INTRODUCTION TO tmQCD AND ITS APPLICATIONS to WMEs

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tmQCD: advantages

1st advantage: IR zero-mode regularization

IR zero-mode regularization

- Wilson fermion matrix $M_W = D_W + m_0$ has spurious zero-modes at small quark mass (lattice artefacts)
- In quenched simulations they cause exceptional configurations which impede simulations at masses lower than, say, half the strange quark mass; $m_0 \leq m_s / 2$
- In un-quenched simulations the fermion determinant suppresses these zero-modes in MC, but an IR cutoff could still be helpful to the approach of the chiral limit
- tmQCD introduces an IR mass cutoff (the twisted mass!) which facilitates the approach to small mass regime

$$\bar{\psi} \; M_W \; \psi \;\; = \;\; (\bar{u} \;\; \bar{d}) \begin{pmatrix} D_W + m_0 + i\mu\gamma_5 & 0 \\ 0 & D_W + m_0 - i\mu\gamma_5 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

IR zero-mode regularization

- Wilson fermion matrix $M_W = D_W + m_q$ has spurious zero-modes at small quark mass (lattice artefacts)
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$$\bar{\psi}\; M_W\; \psi \;=\; (\bar{u}\;\; \bar{d}) \begin{pmatrix} D_W + m_0 + i\mu\gamma_5 & 0 \\ 0 & D_W + m_0 - i\mu\gamma_5 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

$$\begin{aligned} \det M_W &= \det \begin{pmatrix} D_W + m_0 + i\mu\gamma_5 & 0 \\ 0 & D_W + m_0 - i\mu\gamma_5 \end{pmatrix} \\ &= \det \left[(D_W + m_0)^\dagger (D_W + m_0) + \mu^2 \right] \end{aligned}$$

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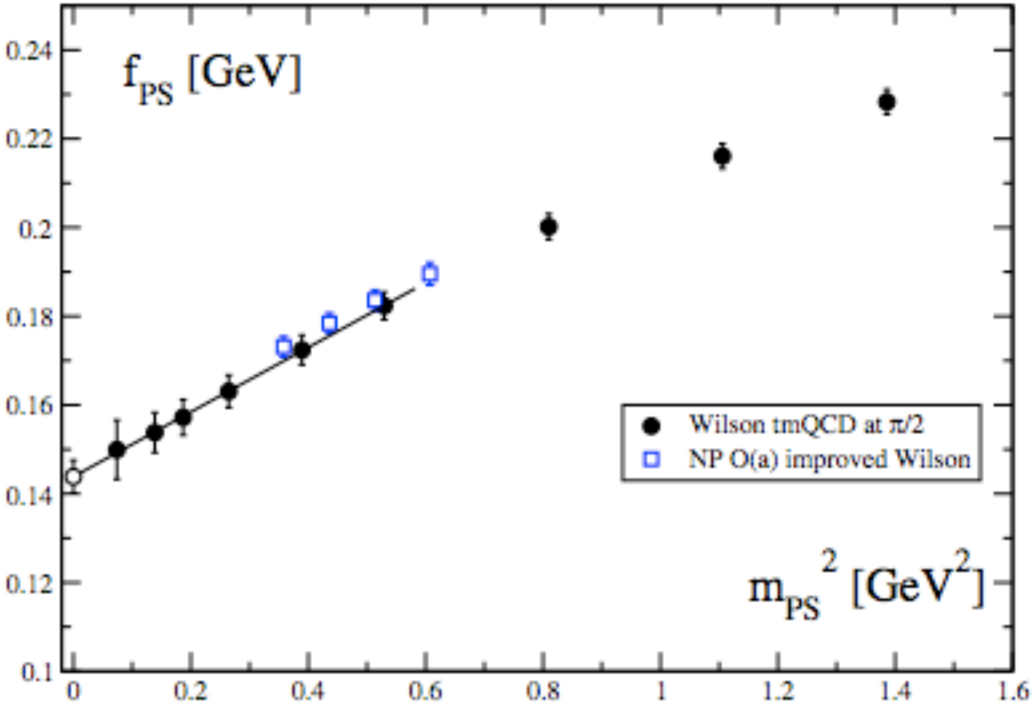

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2nd advantage: simplified renormalization

tmQCD and renormalization

- Wilson fermion renormalization may be complicated due to loss of chiral symmetry
- this may be simplified in tmQCD through judicious choices of the twist angle
- for the moment consider multiplicative renormalization of fermion operators; all *Z's* are logarithmically divergent
- mass-independent renormalization schemes implied throughout
- since (Wilson) standard QCD and tmQCD coincide in the chiral limit (i.e. they both reduce to naive kinetic term + Wilson term) we have the same *Z's* for both lattice theories
- continuum operators (i.e. their correlation functions or MEs) are expressed in terms of Wilson lattice standard QCD and tmQCD:

$$a = 1,2$$

$$\begin{aligned} [A^a_\mu]_{\rm cont} &= Z_A\ [A^a_\mu]_{\rm QCD} + \mathcal{O}(a) \\ &= \cos(\omega)\ Z_A\ [A^a_\mu]_{\rm tmQCD} + \epsilon^{3ab}\ \sin(\omega)\ Z_V\ [V^b_\mu]_{\rm tmQCD} + \mathcal{O}(a) \end{aligned}$$

- NB: valid for isospin indices a,b =1,2
- same Z_A in both regularizations

- maximally tmQCD (ω=π/2) expression significantly simpler

tmQCD and renormalization

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- continuum operators (i.e. their correlation functions or MEs) are expressed in terms of Wilson lattice standard QCD and tmQCD:

$$
\begin{aligned}
\left[P^a\right]_{\mathrm{cont}} &= Z_P \left[P^a\right]_{\mathrm{QCD}} + \mathcal{O}(a) \\
&= Z_P \left[P^a\right]_{\mathrm{tmQCD}} + \mathcal{O}(a)
\end{aligned}
\qquad a = 1, 2
$$

- NB: valid for isospin indices a,b =1,2
- tmQCD expression valid at all ω
- same Z_P in both regularizations

tmQCD and renormalization

- For maximally tmQCD we thus have:

$$\left[A_\mu^a\right]_{\rm cont} \quad = \quad \epsilon^{3ab}\ Z_V\ \left[V_\mu^b\right]_{\rm MtmQCD}\ +\ \mathcal{O}(a)$$

$$\left[P^a\right]_{\rm cont} \quad = \quad Z_P\ \left[P^a\right]_{\rm tmQCD}\ +\ \mathcal{O}(a)$$

- PCVC now reads:

$$Z_V\ \sum_{\vec{x}}\ \nabla_x^\mu\ \langle V_\mu^1(x)\ P^2(0)\ \rangle_{\rm tmQCD} \ = \ -2\mu_q \sum_{\vec{x}} \langle P_\mu^1(x)\ P^2(0)\ \rangle_{\rm tmQCD}$$

- insert complete set of states, take large time separations etc:

$$m_\pi\ Z_V\ \langle 0|V_0(x)|\pi\rangle_{\rm tmQCD} \ = \ 2\mu_q\ \langle 0|P(0)|\pi\rangle_{\rm tmQCD}$$

- NB: the rhs is RGI
- with the point-split, conserved vector current the original VWI is exact

tmQCD and renormalization

- standard definition of pion decay constant is:

$$f_\pi \;=\; \frac{1}{m_\pi}\,\langle 0|A_0(0)|\pi\rangle_{\rm cont}$$

- use maximally tmQCD expression for axial current ...

$$\left[A^a_\mu\right]_{\rm cont} \;=\; \epsilon^{3ab}\,Z_V\,\left[V^b_\mu\right]_{\rm MtmQCD} \;+\; \mathcal{O}(a)$$

- .... to get:

$$f_\pi \;=\; \lim_{a\to 0}\frac{Z_V}{m_\pi}\,\langle 0|V_0(0)|\pi\rangle_{\rm tmQCD}$$

- use PCVC in maximally tmQCD ...

$$m_\pi\;Z_V\;\langle 0|V_0(x)|\pi\rangle_{\rm tmQCD} \;=\; 2\mu_q\;\langle 0|P(0)|\pi\rangle_{\rm tmQCD}$$

- .... to get:

$$f_\pi \;=\; \lim_{a\to 0}\frac{2\mu_q}{m_\pi^2}\,\langle 0|P(0)|\pi\rangle_{\rm tmQCD}$$

tmQCD and renormalization

$$f_\pi \quad = \quad \lim_{a\rightarrow 0} \frac{2\mu_q}{m_\pi^2} \; \langle 0|P(0)|\pi\rangle_{\mathrm{tmQCD}}$$

- The determination of the pion decay constant with standard Wilson fermions requires knowledge of the normalization constant of the axial current

$$f_\pi \quad = \quad \lim_{a\rightarrow 0} \; \frac{1}{m_\pi} \; Z_A \; \langle 0|A_0|\pi\rangle_{\mathrm{QCD}}$$

- with maximally tmQCD this is not the case
- a source of systematic error has been eliminated!

tmQCD and renormalization

- the chiral condensate $< S^0 > = < \bar{\psi}\psi >$
  - multiplicatively renormalizable with chirally symmetric regularization

$$< \bar{\psi}\psi >_R = Z_{s^0} < \bar{\psi}\psi >$$

- the chiral condensate
  - additive renormalization (**cubic** power subtraction) with Wilson fermions

$$< \bar{\psi}\psi >_R = Z_{s^0} \Big[< \bar{\psi}\psi > + \frac{C(g_0^2)}{a^3} \Big]$$

Z_S is log.ly divergent

- Without chiral symmetry, the condensate mixes with the identity operator
- Other terms, quadratically and linearly divergent, must also be subtracted. They are proportional to the quark mass and vanish in the chiral limit.
- Power divergences are "vigorous" and would better be avoided (determination of their finite coefficients - e.g. $C(g_0^2)$ - is a very difficult task in practice, requiring high orders of improvement).

tmQCD and renormalization

- recall that the renormalized condensate insertion is computed from the bare tmQCD theory as:

$$\Big[< \cdots \mathcal{S}^0 \cdots >_{\mathrm{R}} \Big]^{\mathrm{QCD}} = \Big[\cos(\omega) < \cdots S^0 \cdots >_{\mathrm{R}} + i \sin(\omega) < \cdots P^3 \cdots >_{\mathrm{R}} \Big]^{\mathrm{tmQCD}}$$

- the pseudoscalar operator is multiplicatively renormalizable:

$$P^3_R = [\bar{\psi}\tau^3\gamma_5\psi]_R = Z_P P^3$$

- for twist angle ω = π/2 this means that in tmQCD the condensate is obtained from the mult.ly renormalizable pseudoscalar density

$$< \bar{\psi}\psi > = i Z_P \, [< P^3 >]^{\mathrm{tmQCD}}$$

tmQCD and renormalization

- recall that the renormalized condensate insertion is computed from the bare tmQCD theory as:

$$\Big[< \cdots \mathcal{S}^0 \cdots >_{\rm R} \Big]^{\rm QCD} = \Big[\cos(\omega) < \cdots S^0 \cdots >_{\rm R} + i \sin(\omega) < \cdots P^3 \cdots >_{\rm R} \Big]^{\rm tmQCD}$$

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$$< \bar\psi \psi > = i Z_P \, [< P^3 >]^{\rm tmQCD}$$

- above is valid in the chiral limit
- tmQCD: **logarithmic** divergence; standard Wilson: **cubic** divergence
- maximal tmQCD eliminates the power divergence - but this is a chiral limit renormalization pattern

tmQCD and renormalization

- in the chiral limit

$$< \bar{\psi}\psi > = \; iZ_P \; [< P^3 >]^{\mathrm{tmQCD}}$$

- off the chiral limit

$$\langle\bar{\psi}\psi\rangle \;=\; i\,Z_P\Big[\langle P^3\rangle^{\mathrm{tmQCD}} \;+\; \frac{\mu_q C_P(g_0^2)}{a^2} \;+\cdots\;\Big]$$

- these power (quadratic) subtractions, though less vigorous than the standard QCD cubic ones, impede in practice the calculation of the chiral condensate
- similar and very important simplifications of the renormalization patterns of 4-fermion operators will be dealt with in lesson-3

3rd advantage:
automatic improvement

Automatic improvement

- Start with a crash-course on **Symanzik improvement** for a generic lattice action
- close to continuum, lattice action described in terms of a local continuum effective theory

$$
\begin{aligned}
S_{\text{eff}} &= \mathcal{S}_0 + a\,\mathcal{S}_1 + a^2\,\mathcal{S}_2 + \cdots \\
&= \int d^4x\,\mathcal{L}_0 + a\int d^4x\,\mathcal{L}_1 + a^2\int d^4x\,\mathcal{L}_2 + \cdots
\end{aligned}
$$

- each term $\mathcal{L}_0$, , $\mathcal{L}_1$, $\mathcal{L}_2$, ... are dim-4,5,6, ... operators in the continuum; think of them as regularized in some scheme (MS, very fine lattice spacing, ...) and renormalized
- these operators must have the symmetries of the lattice action on the lhs.
- constraining ourselves to on-shell cases (i.e. fields satisfy the equations of motion - Dirac etc.) we have a smaller basis of independent operators
- $\mathcal{L}_0$ is the continuum action - continuum QCD,
- $\mathcal{L}_1$ is proportional to the "magnetic" term $\overline{\psi}\,\sigma_{\mu\nu}\,F_{\mu\nu}\,\psi$ - gives rise to Clover term

Automatic improvement

- this models the cutoff effects generated by the lattice action; how about those generated by operator insertions in correlation functions?
- Consider renormalized connected *n*-point correlation function of a composite field Φ:

$$G(x_1, \cdots, x_n) \quad = \quad [Z_\Phi]^n \ \langle \ \Phi(x_1) \cdots \Phi(x_n) \ \rangle$$

- in the local effective theory the renormalized field Z_Φ Φ is represented by a effective field

$$\Phi_{\rm eff} \quad = \quad \Phi_0 \ + \ a \ \Phi_1 \ + \ a^2 \ \Phi_2 \ + \cdots$$

- each term Φ_0, , Φ_1 , Φ_2 , ... are dim-d,d+1,d+2, ... operators in the continuum
- $\Phi_{\rm eff}$, Φ_0, , Φ_1 , Φ_2 , ... share the same symmetries
- if Φ_0 is the (d=3) axial current $A_\mu(x)$, then Φ_1 is proportional to $\partial_\mu P(x)$ and m $A_\mu(x)$

Automatic improvement

- in terms of the effective theory, the cutoff effect of the renormalized Green functions are written as follows:

$$
\begin{aligned}
G(x_1,\cdots,x_n) &= [Z_\Phi]^n \ \langle\ \Phi(x_1)\cdots\Phi(x_n)\ \rangle \\
&= \langle\ \Phi_{\text{eff}}(x_1)\cdots\Phi_{\text{eff}}(x_n)\ \rangle \\
&= \frac{1}{\mathcal{Z}}\int \mathcal{D}\mu(\bar\psi,\psi,U)\ \Phi_{\text{eff}}(x_1)\cdots\Phi_{\text{eff}}(x_n)\exp(-\mathcal{S}_{\text{eff}})
\end{aligned}
$$

Automatic improvement

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- first line: lattice expectation value w.r.t. lattice action

Automatic improvement

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&= \frac{1}{\mathcal{Z}} \int \mathcal{D}\mu(\bar{\psi}, \psi, U) \ \Phi_{\mathrm{eff}}(x_1) \cdots \Phi_{\mathrm{eff}}(x_n) \exp(-\mathcal{S}_{\mathrm{eff}})
\end{aligned}
$$

- first line: lattice expectation value w.r.t. lattice action
- second, third lines: continuum expectation value w.r.t. effective action

Automatic improvement

- recall Symanzik expansion for effective action and composite field:

$$\begin{aligned} S_{\text{eff}} &= \mathcal{S}_0 + a\,\mathcal{S}_1 + a^2\,\mathcal{S}_2 + \cdots \\ &= \int d^4x\,\mathcal{L}_0 + a\int d^4x\,\mathcal{L}_1 + a^2\int d^4x\,\mathcal{L}_2 + \cdots \end{aligned}$$

$$\Phi_{\text{eff}} = \Phi_0 + a\,\Phi_1 + a^2\,\Phi_2 + \cdots$$

- the Symanzik expansion becomes

$$\begin{aligned} G(x_1,\cdots,x_n) &= \frac{1}{\mathcal{Z}}\int \mathcal{D}\mu(\bar{\psi},\psi,U)\;\Phi_{\text{eff}}(x_1)\cdots\Phi_{\text{eff}}(x_n)\;\exp(-\mathcal{S}_{\text{eff}}) \\ &= \langle\,\Phi_0(x_1)\cdots\Phi_0(x_n)\,\rangle_0 \\ &- a\int d^4y\langle\,\Phi_0(x_1)\cdots\Phi_0(x_n)\;\mathcal{L}_1(y)\,\rangle_0 \\ &+ a\sum_k\langle\,\Phi_0(x_1)\cdots\Phi_1(x_k)\cdots\Phi_0(x_n)\,\rangle_0 + \mathcal{O}(a^2) \end{aligned}$$

- NB: continuum expectation values on rhs, weighted with $\mathcal{L}_0$,

Automatic improvement

- the Symanzik expansion becomes

$$
\begin{aligned}
G(x_1, \cdots, x_n) &= \frac{1}{\mathcal{Z}} \int \mathcal{D}\mu(\bar{\psi}, \psi, U)\ \Phi_{\text{eff}}(x_1) \cdots \Phi_{\text{eff}}(x_n)\ \ \exp(-\mathcal{S}_{\text{eff}}) \\
&= \langle\ \Phi_0(x_1) \cdots \Phi_0(x_n)\ \rangle_0 \\
&- a \int d^4y \langle\ \Phi_0(x_1) \cdots \Phi_0(x_n) \boxed{\mathcal{L}_1(y)} \rangle_0 \\
&+ a \sum_k \langle\ \Phi_0(x_1) \cdots \Phi_1(x_k) \cdots \Phi_0(x_n)\ \rangle_0 + \mathcal{O}(a^2)
\end{aligned}
$$

- generates contact terms; may be absorbed in redefinition of Φ_1
- if we redefine the lattice action, by adding to it a discretization of the d=5 magnetic term (i.e. the Clover term) , and the lattice field, by adding to it the d+1 counter-term, then the lattice correlation function has no O(a) corrections
- these counter-terms have coefficients $c_{sw}(g_0^2)$ and $c_\Phi(g_0^2)$ which must be **tuned**
- this is the standard Symanzik improvement programme

Automatic improvement

- NB: the fully twisted case ω = π/2 is of particular interest. Classically we have:

$$\mathcal{L} = \bar{\psi}\left[\not{D} + M\ \exp[i\frac{\pi}{2}\tau^3\gamma_5]\right]\psi = \bar{\psi}\left[\not{D} + i\,\mu\,\tau^3\gamma_5\right]\psi$$

- the lattice version requires introduction of the Wilson term but also of the critical standard mass *m*cr in order to ensure full twist

$$\mathcal{L} = \bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) + \frac{1}{2}ar\nabla^*_\mu\nabla_\mu + m_{\rm cr} + i\mu_q\tau^3\gamma_5\right]\psi$$

- NB: the concept of twist angle at the quantum level requires renormalized masses:

$$\tan(\omega) = \frac{\mu_{\rm R}}{m_{\rm R}} = \frac{Z_\mu\,\mu_q}{Z_m\left[m_0 - m_{\rm cr}\right]}$$

$$\omega = \frac{\pi}{2} \leftrightarrow m_0 = m_{\rm cr}$$

Automatic improvement

- the study of discretization effects of lattice observables is based on the Symanzik expansion
- the lattice action close to the continuum is described in terms of an effective theory

$$\begin{aligned} S_{\rm eff} &= \mathcal{S}_0 \; + \; a\, \mathcal{S}_1 \; + \; a^2 \; \mathcal{S}_2 \; + \cdots \\ &= \int d^4x\, \mathcal{L}_0 \; + \; a \int d^4x\, \mathcal{L}_1 \; + \; a^2 \int d^4x\, \mathcal{L}_2 \; + \cdots \end{aligned}$$

- for the fully twisted lattice action

$$\mathcal{L} \quad = \quad \bar{\psi} \left[\frac{1}{2} \gamma_\mu (\nabla_\mu + \nabla_\mu^*) \; + \; \frac{1}{2} a r \nabla_\mu^* \nabla_\mu \; + \; m_{\rm cr} \; + \; i \mu_q \tau^3 \gamma_5 \right] \psi$$

- the Symanzik expansion counter-terms are:

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi} \left[\not{D} \; + \; i\, \mu_{\rm R}\, \tau^3\, \gamma_5 \right] \psi \\ \mathcal{L}_1 &= i\, c_{\rm sw} \; (\bar{\psi} \; \sigma \cdot F \; \psi) \; + \; c_\mu \; \mu_{\rm R}^2 \; (\bar{\psi} \psi) \end{aligned}$$

- the Symanzik expansion for a lattice field operator is:

$$\Phi_{\rm Latt} = \Phi_0 \; + \; a\, \Phi_1 \; + \; \cdots$$

Automatic improvement

- the study of discretization effects of lattice observables is based on the Symanzik expansion
- the lattice action close to the continuum is described in terms of an effective theory

$$\begin{aligned} S_{\rm eff} &= \mathcal{S}_0 + a\,\mathcal{S}_1 + a^2\,\mathcal{S}_2 + \cdots \\ &= \int d^4x\,\mathcal{L}_0 + a\int d^4x\,\mathcal{L}_1 + a^2\int d^4x\,\mathcal{L}_2 + \cdots \end{aligned}$$

- for the fully twisted lattice action

$$\mathcal{L} = \bar\psi\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) + \frac{1}{2}ar\nabla^*_\mu\nabla_\mu + m_{\rm cr} + i\mu_q\tau^3\gamma_5\right]\psi$$

- the Symanzik expansion counter-terms are:

$$\begin{aligned} \mathcal{L}_0 &= \bar\psi\left[\not{D} + i\,\mu_{\rm R}\,\tau^3\,\gamma_5\right]\psi \\ \mathcal{L}_1 &= i\,c_{\rm sw}\,(\bar\psi\,\sigma\cdot F\,\psi) + c_\mu\,\mu_{\rm R}^2\,(\bar\psi\psi) \end{aligned}$$

- the Symanzik expansion for a lattice field operator is:

$$\boxed{\Phi_{\rm Latt} = \Phi_0} + a\,\Phi_1 + \cdots$$

dimension d

Automatic improvement

- the study of discretization effects of lattice observables is based on the Symanzik expansion
- the lattice action close to the continuum is described in terms of an effective theory

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\begin{aligned}
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\end{aligned}
$$

- for the fully twisted lattice action

$$
\mathcal{L} = \bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) + \frac{1}{2}ar\nabla^*_\mu\nabla_\mu + m_{\rm cr} + i\mu_q\tau^3\gamma_5\right]\psi
$$

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$$
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\mathcal{L}_0 &= \bar{\psi}\left[\not{D} + i\,\mu_{\rm R}\,\tau^3\,\gamma_5\right]\psi \\
\mathcal{L}_1 &= i\,c_{\rm sw}\,(\bar{\psi}\,\sigma\cdot F\,\psi) + c_\mu\,\mu_{\rm R}^2\,(\bar{\psi}\psi)
\end{aligned}
$$

- the Symanzik expansion for a lattice field operator is:

$$
\underbrace{\Phi_{\rm Latt} = \Phi_0}_{\text{dimension d}} + a\,\underbrace{\Phi_1}_{\text{dimension d+1}} + \cdots
$$

Automatic improvement

- automatic improvement is based on the following field transformations
- discrete "chiral" transformations R_5^1 :
- lattice action terms transform as R_5^1 eigenstates

$$\begin{aligned} \psi &\longrightarrow i\,\gamma_5\,\tau^1\,\psi \\ \bar\psi &\longrightarrow \bar\psi\, i\,\gamma_5\,\tau^1 \end{aligned}$$

$$\mathcal{L} \;=\; \bar\psi\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla_\mu^*) \;+\; \frac{1}{2}ar\nabla_\mu^*\nabla_\mu \;+\; m_{\rm cr} \;+\; i\mu_q\tau^3\gamma_5\right]\psi$$

$\mathcal{L} - \text{terms}$	R_5^1		
$\bar\psi\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla_\mu^*)\right]\psi$	$+$		
$\bar\psi\left[\frac{1}{2}ar\nabla_\mu^*\nabla_\mu\right]\psi$	$-$		
$\bar\psi\, m_{\rm cr}\psi$	$-$		
$\bar\psi\left[i\mu_q\tau^3\gamma_5\right]\psi$	$+$		

Automatic improvement

- automatic improvement is based on the following field transformations
- operator dimensionality transformations *D* :
- lattice action terms transform as *D* eigenstates

$$\begin{aligned} U_\mu(x) &\rightarrow U_\mu^\dagger(-x-a\hat{\mu}) \\ \psi(x) &\rightarrow \exp[3i\pi/2]\,\psi(-x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(-x)\exp[3i\pi/2] \end{aligned}$$

$$\mathcal{L} \quad = \quad \bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla_\mu^*) \;+\; \frac{1}{2}ar\nabla_\mu^*\nabla_\mu \;+\; m_{\rm cr} \;+\; i\mu_q\tau^3\gamma_5\right]\psi$$

$\mathcal{L}-\text{terms}$	R_5^1	$\mathcal{D}$	
$\bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla_\mu^*)\right]\psi$	+	+	
$\bar{\psi}\left[\frac{1}{2}ar\nabla_\mu^*\nabla_\mu\right]\psi$	−	−	
$\bar{\psi}\,m_{\rm cr}\psi$	−	−	
$\bar{\psi}\left[i\mu_q\tau^3\gamma_5\right]\psi$	+	−	

Automatic improvement

- automatic improvement is based on the following field transformations
- twisted mass sign flip μ → −μ
- lattice action terms transform as *sign flip* eigenstates

$$\mathcal{L} \quad = \quad \bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) \;+\; \frac{1}{2}ar\nabla^*_\mu\nabla_\mu \;+\; m_{\rm cr} \;+\; i\mu_q\tau^3\gamma_5\right]\psi$$

$\mathcal{L}-\mathrm{terms}$	R_5^1	$\mathcal{D}$	$\mu \longrightarrow -\mu$
$\bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu+\nabla^*_\mu)\right]\psi$	$+$	$+$	$+$
$\bar{\psi}\left[\frac{1}{2}ar\nabla^*_\mu\nabla_\mu\right]\psi$	$-$	$-$	$+$
$\bar{\psi}\,m_{\rm cr}\psi$	$-$	$-$	$+$
$\bar{\psi}\left[i\mu_q\tau^3\gamma_5\right]\psi$	$+$	$-$	$-$

- NB: massive theory invariant under $R_5^1 \otimes D \otimes [\mu \rightarrow -\mu]$

Automatic improvement

- the lowest- order Symanzik expansion for the vev of an operator Φ is:

$$< \Phi > = < \Phi_0 >_0 \; + a \; < \Phi_1 >_0 \; -a \int d^4y < \Phi_0 \; \mathcal{L}_1 >_0$$

- the LHS is a lattice vev, determined by the lattice fully twisted action
- for an operator Φ with positive R_5^1 parity and even dimension d, the LHS is invariant under $R_5^1 \otimes D \otimes [\mu \rightarrow -\mu]$
- also the RHS must be invariant under $R_5^1 \otimes D \otimes [\mu \rightarrow -\mu]$
- the RHS operators and vev are continuum quantities determined by the continuum tmQCD action with positive R_5^1 parity, $\mathcal{L}_0$

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi} \left[\not{D} \; + \; i \, \mu_{\rm R} \, \tau^3 \, \gamma_5 \right] \psi \\ \mathcal{L}_1 &= i \, c_{\rm sw} \; (\bar{\psi} \; \sigma \cdot F \; \psi) \; + \; c_\mu \; \mu_{\rm R}^2 \; (\bar{\psi} \psi) \end{aligned}$$

- the continuum operator Φ_0 has positive R_5^1 parity and even dimension d
- the continuum operator Φ_1 has odd dimension $d+1$ and therefore negative R_5^1 parity
- thus $< \Phi_1 >_0$ vanishes as it is R_5^1 -odd, weighted by an R_5^1 -even action $\mathcal{L}_0$

Automatic improvement

- the lowest- order Symanzik expansion for the vev of an operator Φ is:

$$< \Phi > = < \Phi_0 >_0 \ + a \ < \Phi_1 >_0 \ - a \int d^4 y < \Phi_0 \ \mathcal{L}_1 >_0$$

- the LHS is a lattice vev, determined by the lattice fully twisted action
- for an operator Φ with positive R_5^1 parity and even dimension *d*, the LHS is invariant under $R_5^1 \otimes D \otimes [\mu \rightarrow -\mu]$
- also the RHS must be invariant under $R_5^1 \otimes D \otimes [\mu \rightarrow -\mu]$
- the RHS operators and vev are continuum quantities determined by the continuum tmQCD action with positive R_5^1 parity, $\mathcal{L}_0$

$$\begin{aligned} \mathcal{L}_0 &= \bar\psi \left[\not{D} \ + \ i \, \mu_{\rm R} \, \tau^3 \, \gamma_5 \right] \psi \\ \mathcal{L}_1 &= i \, c_{\rm sw} \ (\bar\psi \ \sigma \cdot F \ \psi) \ + \ c_\mu \ \mu_{\rm R}^2 \ (\bar\psi \psi) \end{aligned}$$

- the continuum *O*(a) counter-term of the action is $\mathcal{L}_1$, with negative R_5^1 parity
- thus $< \Phi_0 \mathcal{L}_1 >_0$ vanishes as it is R_5^1 -odd, weighted by an R_5^1 -even action $\mathcal{L}_0$
- fully twisted QCD action is not improved, but has automatically improved vev !!!!!

Automatic improvement

- the lowest- order Symanzik expansion for the vev of an operator Φ is:

$$< \Phi > = < \Phi_0 >_0 \; + a \; < \Phi_1 >_0 \; -a \int d^4y < \Phi_0 \; \mathcal{L}_1 >_0$$

- NB (subtlety): the proof rests on the vanishing of the continuum vev

$$< \Phi_1 >_0 \qquad\qquad < \Phi_0 \mathcal{L}_1 >_0$$

- they vanish due to their breaking of the discrete “chiral” symmetry $R_5{}^1$, which is a symmetry of the continuum tmQCD $\mathcal{L}_0$
- BUT: is this true, i.e. do these “chiral condensates” vanish in a theory with SSB?
- YES! because the term generating SSB is the twisted mass term, while O(a) counter-terms are generated by the “chirally orthogonal” Wilson term

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi} \left[\not{D} \; + \; i \, \mu_{\rm R} \, \tau^3 \, \gamma_5 \right] \psi \\ \mathcal{L}_1 &= i \, c_{\rm sw} \; (\bar{\psi} \; \sigma \cdot F \; \psi) \; + \; c_\mu \; \mu_{\rm R}^2 \; (\bar{\psi} \psi) \end{aligned}$$

Automatic improvement

- stated differently, the discrete “chiral symmetry” R_5^1 is a specific **vector** rotation $\boldsymbol{U_V^2(1)}$, which does not generate SSB

Automatic improvement

- stated differently, the discrete "chiral symmetry" R_5^1 is a specific **vector** rotation $U_V^2(1)$, which does not generate SSB
- recall twisted vector symmetry (isospin) is $SU_V(2)_\omega$ broken to $U_V^3(1)$ by Wilson term

$$\mathcal{L} \quad = \quad \bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) \;+\; \frac{1}{2}ar\nabla^*_\mu\nabla_\mu \;+\; m_{\mathrm{cr}} \;+\; i\mu_q\tau^3\gamma_5\right]\psi$$

Automatic improvement

- stated differently, the discrete “chiral symmetry” $R_5{}^1$ is a specific **vector** rotation ***$U_V{}^2(1)$***, which does not generate SSB
- recall twisted vector symmetry (isospin) is ***$SU_V(2)_\omega$*** broken to ***$U_V{}^3(1)$*** by Wilson term

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- at ω = π/2 different subgroups of ***$SU_V(2)_{\pi/2}$*** remain (un)broken by either the (twisted) mass term or the Wilson term
- ***$U_V{}^1(1)$*** and ***$U_V{}^2(1)$*:** unbroken by μ-mass term; hard-broken by Wilson and **m_{cr}**-terms

$$\psi \quad \rightarrow \quad \exp\left[i\frac{\alpha^1}{2}\gamma_5\tau^2 \right] \; \psi$$

$$\bar\psi \quad \rightarrow \quad \bar\psi \; \exp\left[i\frac{\alpha^1}{2}\gamma_5\tau^2 \right]$$

- this is ***$U_V{}^1(1)$*** ; similarly for ***$U_V{}^2(1)$***
- i.e. **vector** symmetry in fully tmQCD has an **axial** form
- it is a **vector** (flavour) symmetry as it is preserved by the mass term

Automatic improvement

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- ***$U_V^1(1)$*** and ***$U_V^2(1)$*:** unbroken by μ-mass term; hard-broken by Wilson and **m_{cr}**-terms

$$\psi \quad \rightarrow \quad \exp\left[i\frac{\alpha^1}{2}\gamma_5\tau^2\right]\,\psi$$

$$\bar{\psi} \quad \rightarrow \quad \bar{\psi}\,\exp\left[i\frac{\alpha^1}{2}\gamma_5\tau^2\right]$$

- this is ***$U_V^1(1)$*** ; similarly for ***$U_V^2(1)$***

- for $\alpha^2 = \pi$, ***$U_V^2(1)$*** reduces to R_5^1

Automatic improvement

- stated differently, the discrete "chiral symmetry" R_5^1 is a specific **vector** rotation ***$U_V^2(1)$***, which does not generate SSB

- recall twisted vector symmetry (isospin) is ***$SU_V(2)_\omega$*** broken to ***$U_V^3(1)$*** by Wilson term

$$\mathcal{L} \quad = \quad \bar{\psi}\left[\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) \; + \; \frac{1}{2}ar\nabla^*_\mu\nabla_\mu \; + \; m_{\mathrm{cr}} \; + \; i\mu_q\tau^3\gamma_5 \;\right]\psi$$

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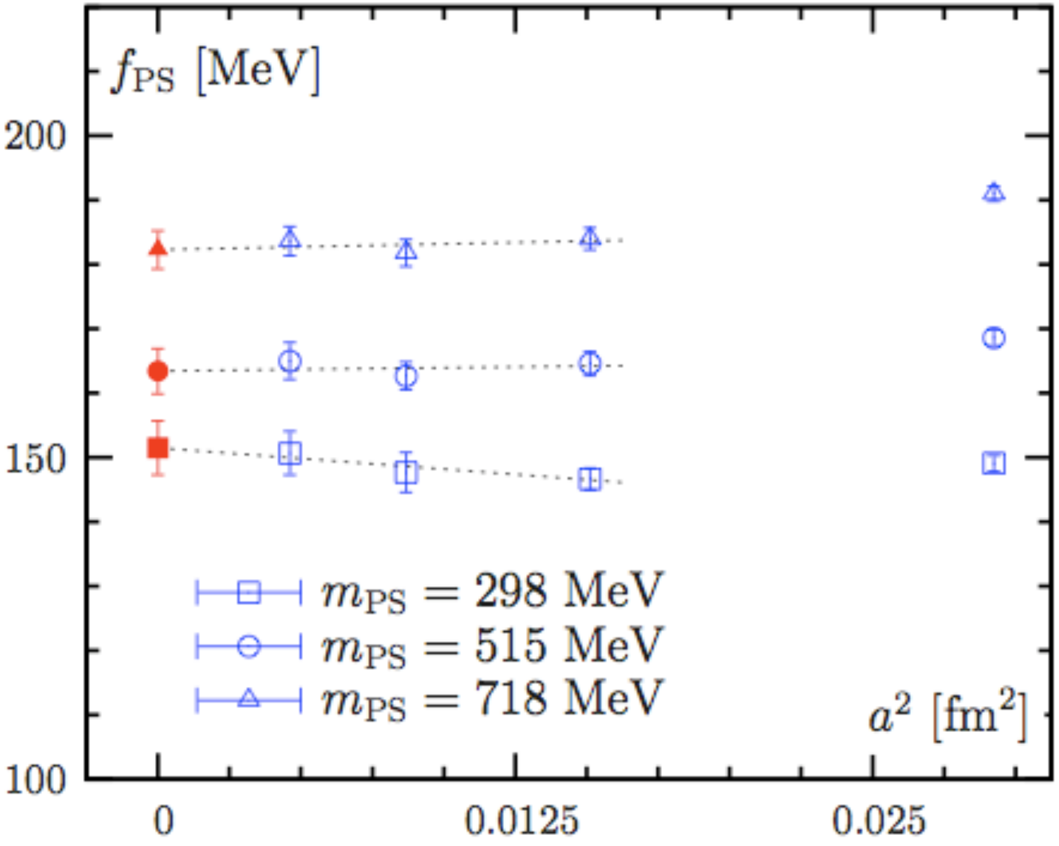
$$\bar{\psi} \quad \rightarrow \quad \bar{\psi}\,\exp\left[i\frac{\alpha^1}{2}\gamma_5\tau^2\right]$$

- NB: in simulations we must take extra care that the continuum limit is approached before the chiral limit; the chiral phase of the vacuum must be driven by the mass term and not by the Wilson term

$$\mu \; > \; a\Lambda^2_{\mathrm{QCD}}$$

Automatic improvement

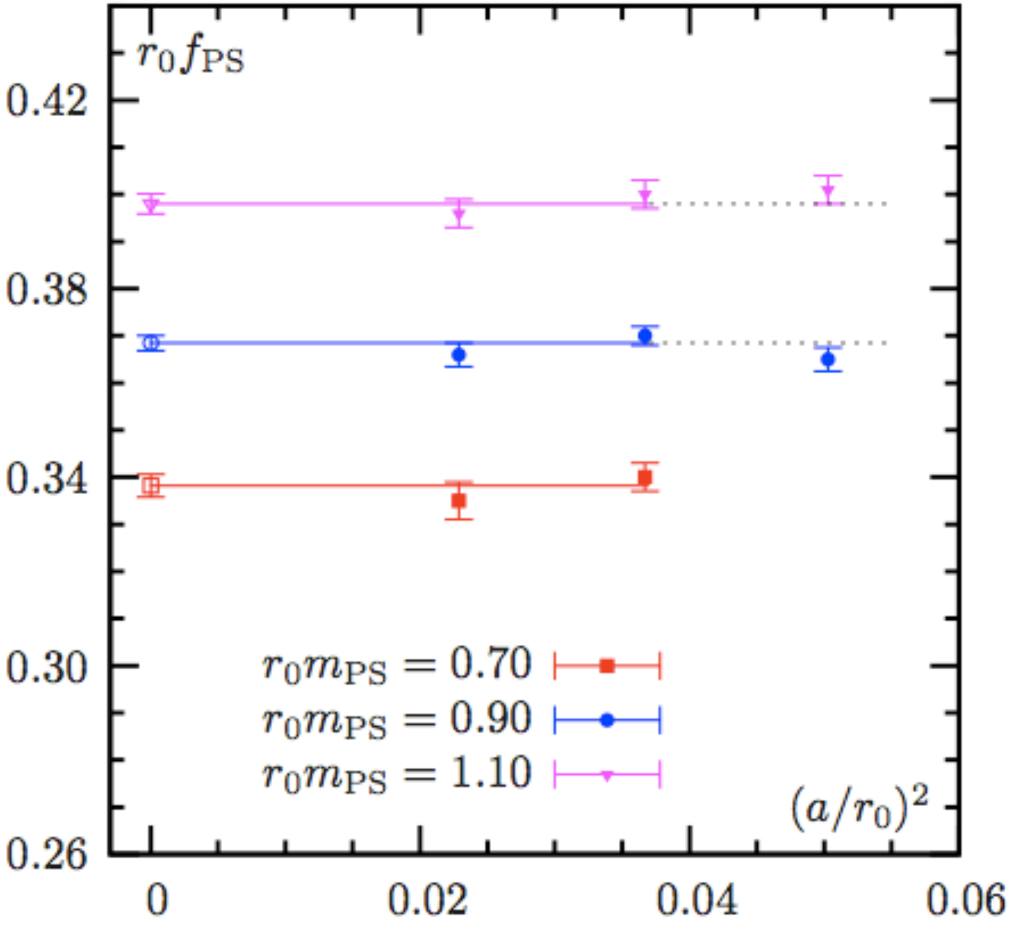
- numerical evidence supports automatic improvement
- result shown is quenched



χLF Collab. K. Jansen, M. Papiutto, A. Shindler,C. Urbach, I. Wetzorke JHEP09 (2005) 071

Automatic improvement

- numerical evidence supports automatic improvement
- result shown is unquenched $N_f = 2$



ETMC Collab.,C. Urbach, PoS(LAT2007) 022

Concluding remarks

Concuding remarks

- tmQCD has several advantages, making it an efficient alternative to GW-type computations with dynamical fermions; novel results have been obtained at $N_f = 2$
- the simplest quantities (pseudoscalar masses, decay constants, quark masses etc.) have currently been measured, close to the "physical regime" (lightish pions ~ 300MeV)
- it is possible to overcome the limitation of two degenerate flavours in the formalism without losing tmQCD advantages; $N_f = 2+1+1$ simulations are well under way
- the main drawback is the lack of flavour symmetry at finite UV cutoff
    - early quenched studies have shown that these effects vanish in the ocntinuum
    - detailed unquenched studies are still to be carried out; the overall qualitative behaviour points out to the vanishing of such effects in the continuum
- one must also be aware that tuning of the theory bare parameters to maximal twist is an issue requiring extra care (not covered here)
- the phase diagramme of tmQCD has also been under study in order to gain further insight to the issues related to symmetry breaking and their restoration in the continuum (not covered here)
- an important theoretical issue is the combination of tmQCD with Schrödinger Functional: S. Sint PoS (LAT2005) 235 (not covered here)