|Z>y} \frac{e^{-0.5Z^2}}{e^{-0.5(Z-\mu)^2}} \mid Z \right)$$

Finally

$$E(1_{Y>y}) = E_\mu\left(1_{Y_\mu > y} \frac{e^{-0.5Z^2}}{e^{-0.5(Z-\mu)^2}} \right),$$

where E_μ means sampling $Y_\mu = \sum_k a_k Y_{k\mu},\ Y_{k\mu} = 1_{\beta_k \tilde{Z}_\mu + \sqrt{1-\beta_k^2}\eta_k < b_k}, \tilde{Z}_\mu \sim N(\mu,1), \eta_k \sim N(0,1)$

Importance sampling algorithm

Brute force Monte Carlo

Generate the economy state Z_n ~ N(0,1)

Compute the conditional PD:

$$p_{kZ} = N\left(\frac{N^{-1}(p_k) - \beta_k Z}{\sqrt{1-\beta_k^2}}\right)$$

Generate Monte Carlo losses by sampling from the uniform distribution *U* and comparing with p_{kZ}

Compute

$$T(y) = E(1_{Y>y}) \approx \frac{1}{L}\sum_{n=1}^{L} 1_{Y_n>y}$$

Importance sampling

Generate the world state Z_n ~ N(μ,1)

Compute the conditional PD:

$$p_{kZ} = N\left(\frac{N^{-1}(p_k) - \beta_k Z}{\sqrt{1-\beta_k^2}}\right)$$

Generate Monte Carlo losses by sampling from the uniform distribution *U* and comparing with p_{kZ}

Compute

$$T(y) = E_\mu\left(1_{Y_\mu>y}\frac{e^{-0.5Z^2}}{e^{-0.5(Z-\mu)^2}}\right) \approx \frac{1}{L}\sum_{n=1}^{L}\frac{e^{-0.5Z_n{}^2}}{e^{-0.5(Z_n-\mu)^2}} 1_{Y_n>y}$$

Kalkbrener Monte Carlo

M. Kalkbrener et al, 'Sensible and efficient capital allocation for credit portfolios', Risk, 2004

Problem: reduction of the variance of the expected shortfall

$$\mathrm{ES}(\alpha) = E(Y \mid Y > VaR_\alpha) = \frac{E(Y1_{Y>y})}{1-\alpha} \approx \frac{1}{1-\alpha}\frac{1}{L}\sum_{n=1}^{L} Y_n 1_{Y_n>y}$$

$$\mathrm{var} = \frac{1}{(1-\alpha)^2}\left(E(Y^2 1_{Y>y}) - \left(E(Y1_{Y>y})\right)^2\right)$$

Solution: The shift has been computed by matching the initial portfolio with an analytically tractable portfolio and then optimising the shift for it

Glasserman Monte Carlo

P. Glasserman et al 'Importance sampling for Portfolio Credit Risk' Management Science, 2005

Problem: reduction of the variance for the tail probability

$$T(y) = P(Y > y) = E(1_{Y>y}) \approx \frac{1}{L}\sum_{n=1}^{L} 1_{Y>y}$$

$$\text{var} = E(1_{Y>y}) - \left(E(1_{Y>y})\right)^2$$

The initial problem is: $E(1_{Y>y}) = \int dZ \frac{1}{\sqrt{2\pi}} e^{-0.5Z^2} E(1_{Y|Z>y} \mid Z)$

Instead of sampling from $e^{-0.5Z^2} E(1_{Y|Z>y} \mid Z)$ sample from a normal distribution N(z_{max},1) with the shift

$$z_{\max} = \max_{z} e^{-0.5Z^2} E(1_{Y|Z>y} \mid Z)$$

Glasserman Monte Carlo (cont.)

This is however not feasible to resolve and therefore an upper bound of this problem is optimised to compute the shift

Solution: combination of two steps

- Shift the distribution of the systematic risk factor *Z*
  - **Note:**
    - Leads to the variance reduction only for strongly correlated portfolios
    - The shift depends on the loss at which the importance sampling is defined
- Amplify the conditional PD
  - **Note:** leads to the variance reduction only for weakly correlated portfolios

Ensemble Monte Carlo

K. Thompson et al 'Accelerated Ensemble Monte Carlo simulation' Risk, 2009

Problem: reduction of the variance for the tail probability

Ensembles: An ensemble of portfolios is defined as a set of identical portfolios. It is similar to Gibbs ensembles

K. Thompson et al 'Credit Ensembles' Risk, 2003

Solution:

1. Consider a constrained Ensemble of copies of the same portfolio with the constraint being the **Ensemble average loss**
2. For such a system it is possible to work out:
   - the distribution of the economy states *Z* in the constrained Ensemble given the value of the constraint
   - the conditional probabilities of default given the value of the constraint
3. Also it is possible to define a transform for the risk measures from the constrained Ensemble to the unconstrained
4. This framework defines an importance sampling method where quantities are simulated with the Ensemble constraint and then the constraint is lifted by reweighting the simulations paths using the transform

Markov Chain Monte Carlo

T. Reitan et al 'A new robust importance-sampling method for measuring VaR and ES allocations for credit portfolios' The journal of Credit Risk, 2010/11 Volume 6, number 4

Problem: reduction of the variance for the tail probability

$$T(y) = P(Y > y) = E(1_{Y>y}) \approx \frac{1}{L}\sum_{n=1}^{L} 1_{Y>y}$$

$$\mathrm{var} = E(1_{Y>y}) - \left(E(1_{Y>y})\right)^2$$

Solution:

- Recall that $Y = \sum_k a_k Y_k,\ Y_k = 1_{\beta_k Z_\mu + \sqrt{1-\beta_k^2}\eta_k < b_k}, Z_\mu \sim N(0,1), \eta_k \sim N(0,1)$

- Then the problem can be formulated as

$$E(1_{Y>y}) = \int dZ \left[\prod_{k=1}^{N} d\eta_k\right] 1_{Y|Z,\eta>y} \frac{1}{\sqrt{2\pi}} e^{-0.5Z^2} \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-0.5\eta_k^2}$$

- Use the Metropolis algorithm to sample from $1_{Y|Z,\eta>y} e^{-0.5Z^2} \prod_{k=1}^{N} e^{-0.5\eta_k^2}$ to identify the optimal shift

- Use this shift with the normal distribution of the economy state random variable *Z*

Extensions and unanswered questions

- We have discussed only the simplest default/no-default model where each obligor can be either in the default or no default state. Reality is more complicated as the obligors have credit ratings and the default state is only the last notch in the rating table
- We have only considered a portfolio of loans. Banks have portfolios of more complicated derivative contracts where the exposure itself is stochastic
- We have considered a very simple model for default correlations where the defaults were linked through a single state of the economy
- We have considered the probability density function of the portfolio losses only over one period. A better way is to ask how it would evolve over many periods
- We have considered only one type of the risk i.e. credit default risk. What about market movements of the portfolio, operational risk inherent in the financial institution etc.
- And we have not answered many other problems e.g. what is the most optimal portfolio given certain constraints and the risk measures...

Recap

- The financial markets are too complicated to rely on intuition. Therefore modelling is used to assess and quantify the risks
- The credit crunch of 2007-2008 has shown that the banking industry requires better and more prudent risk practices
- There is a growing demand to understand and quantify the risk associated with a counterparty’s default
- Better Monte Carlo and analytical algorithms can significantly improve the risk management tools which provide senior management with information to assess the performance of an institution, identify potentially dangerous risk concentrations and define future strategy

Skills and qualifications required

Required

- Experience in Fortran/Pascal/C/C++ and willingness to implement models in computer languages for numerical applications is essential
- Proven experience of modelling of complex systems.
- Good communication skills – both verbal reasoning skills (including the ability to communicate complex ideas effectively) and written skills
- Ability to work in a team – sharing ideas and support all other team members
- The ability to manage own work and time effectively
- The candidate will be highly articulate with an outstanding level of written and spoken English

Qualifications

- Essential first class degree in applied mathematics/engineering/physics or related discipline and ideally post-graduate qualifications (MSc or PhD)

Desirable

- Good practical knowledge of advanced modelling techniques such as saddle-point methods, ensemble theory, Fourier transforms, characteristic functions, asymptotic expansions, factor analysis, cluster analysis, PCA, regression techniques, stochastic calculus, etc. (this must be real-world experience)
- Thorough knowledge and experience of Monte Carlo techniques including variance reduction techniques
- Evidence of published articles or new thought leadership
- Knowledge of Numerical Algorithms Group (NAG) Libraries
- Knowledge of STL, R

Many thanks for inviting me to the school!!!

Contact details:

Kevin Thompson is a head of Credit Portfolio Management and Modelling (CPM):

kevin.thompson@kpmg.co.uk

Stanislav Shcheredin is an executive advisor at CPM
S.Shcheredin@kpmg.co.uk

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