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we define consulting

Valuation
of Equity Derivatives

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What‘s a derivative?

More complex financial products are „derived“ from simpler products

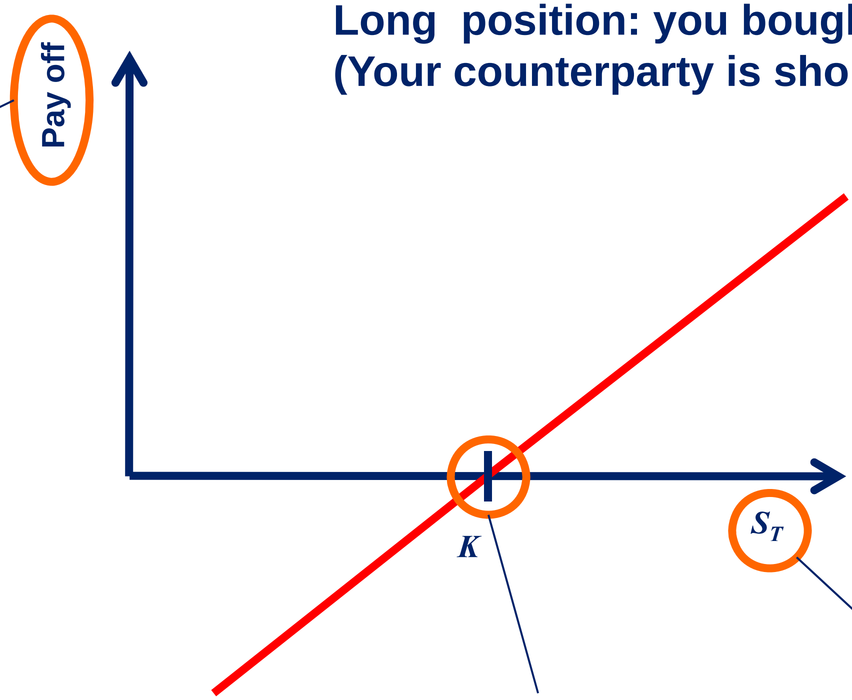
- What‘s **not** a derivative?
  - Stocks, interest rates, FX rates, oil prices, …
- Derivatives are pay off claims somehow based on prices of simpler products or other derivatives
- Derivatives may be traded via an exchange or directly between two counterparties (OTC: over-the-counter)
- OTC-Derivatives are based on freely defined agreements between counterparties and may be arbitrarily complex

Example I: Equity Forward

Buying (or selling) stocks at some future date

Long position: you bought the stock (Your counterparty is short)

Pay off $S_T - K$ at maturity (expiry) T



Physical Settlment: get Stock, pay K

Cash Settlment: get $S_T - K$

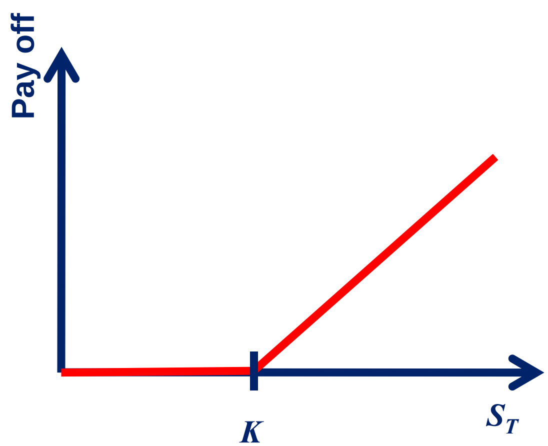
Forward Price K or Strike Price

Stock Price S_T at maturity (expiry) T

Example II: Plain Vanilla Option

Most simple and liquidly traded options

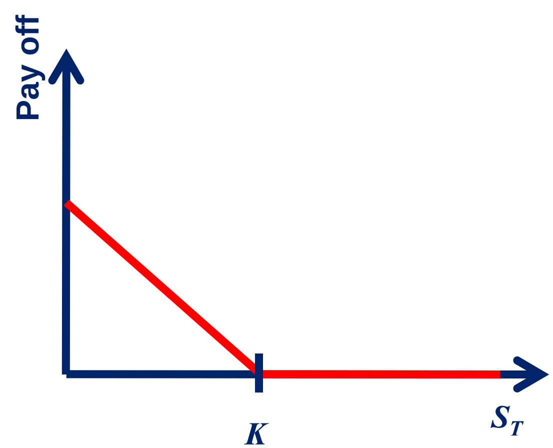
(Plain Vanilla) Call Option



Pay off:

$$\max(S_T - K, 0) \equiv (S_T - K)^+$$

(Plain Vanilla) Put Option

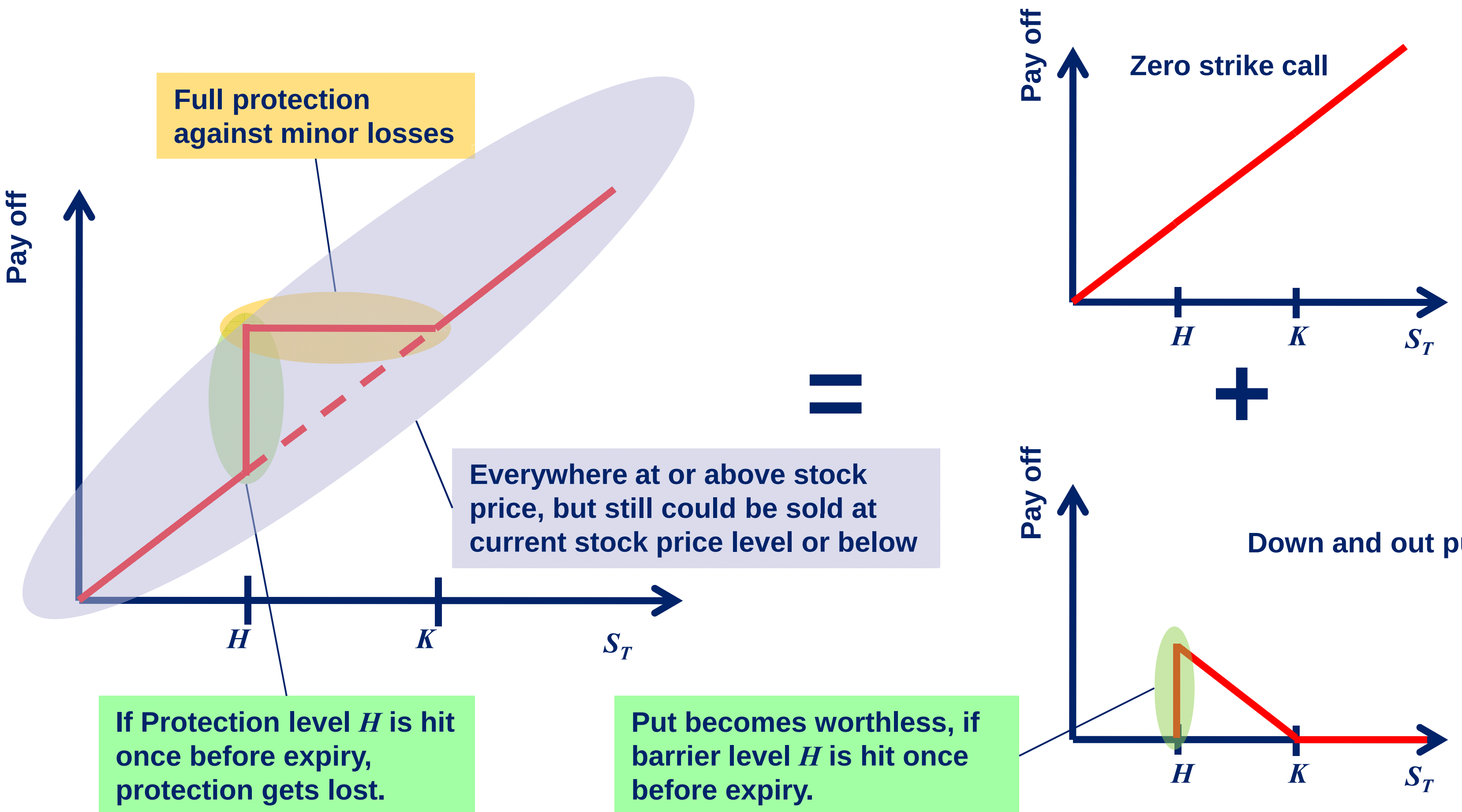


Pay off:

$$\max(K - S_T, 0) \equiv (K - S_T)^+$$

Example III: Bonus Certificate

Getting more than you might expect



Even more complex structures

There are no limits to complexity

- Baskets as underlying
  - Simple basket products: Pay off depends on total value of basket of stocks
  - Correlation basket products: Pay off depends on performance of single stocks within the baskets, e.g. the stock that performs worst or best, etc.
  - Simulation of trading strategies
- Quantos
  - Pay off in a currency different from the stock currency
- Combination with other risk factors (hybrid derivatives)
  - E.g. Convertible Bonds (bonds that could be converted into stocks)

Stocks

Know your underlying

- A Stock or Share is the certification of the ownership of a part of a company, the community of shareholders is the owner of the whole company
- Issued stock is equivalent to tier 1 equity capital of the corporation
- The stock may pay a dividend
- The company can be listed at one or more stock exchanges

Stock process

The Geometric Brownian motion of stocks

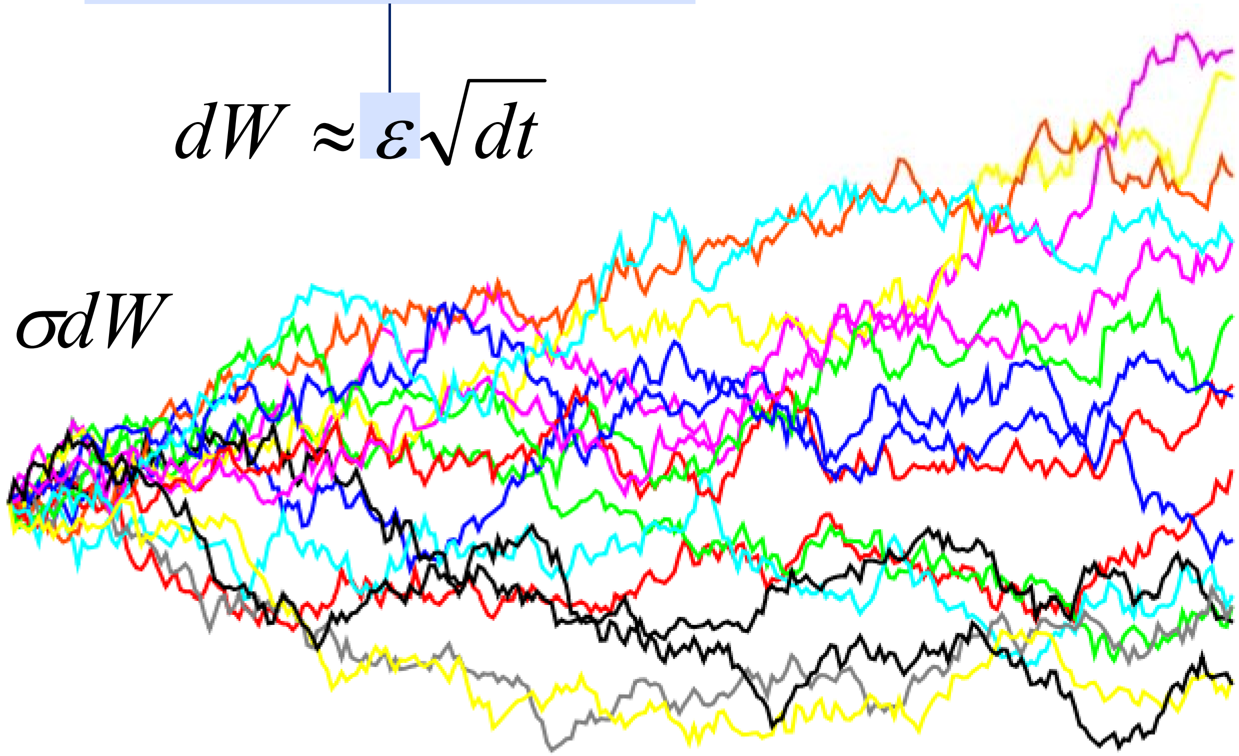
Drift

Volatility

Standard normal distributed random number

$$dS = \mu S dt + \sigma S dW \qquad dW \approx \varepsilon \sqrt{dt}$$

$$d \ln S = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW$$

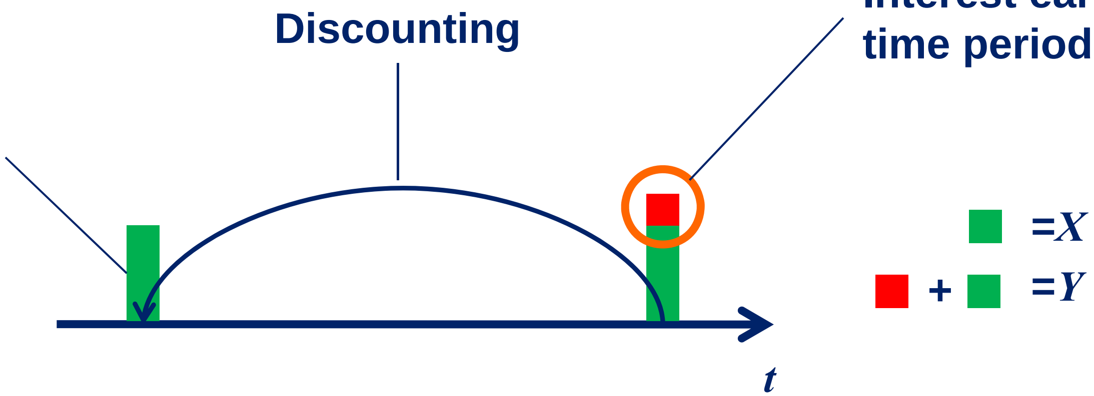


Divends are neglected!

Time value of money

Time is money. But how much money is it?

- Money today is worth more than the same amount in some distant future
  - Risk of default
  - Missing earned (risk free) interest



A note on notation I

Compounding of interest rates

- Usually, interest is paid on a regular basis, e.g. monthly, quarterly or annually
- If re-invested, the compounding effect is significant

Number of years

$$\underbrace{\left(1+\frac{r}{m}\right)\left(1+\frac{r}{m}\right)\cdots\left(1+\frac{r}{m}\right)}_{nm\ \text{times}}=\left(1+\frac{r}{m}\right)^{nm}$$

Compoundings per year

A note on notation II

The Continuous compounding limit

Continuous compounding is the limit of compounding in infinitesimal short time periods

$$\lim_{\substack{m\to\infty \\ n=\text{const.}}} \left(1+\frac{r}{m}\right)^{nm} = e^{rT}, \quad T = n$$

e^{-rT}

Value of one unit at future time T as of today $t=0$.
Also: discount factor or zero bond
r: continuous compounding rate

This mathematically convenient notation is used throughout the rest of the talk. It is also most often assumed in papers on finance.

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Example I: valuation of the Forward contract

First Try: Forward value based on expectation

1. Step: Calculate expectation of forward pay off

$$\mathrm{E}\left(S_T - K\right) = \mathrm{E}\left(S_T\right) - K = e^{\mu T} S_0 - K$$

2. Step: Discount expected pay off to today

$$V_{\text{Forward}}^{\mathrm{E}} = e^{(\mu - r)T} S_0 - e^{-rT} K$$

Fair price for new contract: $K = e^{\mu T} S_0$

To avoid losses, bank would have either

1. sell as many Forwards with same strike and maturity as they buy
2. or have to rely on the correctness of the above formula on average

Arbitrage

Making money out of nothing

- Arbitrage is the art of earning money (immediately) without taking risk
- Example: Buy stock at 10 and sell at 15
  - Because of bid/ask-spreads, broker buys at 10 and sells at 10.05
  - 0.05 in this example is the broker fee
- Since money earned by arbitrage is easy money, market participants will take immediate advantage of arbitrage opportunities
- If the markets are efficient, there are **no** opportunities for arbitrage
  - If you can replicate a pay off with a strategy built on liquidly traded instruments, this must be the fair value of the pay off –> **there is no free lunch!**

Example I: valuation of the Forward contract I

Replicating the Forward agreement

Assumption: $S_0 < e^{-rT} K$

1. **Step: Borrow at interest rate r for term T the money amount** $B = e^{-rT} K$
2. **Step: Buy the stock and put the rest of the money aside:** $A = e^{-rT} K - S_0$
3. **Step: At time T, loan has compounded to K:** $e^{rT} B = K$
4. **Step: Exchange stock with strike K and pay back loan**

Amount A has been earned arbitrage free!

To avoid arbitrage, the fair strike must be $K = e^{rT} S_0$

Example I: valuation of the Forward contract II

Lessons learned

$$V_{\text{Forward}} = S_0 - e^{-rT} K$$

- The real world expectation of S at future time t doesn‘t matter at all!
- Hedged counterparties face no market risk
  - Credit risk remains
- Value of the Forward is equal to the financing cost
  - No fee for bearing market risk
- Required assumptions:
  - No arbitrage
  - Possible to get loan at risk-free interest rate

Adding optionality

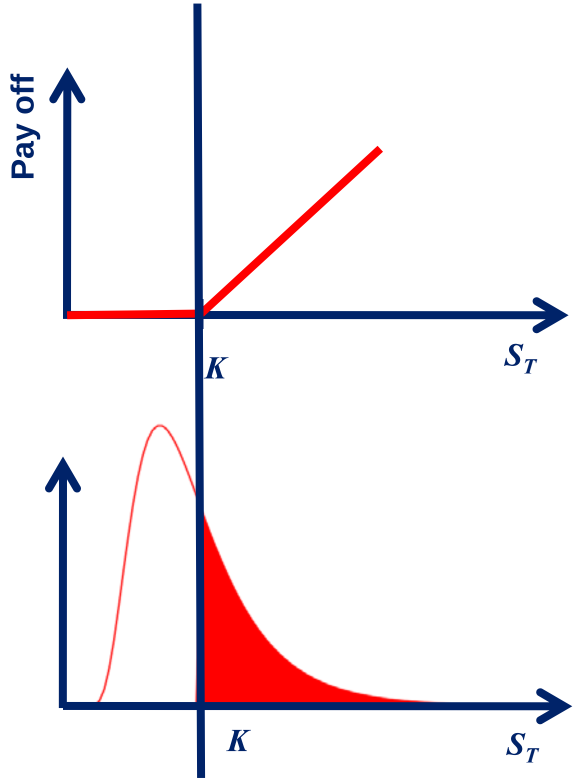
For options, the distribution function matters

Plain Vanilla option: cut off distribution function at strike K

European Call option pay off:

$$\max(S_T - K, 0) \equiv (S_T - K)^+$$

Q: Is there any arbitrage free replication strategy to finance these pay offs?



Ito‘s lemma

The stochastic process of a function of a stochastic process

Process of underlying: $dS = \mu S dt + \sigma S dW$

Fair value V of option is function of S: $V = V(S)$

Ito‘s lemma:

$$dV = \left(\frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dW$$

Caused by stochastic term $\sim \sqrt{dt}$

Replication portfolio for general claims

Replicate option pay off by holding portfolio of cash account and stock

- Ansatz: $V = B + xS$ with $dB = rBdt$
- Changes in option fair value V

$$dV = \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S}\sigma S dW = rBdt + x\mu S dt + x\sigma S dW$$

- Choose $x = \frac{\partial V}{\partial S}$ and insert for $B = V - xS$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rB = rV - rS\frac{\partial V}{\partial S}$$

With this choice of x, the stochastic term vanishes

A closer look at the Black-Scholes PDE

The arbitrage free PDE of general claims

$$rV = \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

Final equation does not depend on μ

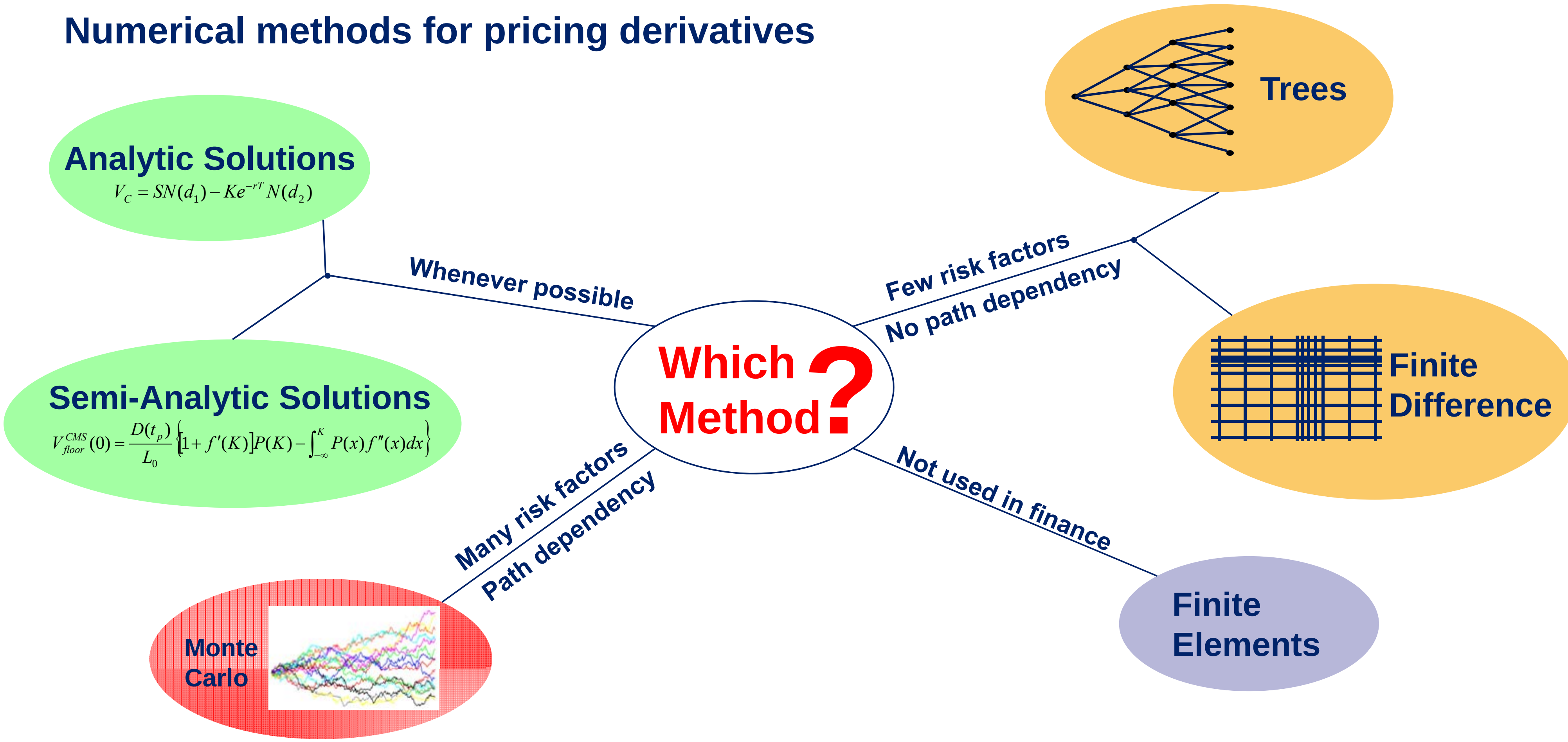
Replication portfolio is self-financing

Additional terms for deterministic dividends have been neglected

With time dependent $r=r(t)$ or $\sigma=\sigma(t)$, the PDE is still valid

Solving the Black-Scholes PDE

Numerical methods for pricing derivatives



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The famous formula of Black and Scholes

Analytic Solution of Black-Scholes PDE for Call options

Solve the Black-Scholes PDE for Plain Vanilla Call options

$$rV = \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

Specific products can be defined as set of end and boundary conditions of the Black-Scholes PDE

$$V(T,S) = \left(S(T) - K\right)^+$$

$$V(t,0) = 0$$

$$\lim_{S\to\infty} V(t,S) = S - K$$

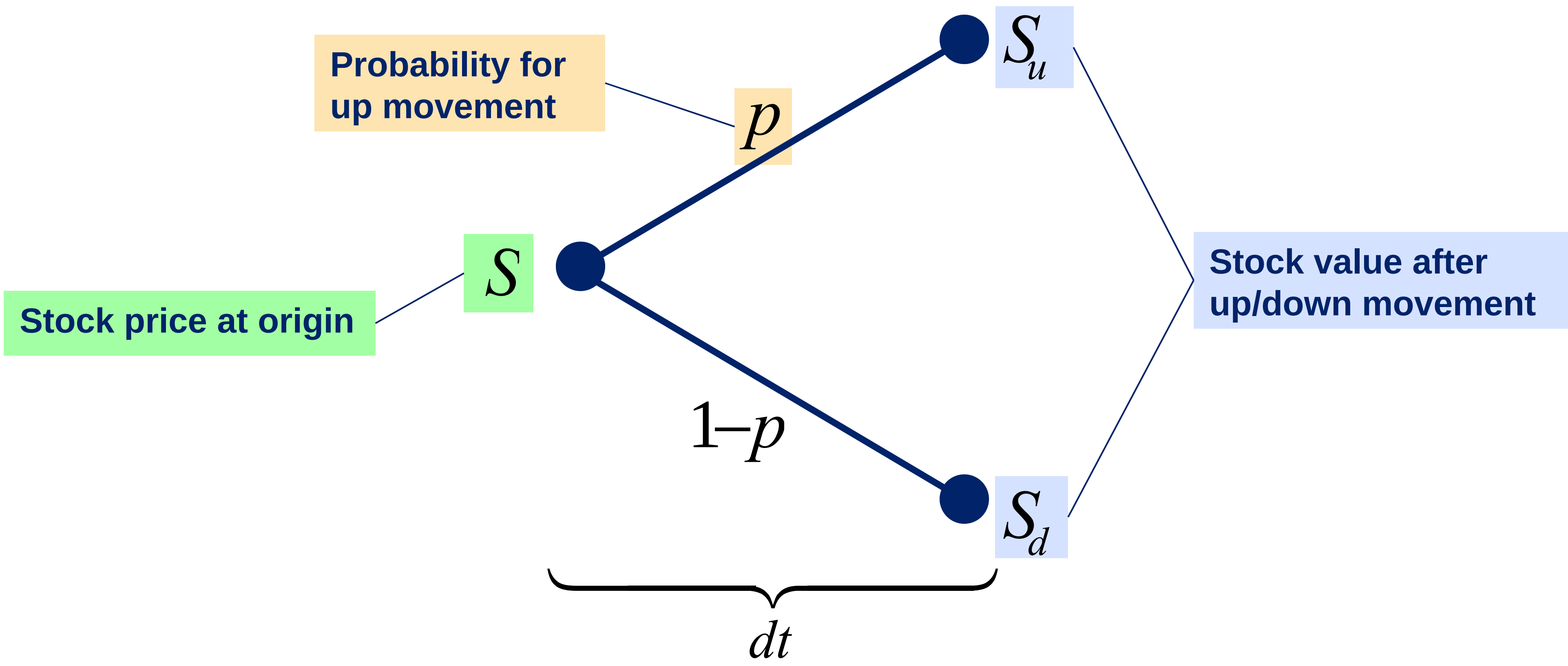
Solution: the famous result of Black and Scholes

$$V_{\text{Call}} = SN(d_1) - Ke^{-rT}N(d_2)$$

$$d_{1,2} = \frac{\ln(S/K) - rT}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma^2 T$$

Binomial tree – single step

Simulating the stochastic process on a tree like grid



Binomial tree – parameters

Four parameters and two equations

Parameters:

S_u, S_d: **Value of stock after up/down move**

p: **Probability of up move**

dt: **Length of time step (determines the accuracy of the calculation)**

Equations:

$$\mathrm{E}\left[S(dt)\right] = pS_u + (1-p)S_d \equiv Se^{rdt}$$

$$\mathrm{var}\left[S(dt)\right] = p(1-p)\left(S_u - S_d\right)^2 \equiv S^2 e^{2rdt}\left(e^{\sigma^2 dt} - 1\right)$$

Derived from lognormal stock price process

Binomial tree – choices

Popular choices to determine parameters

Cox-Ross-Rubinstein choice: $S_u = uS$ and $S_d = S / u$

$$u \approx e^{\sigma \sqrt{dt}}$$

$$p \approx \frac{ue^{rdt} - 1}{u^2 - 1}$$

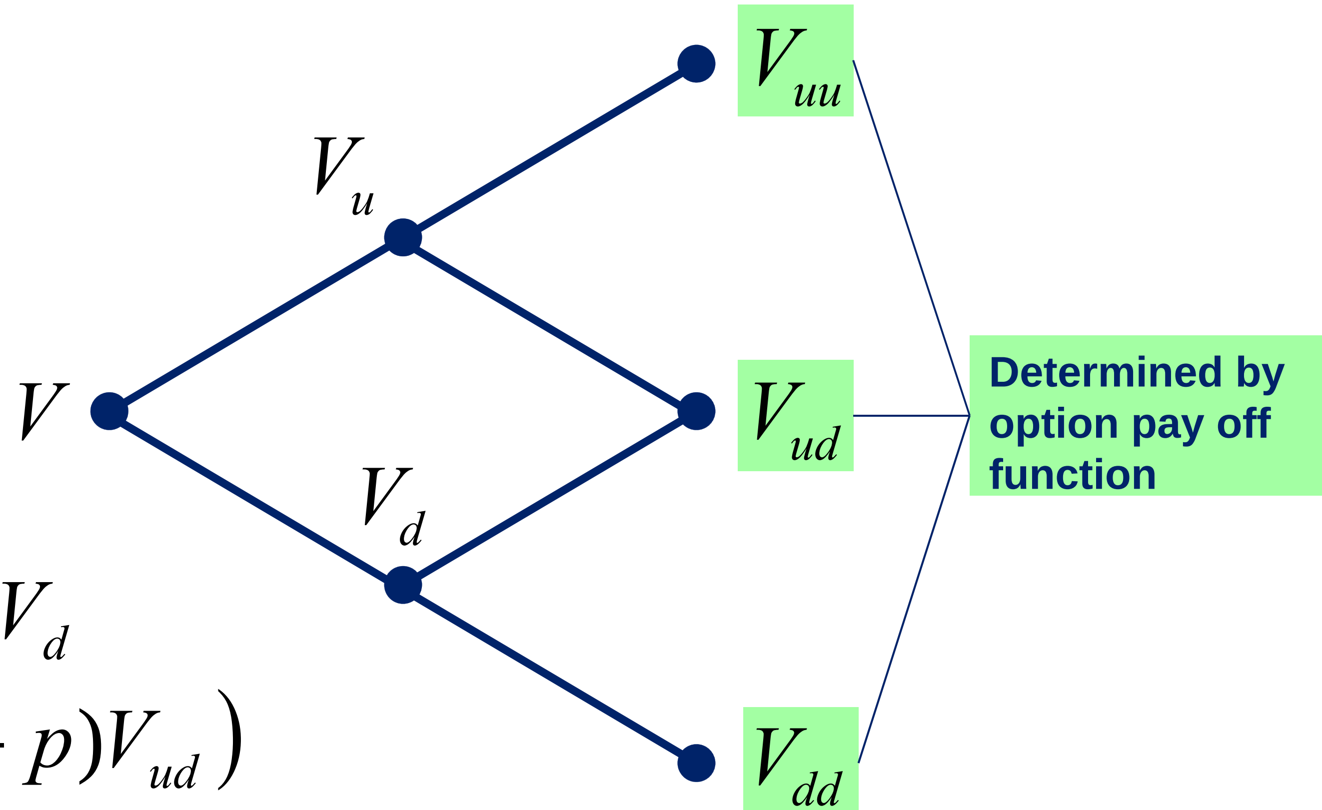
Jarrow-Rudd-choice: $p = 0.5$

$$S_u \approx Se^{\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\sqrt{dt}}$$

$$S_d \approx Se^{\left(r - \frac{1}{2}\sigma^2\right)dt - \sigma\sqrt{dt}}$$

Binomial trees – application

Applied decision trees

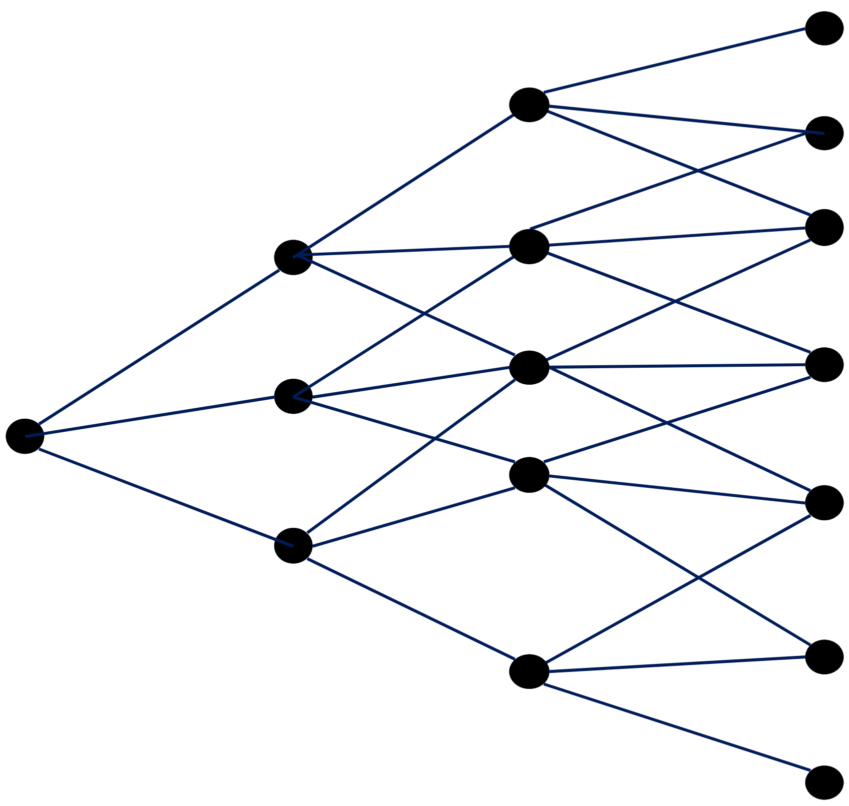


Calculate option price by “backward induction”

$$
\begin{aligned}
V &= pV_u + (1-p)V_d \\
&= p\left(pV_{uu} + (1-p)V_{ud}\right) \\
&\quad + (1-p)\left(pV_{ud} + (1-p)V_{dd}\right)
\end{aligned}
$$

Trinomial trees

Two-step binomial tree or explicit finite difference scheme



Key features of trinomial trees

- **6 free parameters (three different states, two probabilities, one time step)**
- **Otherwise, approach is similar to binomial trees**
- **More flexibility than binomial tree**
- **Faster convergence (two time steps in one)**
- **For certain geometries, trinomial trees are identical to explicit finite difference method**

Finite differences

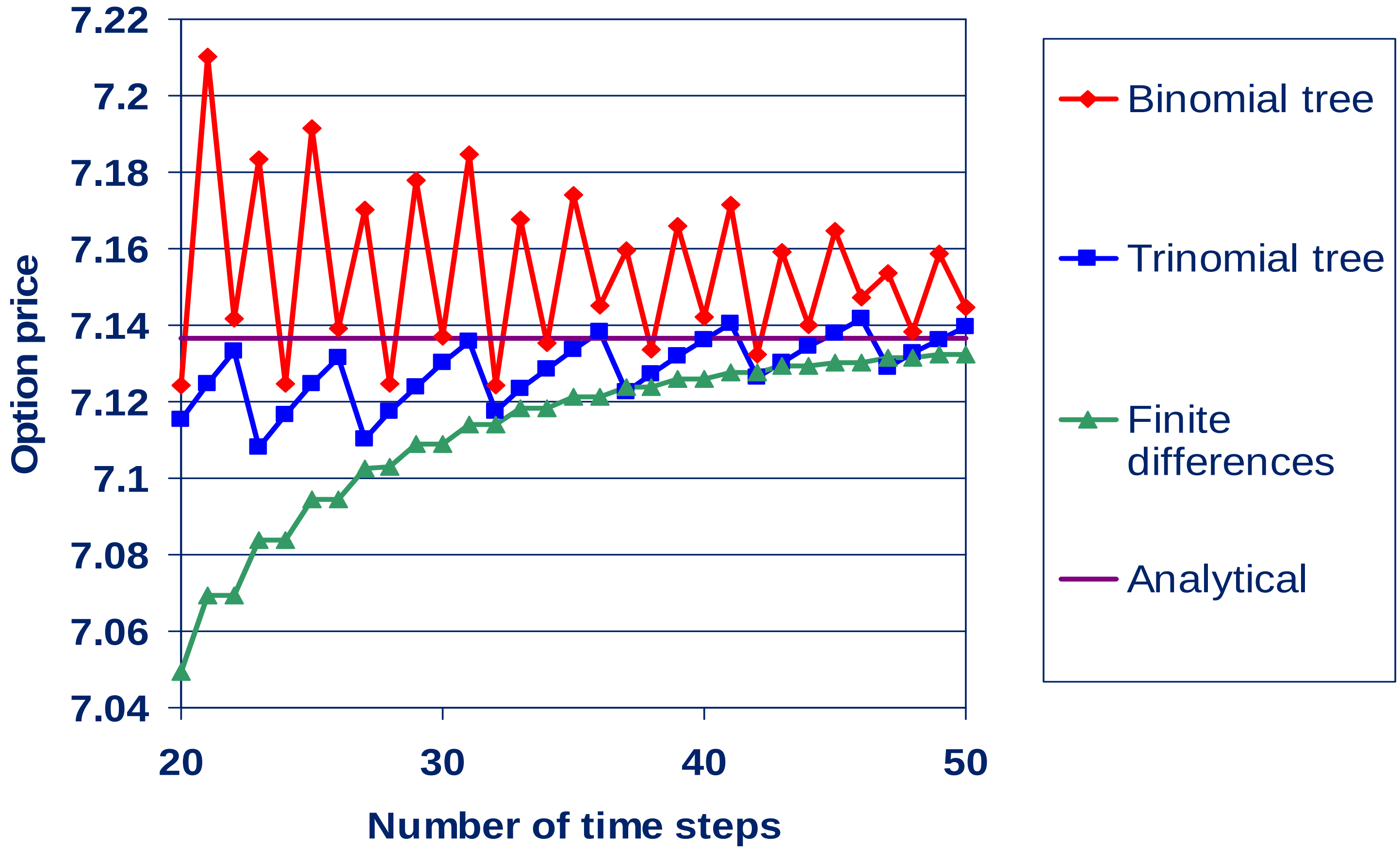
Solving the PDE on a rectangular grid

Application of finite difference method

1. **Choose one of the well established finite difference schemes for PDEs of the parabolic type (diffusion equation), e.g. explicit or implicit Euler scheme or Crank Nicholson scheme**
2. **Discretise PDE on a rectangular lattice according to finite difference scheme**
3. **Apply pay off function as final boundary condition**
4. **Depending on option type, apply Dirichlet or (generalised) von Neumann (e.g. second order derivative is zero) conditions on upper and lower boundaries**
5. **Roll back throw lattice to get solution**

Comparison of numerical methods

Instead of trees, use finite difference method!



Monte Carlo simulation

The method of last means

- If everything else false, use Monte Carlo simualtion
  - Easy to implement, but bad performance
- Typical applications:
  - Underlying is basket of underlying (e.g. many dimension)
  - Multi-factor problems
  - Path dependent problems
- Implementation methods
  - Simulate stochastic differential using small time steps
  - Better: Integrate of longer time period and draw random numbers directly from log-normal distribution function
  - Calculate pay off based on simulated stock prices and discount to today

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Assumptions

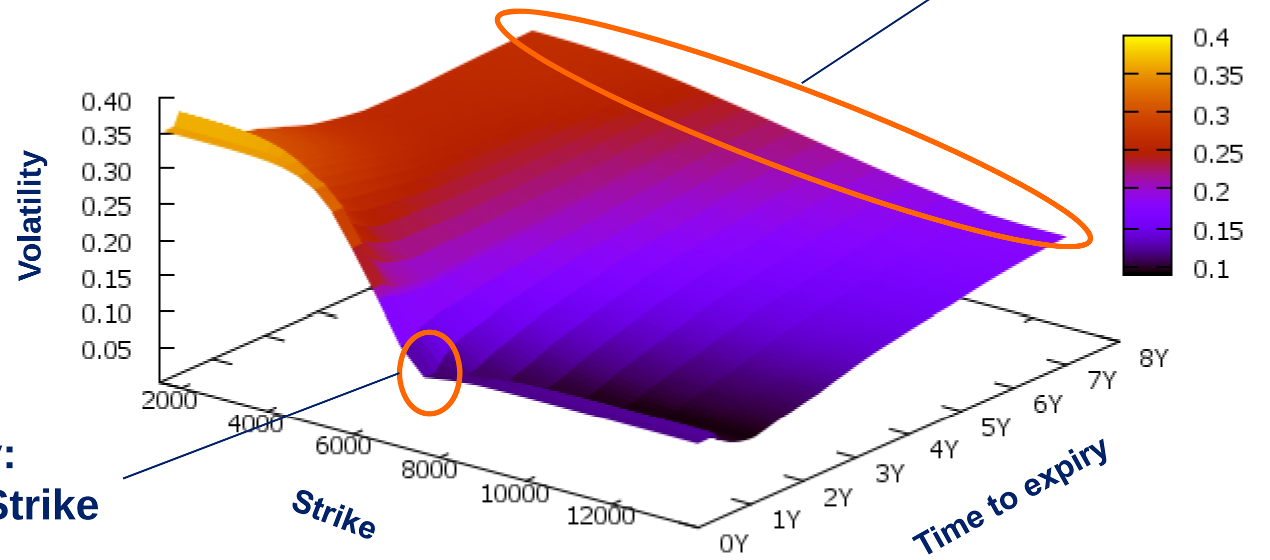
Which of these assumptions hold in reality?

▪ There are no transaction costs	No, there are bid-ask Spreads
▪ Continuous trading is possible	No, due to technical limitations
▪ Markets have infinite liquidity	No, problem for small caps
▪ **Dividends are deterministic**	**No, company performance dependent**
▪ Markets are arbitrage free	Almost, because of transaction costs
▪ **Volatility has only termstructure**	**No, volatility depends on strike and term**
▪ Stocks follow log-normal Brownian motion	No, only approximately, problems of „fat tails“
▪ **There is no counterparty risk**	**No, as the last crisis has shown**
▪ Everybody can finance at risk-free rate	No, financing depends varies broadly

Volatility smile

Volitility depends on strike („moneyness“) and expiry

Smile vanishes for long expiries



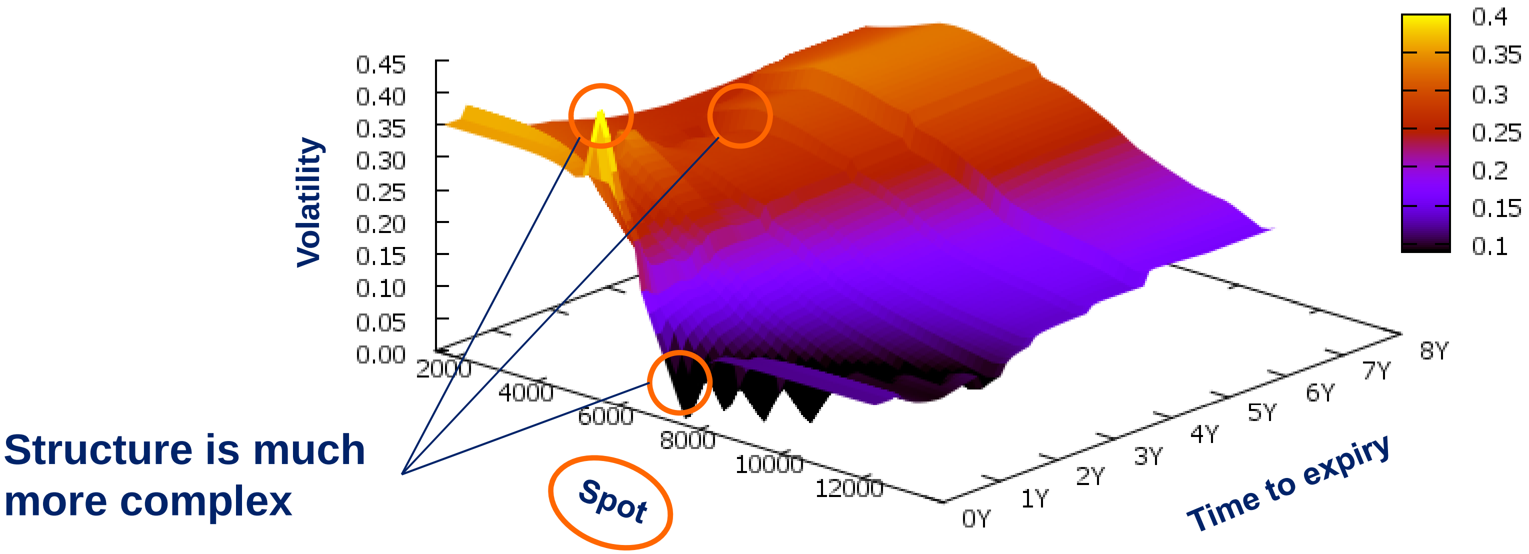
At-the-money: Spot equals Strike

Using Black-Scholes: putting the wrong number (i.e. volatility) into the wrong formula to get the right price.

Local volatility surface

Transform $\sigma(K,T)$ into $\sigma(S,T)$

When S moves, local volatility moves in wrong direction



Local Volatility: Allows for fit to the whole volatility surface, but behaves badly. Still, it is widely used.

Examples of other methods of modelling volatility I

More advanced volatility models

- Displaced diffusion
  - Assume $S+d$ instead of S to follow the lognormal process

$$d(S+d) = r(S+d)dt + \sigma(S+d)dW$$

- Jumps
  - Add additional stochastic Poisson process to spot process

$$dS = (r - \lambda\gamma)Sdt + \sigma SdW + \gamma SdJ$$

Poisson process

Rate of poisson process or jump intensity

Relative jump size

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Other methods of modelling volatility II

More advanced volatility models

- Stochastic Volatility
  - Model volatility as second stochastic factors, e.g. Heston model

Instantaneous variance

Wiener processes with correlation ρ

$$dS = rSdt + \sqrt{\nu}\, S dW_S$$

$$d\nu = \kappa(\theta - \nu)dt + \xi\sqrt{\nu}\, dW_\nu$$

Mean reversion rate

Long term average variance

Volatility of volatility

- Local-Stoch-Vol
  - Combination of local volatility and stochastic volatility

Dividends I

Dividend payments: interest equivalent for equities

- Dividends compensate the shareholder for providing money (equity)
- Some companies don't pay dividends, most pay annually, some even more often
- In general, dividend is paid a few days after the (annual) shareholder meeting
- Dividend payment amount is loosely related to the company's P&L
- Regardless of the above, most models assume deterministic dividends
  - i.e. Dividend amount or rate and payment date are known and fixed

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Dividends II

Three methods of modeling dividends

- Continuous dividend yield q: continuous payment of dividend payment proportional to current stock price S
  - Unrealistic, but mathematically easy to handle
- Discrete proportional dividends: dividend is paid at dividend payment date, amount of dividend is proportional to stock price S
  - Tries to model dividend's loose dependency on P&L (assuming P&L and stock price to be strongly correlated)
- Discrete fixed dividends: fixed dividend amount is paid at dividend payment date
  - Causes headaches for the quant (i.e. the person in charge of modelling the fair value of the derivative)

Dividends III

Impact of dividends on stock process I

- Continuous dividend yield:

$$dS = (\mu - q)Sdt + \sigma S dW$$

- Discrete dividends
  - 1. Method: Subtract dividend value from S and model S without dividends
    - proportional dividends

$$S^* = S\left(1 - \sum_{i=1}^{n} D_i\right)$$

    - fixed dividends

$$S^* = S - \sum_{i=1}^{n} e^{-rt_i} D_i$$

Dividends III

Impact of dividends on stock process II

- Discrete dividends
  - 2. Method: Modelling (deterministic) jumps in the stochastic process with jump conditions defined as
    - Proportional dividend

$$S(t_i^-) = S(t_i^+)(1 + D_i)$$

    - Fixed dividend

$$S(t_i^-) = S(t_i^+) + D_i$$

Methods 1 and 2 assume different stochastic processes, i.e. the volatilities are different!

Example Ia: Forward contract with dividends

Replicating the Forward agreement

1. Step: Borrow the money to buy the stock at price S_0

2. Step: Split loan into two parts
   a) For time period t_D at rate r_D the amount $e^{-r_D t_D} D$

   b) For time period T at rate r the rest $S_0 - e^{-r_D t_D} D$

3. Step: At dividend payment date t_D, receive dividend D and pay back first loan which is now worth D

4. Step: At expiry, the second loan amounts to $e^{rT}\left(S_0 - e^{-r_D t_D} D\right)$

In order to make the Forward contract be arbitrage free, the fair strike must be $K = e^{rT}\left(S_0 - e^{-r_D t_D} D\right)$

Example Ia: Forward Contract with dividends

Formulas for Forward agreements with dividends

- Continuous dividend yield
  - Fair strike:

$$K = e^{(r-q)T} S_0$$

  - Fair value:

$$V_{\text{Forward}} = e^{-qT} S_0 - e^{-rT} K$$

- Proportional discrete dividends
  - Fair strike:

$$K = e^{rT} \left(1 - \sum_{i=1}^{n} D_i \right)$$

  - Fair value:

$$V_{\text{Forward}} = S_0 \left(1 - \sum_{i=1}^{n} D_i \right) - e^{-rT} K$$

Example Ia: Forward Contract with dividends

Formulas for Forward agreements with dividends

- Fixed discrete dividends
  - Fair strike:

$$K = e^{rT}\left(S_0 - \sum_{i=1}^{n} e^{-rt_i} D_i\right)$$

  - Fair value:

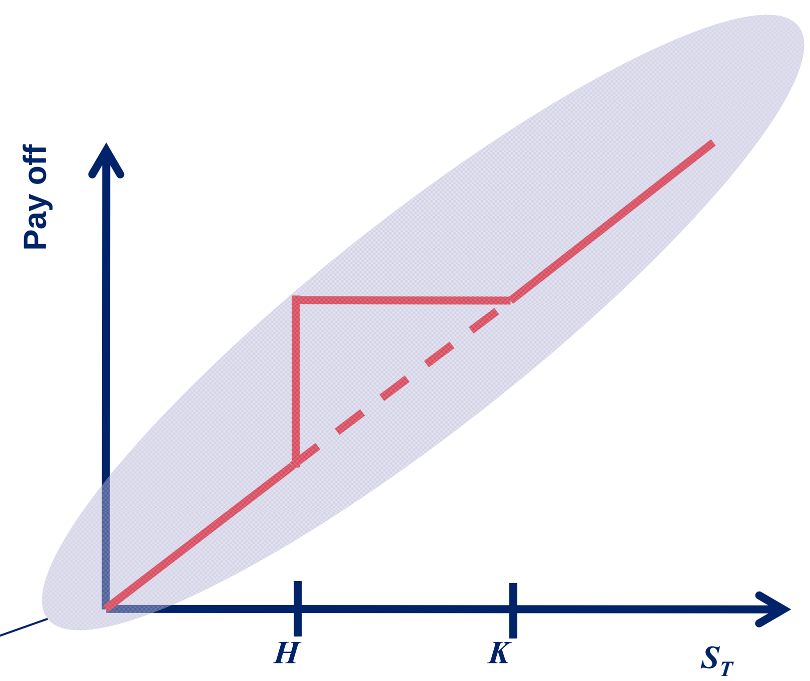
$$V_{\text{Forward}} = S_0 - \sum_{i=1}^{n} e^{-rt_i} D_i - e^{-rT} K$$

Dividends reduce the forward value by reducing the financing cost of the replication strategy!

Impact of dividends on derivative prices

Currently, no well-established model for stochastic dividends

- Dividends can be hedged, but not (easily) modelled stochastically
- Impact on option price is significant
  - Trick: use dividends to hide option costs



Everywhere at or above stock price, but still could be sold at current stock price level…

…since between today and T lies a stream of dividend payments!

Hedging derivative in practice

Hedging makes the difference

- Hedging: trading the replication portfolio
- Reduce transaction cost by
  - Frequent, but discrete hedging
  - Hedging derivative portfolio as a whole
- Improve Hedging performance
  - Hedging based on „Greeks“

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Greeks

Partial derivatives named by Greek letters

$$\Delta \equiv \frac{\partial V}{\partial S}$$ **Delta**

$$\Gamma \equiv \frac{\partial^2 V}{\partial S^2}$$ **Gamma**

$$\Theta \equiv \frac{\partial V}{\partial t}$$ **Theta**

$$\rho \equiv \frac{\partial V}{\partial r}$$ **Rho**

Most important Greeks

„Bucketed" Sensitivities

$$\frac{\partial V}{\partial \sigma}$$ **Vega**

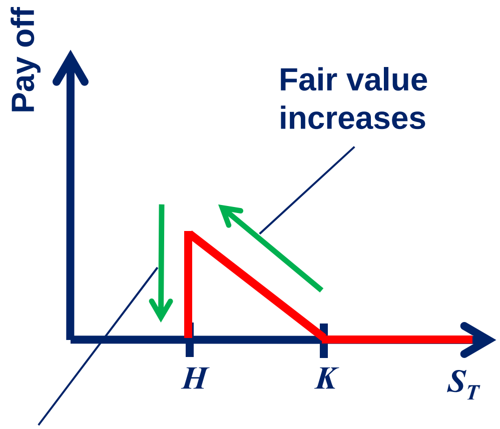
$$\frac{\partial^2 V}{\partial \sigma^2}$$ **Volga**

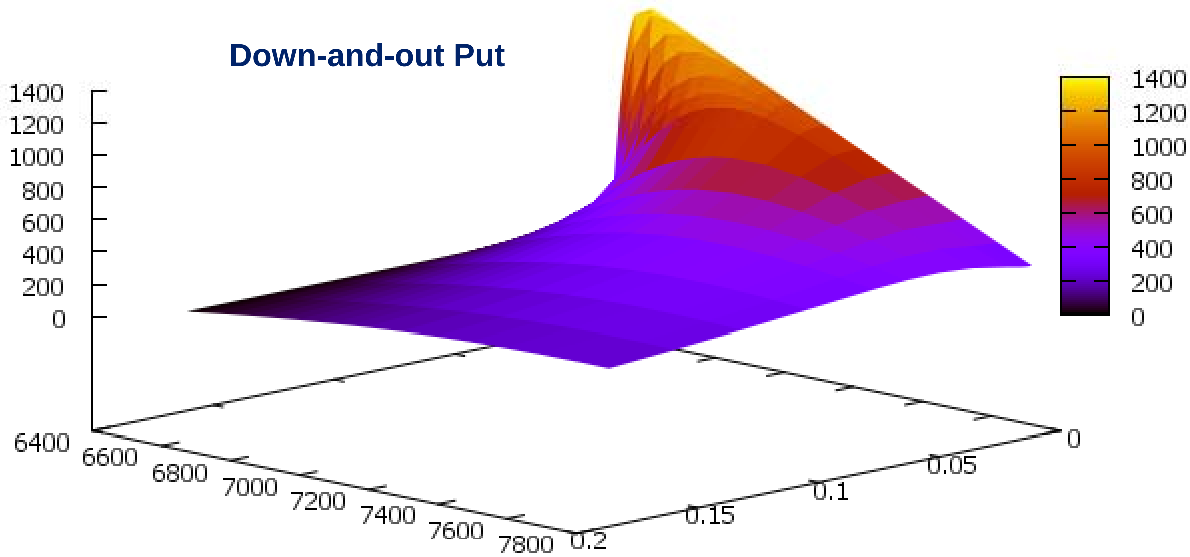
$$\frac{\partial^2 V}{\partial S \partial \sigma}$$ **Vanna**

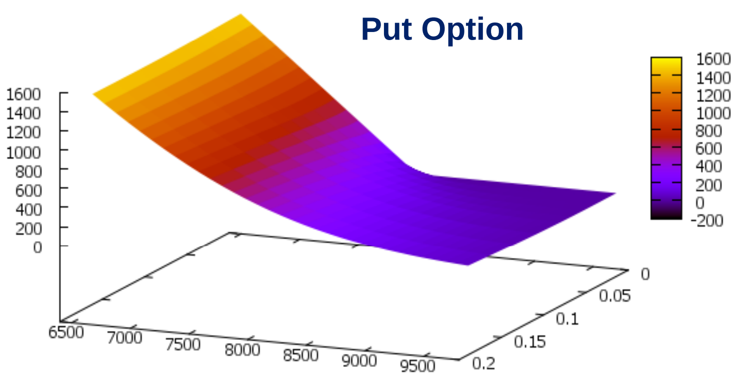
Type of hedging strategy often named after Greeks hedged, e.g. „Delta-Hedging"

Example IIIb: down-and-out Put

Present Value of Down-and-out Put option



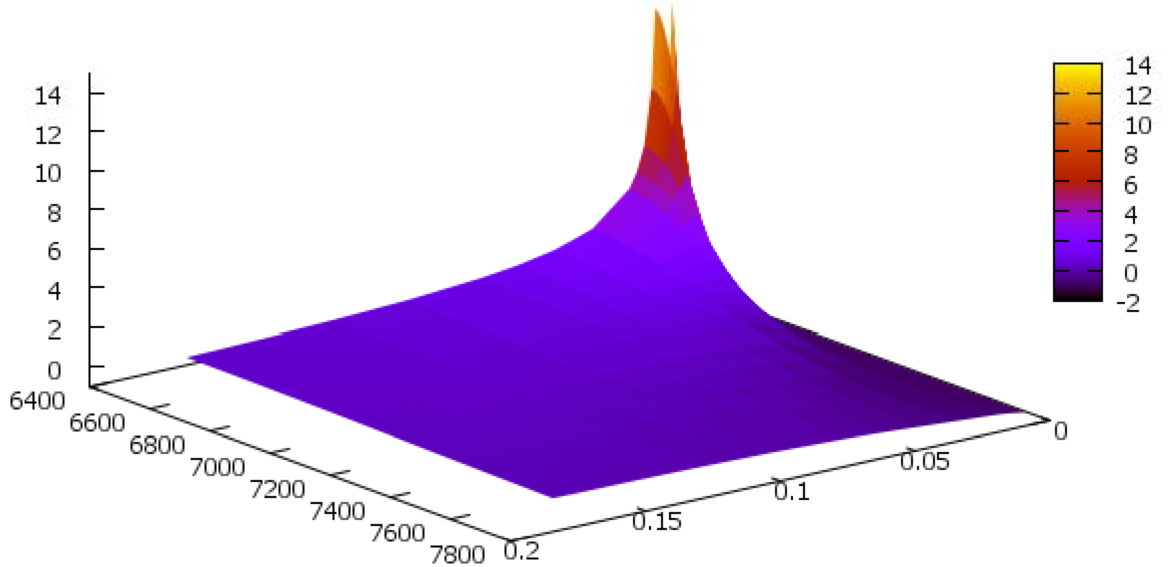




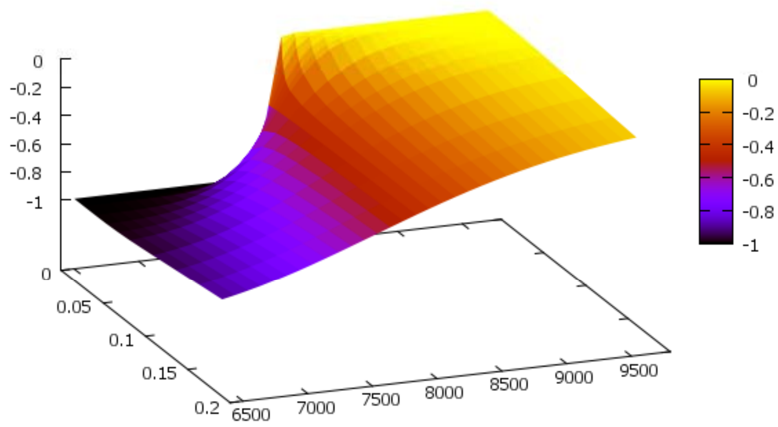
Example IIIb: down-and-out Put

Delta of Down-and-out Put option

Delta Down-and-out Put



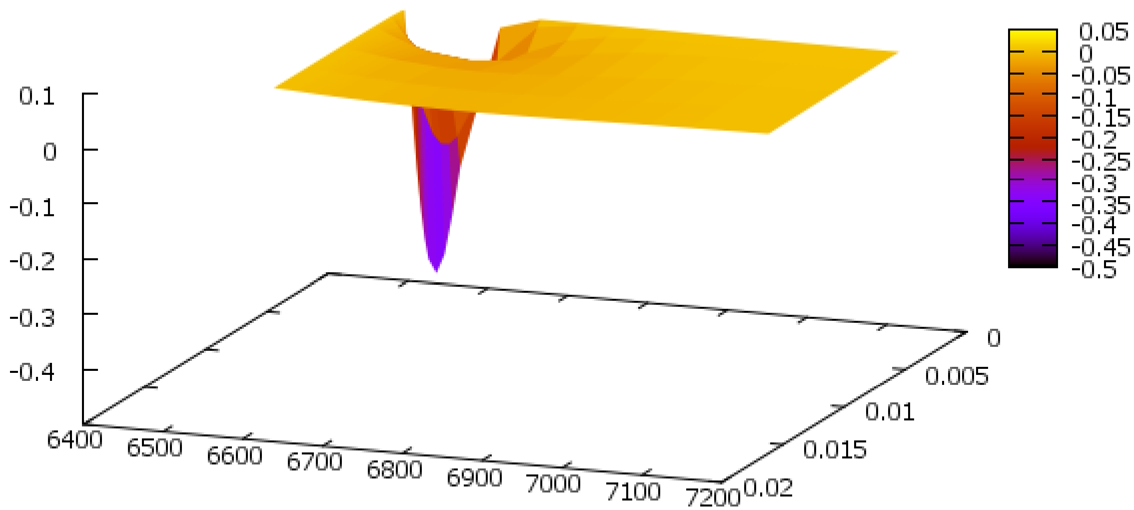
Delta Put Option



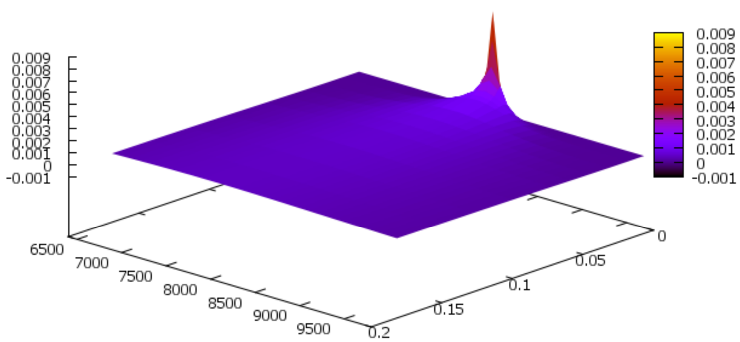
Example IIIb: down-and-out Put

Gamma of Down-and-out Put option

Gamma Down-and-out Put



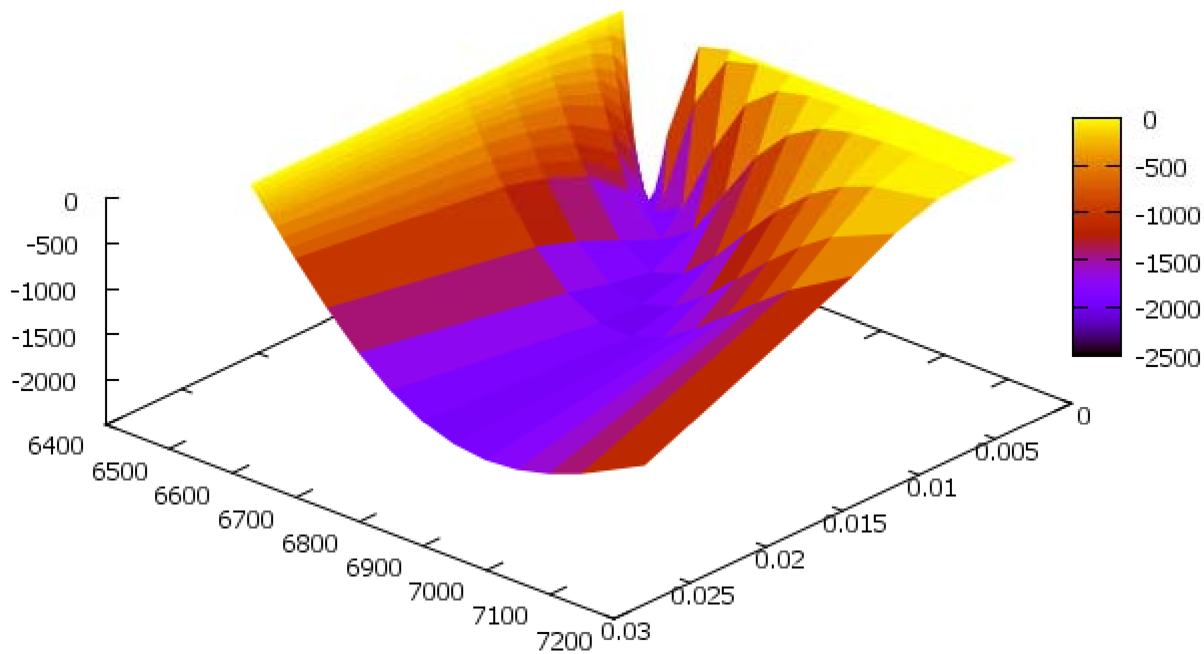
Gamma Put Option



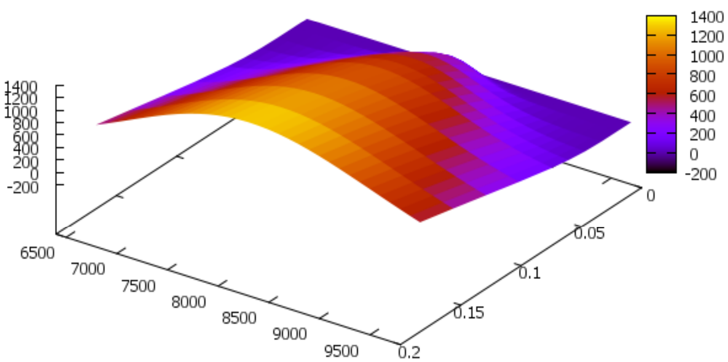
Example IIIb: down-and-out Put

Vega of Down-and-out Put option

Vega Down-and-out Put



Vega Put Option



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Counterparty default risk

Calculation of the Credit Value Adjustment (CVA)

$$V^D = V - (1 - R)\int_0^T h e^{-ht} e^{-rt}\,\mathrm{E}\left[V^+(t)\right]dt$$

Defaultable Fair Value (V^D)

Recovery Rate (R)

Hazard rate: Probability of default in time intervall dt (h)

Survival probability until time t (e^{-ht})

Positve part of derivative pay off ($V^+(t)$)

Counterparty risk can be traded by means of Credit Default Swaps – paying periodically a premium for insurance against default of specific counterparty

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Forward fair value with CVA

Counterparty risk adds option feature

Forward with CVA:

$$V^D_{\text{Forward}} = S - e^{-rT}K - (1-R)\left(1-e^{-hT}\right)V_{\text{Call}}$$
$$= S - e^{-rT}K - (1-R)\left(1-e^{-hT}\right)\left(SN(d_1) - Ke^{-rT}N(d_2)\right)$$

Call option with CVA:

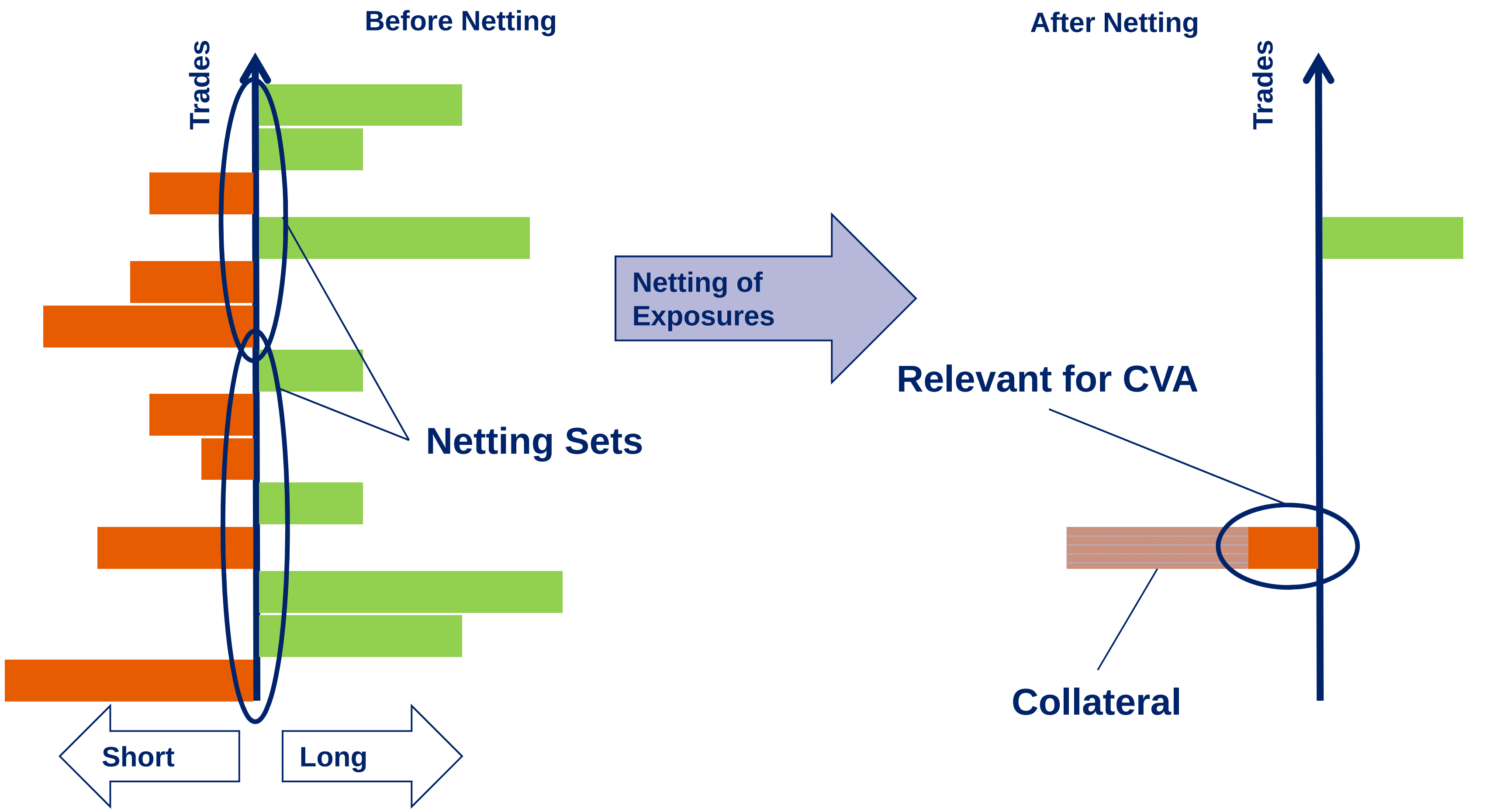
$$V^D_{\text{Call}} = V_{\text{Call}} - (1-R)V_{\text{Call}}\left(1-e^{-hT}\right)$$

Special case R=0:

$$V^D_{\text{Call}} = e^{-hT}\left(SN(d_1) - Ke^{-rT}N(d_2)\right)$$

CVA for the bank's derivatives portfolio

Effect of Netting Agreements and Collateral Management



Summary

What you might have learnt

- What‘s a derivative
- Typical equity derivatives and how they work
- General idea behind the method of arbitrage-free pricing
- The assumptions of the theory and their validity in reality
- What‘s missing in the theoretical framework
- Impact of counterparty default risk

Literature

- “Financial Calculus : An Introduction to Derivative Pricing”, Martin Baxter & Andrew Rennie, Cambridge University Press, 1996
- “Options, Futures & Other Derivatives”, John C. Hull, 7th edition, Prentice Hall India, 2008
- “Derivatives and Internal Models”, Hans-Peter Deutsch, 4th edition, Palgrave McMillan, 2009
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Glossar

Frequently used expressions

S	Underlying asset/stock price
r	(default) risk free interest rate
σ	Volatility
K	Strike price
T	Expiry Date
μ	Drift of underlying
$V(t)$	Value of derivative at time t
dW	Stochastic differential $\sim \sqrt{dt}$

Stock price process	$dS = \mu S dt + \sigma S dW$
Risk neutral stock price process	$dS = rSdt + \sigma S dW$
Pay off function of call/put option	$\max(S_T - K, 0) \equiv (S_T - K)^+$
Discount factor = value of zero bond	$e^{-r(T-t)}$
Black-Scholes equation	$rV = \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$
Short position	Position sold
Long position	Position bought