Chiral perturbation theory

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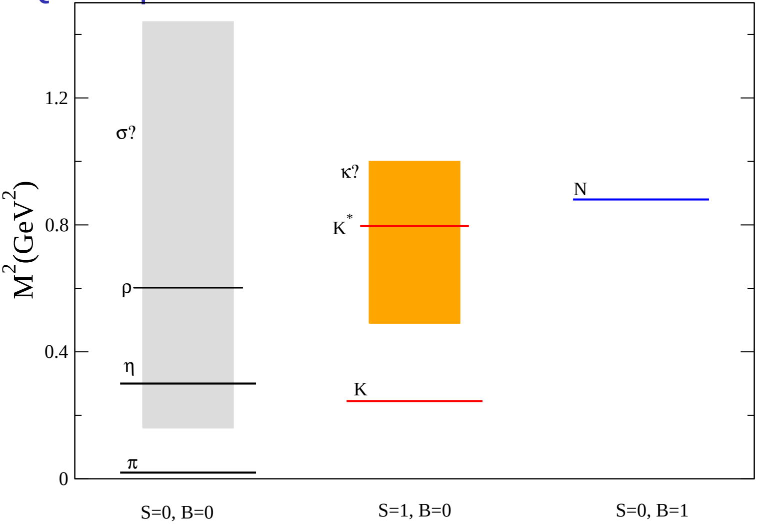
Lecture I: Foundations of chiral perturbation theory

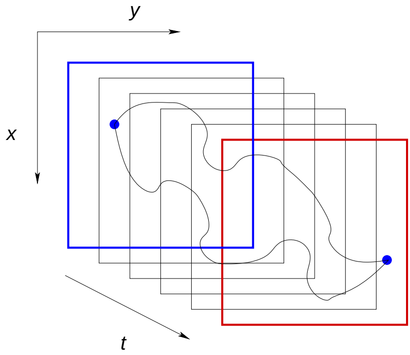
The QCD spectrum



The QCD spectrum – on the lattice

$$C(t) = \int d^3x \, \langle [\bar{q}\gamma_5 q(x)]\,[\bar{q}\gamma_5 q(0)] \rangle \, e^{i\mathbf{px}} \xrightarrow{t\to\infty} \sum_{n=0}^{\infty} c_n \, e^{-E_n t}$$

$$M_\pi = \lim_{\substack{L\to\infty \\ a\to 0}} M_\pi(L,a) \qquad M_\pi(L,a) = E_n(\mathbf{p}=0)$$



- $\bar{q}\,\gamma_5\, q, \quad \bar{q}\,\Gamma\, q$

fermion propagator

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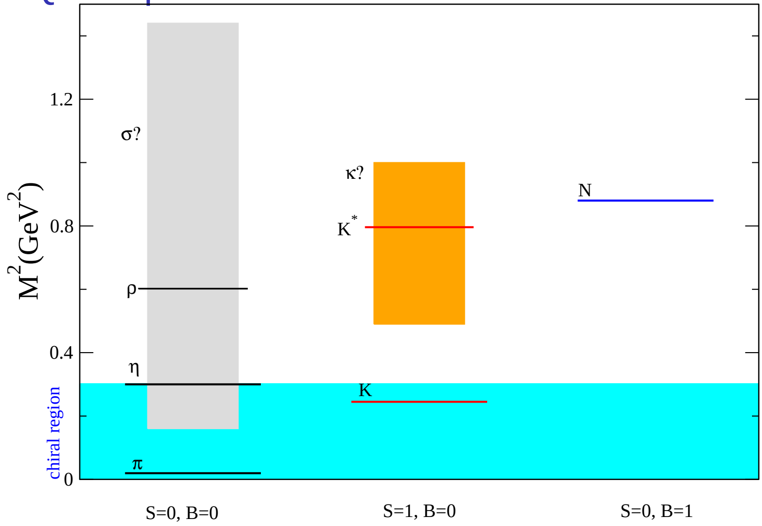

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- the positions of the low-lying resonances is more difficult to determine and understand (lattice? χPT+ dispersion relations?)

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- they set the limit of validity of the chiral expansion

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- effective Lagrangian: systematic method to construct this expansion, respecting symmetry and all the general principles of quantum field theory Weinberg (79)
- The method leads to predictions – even very sharp ones

Quantum Chromodynamics in the chiral limit

$$\mathcal{L}_{\mathrm{QCD}}^{(0)} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Large global symmetry group:

$$SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

1. $U(1)_V \Rightarrow$ baryonic number
2. $U(1)_A$ is anomalous
3. $$SU(3)_L \times SU(3)_R \Rightarrow SU(3)_V$$

   $\Rightarrow$ Goldstone bosons with the quantum numbers of pseudoscalar mesons will be generated

Quark masses, chiral expansion

In the real world quarks are not massless:

$$\mathcal{L}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \mathcal{L}_m, \qquad \mathcal{L}_m := -\bar{q}\mathcal{M}q$$

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

the mass term $\mathcal{L}_m$ can be considered as a small perturbation $\Rightarrow$
Expand around $\mathcal{L}^{(0)}_{QCD}$ $\equiv$ Expand in powers of m_q

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Chiral perturbation theory, the low-energy effective theory of QCD, is a simultaneous expansion in powers of momenta and quark masses

Quark mass expansion of meson masses

General quark mass expansion for the P particle:

$$M_P^2 = M_0^2 + \langle P|\bar{q}\mathcal{M}q|P\rangle + O(m_q^2)$$

For the pion $M_0^2 = 0$:

$$M_\pi^2 = -(m_u + m_d)\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle + O(m_q^2)$$

where we have used a Ward identity:

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0$$

$\langle 0|\bar{q}q|0\rangle$ is an order parameter for the chiral spontaneous symmetry breaking

Gell-Mann, Oakes and Renner (68)

Quark mass expansion of meson masses

Consider the whole pseudoscalar octet:

$$
\begin{aligned}
M_\pi^2 &= (m_u + m_d)B_0 + O(m_q^2) \\
M_{K^+}^2 &= (m_u + m_s)B_0 + O(m_q^2) \\
M_{K^0}^2 &= (m_d + m_s)B_0 + O(m_q^2) \\
M_\eta^2 &= \frac{1}{3}(m_u + m_d + 4m_s)B_0 + O(m_q^2)
\end{aligned}
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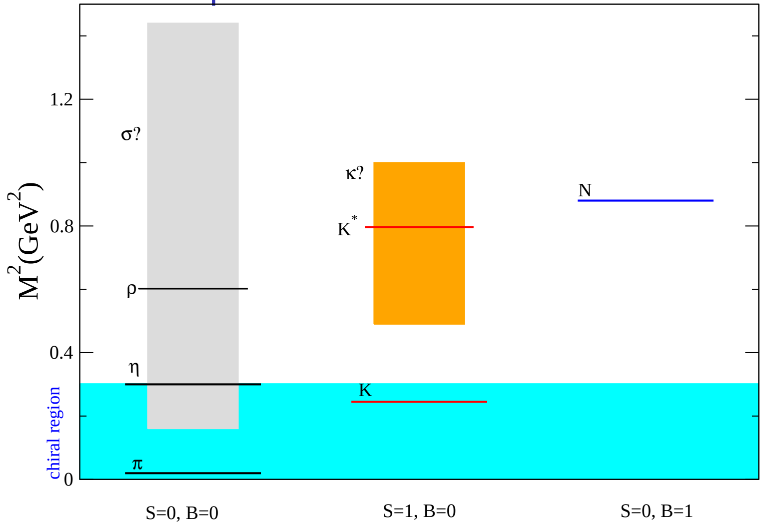
Consequences: $(\hat{m} = (m_u + m_d)/2)$

$$\begin{aligned} M_K^2/M_\pi^2 &= (m_s + \hat{m})/2\hat{m} && \Rightarrow m_s/\hat{m} = 25.9 \\ M_\eta^2/M_\pi^2 &= (2m_s + \hat{m})/3\hat{m} && \Rightarrow m_s/\hat{m} = 24.3 \\ 3M_\eta^2 &= 4M_K^2 - M_\pi^2 && \text{Gell-Mann–Okubo (62)} \\ (0.899 &= 0.960)\ \text{GeV}^2 \end{aligned}$$

Quark mass expansion of meson masses



Goldstone theorem

Hamiltonian $\mathcal{H}$ symmetric under the group of transformations G:
[Q_i are the generators of G]

$$[Q_i, \mathcal{H}] = 0 \qquad\qquad i = 1, \ldots n_G$$

Ground state not invariant under G, i.e. for some generators X_i

$$X_i|0\rangle \neq 0$$

$$\{Q_1, \ldots, Q_{n_G}\} = \{H_1, \ldots, H_{n_H}, X_1, \ldots, X_{n_G - n_H}\}$$

Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \dots n_G \ , \qquad X_i|0\rangle \neq 0 \ , \qquad H_i|0\rangle = 0$$

1. The subset of generators H_i which annihilate the vacuum forms a subalgebra

$$[H_i, H_k]|0\rangle = 0 \qquad\qquad i, k = 1, \dots n_H$$

2. The spectrum of the theory contains $n_G - n_H$ massless excitations

$$X_i|0\rangle \qquad\qquad i = 1, \dots n_G - n_H$$

from $[X_i, \mathcal{H}] = 0$ follows that $X_i|0\rangle$ is an eigenstate of the Hamiltonian with the same eigenvalue as the vacuum

Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \ldots n_G \;, \qquad X_i|0\rangle \neq 0 \;, \qquad H_i|0\rangle = 0$$

- $X_i|0\rangle$ are the Goldstone boson states
- the X_i are generators of the quotient space G/H
- the Goldstone fields are elements of the space G/H
- their transformation properties under G are fully dictated: they transform nonlinearly
- the dynamics of the Goldstone bosons at low energy is strongly constrained by symmetry

Matrix elements of conserved currents

Goldstone's theorem also asserts that:

the transition matrix elements between the conserved currents associated with the generators Q_i and the pions*

$$\langle 0|J_i^\mu|\pi^a(p)\rangle = iF_i^a p^\mu$$

is an $n_G \times (n_G - n_H)$ matrix F_i^a of rank $N_{GB} = n_G - n_H$

*We have introduced the symbol π for the Goldstone boson fields, and will call them "pions", as in strong interactions. The discussion however, remains completely general

Pions do not interact at low energy

Current conservation implies

$$p_\mu \langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\ldots\text{out}|J_i^\mu|0\rangle = 0 \qquad p^\mu = p_1^\mu + p_2^\mu + \ldots$$

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Consider the amplitude for pair creation

$$\langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\mathrm{out}|J_i^\mu|0\rangle = \frac{p_3^\mu}{p_3^2}\sum_{a_3} F_i^{a_3} v_{a_1 a_2 a_3}(p_i) + \ldots$$

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Current conserv. $\Rightarrow \sum_{a_3} F_i^{a_3} v_{a_1a_2a_3}(0) = 0 \Rightarrow v_{a_1a_2a_3}(0) = 0$

Lorentz invariance $\Rightarrow$ $v_{a_1a_2a_3}(p_1,p_2,p_3)$ can only depend on p_1^2, p_2^2, p_3^2: on the mass shell it is always zero

Pions do not interact at low energy

Amplitude for three–pion creation from a conserved current

$$\langle \pi^{a_1} \pi^{a_2} \pi^{a_3} \text{out} | J_i^\mu | 0 \rangle = \frac{p_4^\mu}{p_4^2} \sum_{a_4} F_i^{a_4} v_{a_1 a_2 a_3 a_4}(p_i) + \ldots$$

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In this case the vertex function can depend on two Lorentz scalars, s and t, and we can do a Taylor expansion:

$$v_{a_1a_2a_3a_4}(p_1,p_2,p_3,p_4) = c^1_{a_1a_2a_3a_4} s + c^2_{a_1a_2a_3a_4} t + \ldots$$

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- Effective Lagrangian for Goldstone Bosons = χPT

Weinberg (79)

Transformation properties of the pions

The pion fields transform according to a representation of G

$$g \in G : \vec{\pi} \to \vec{\pi}' = \vec{f}(g, \vec{\pi})$$

where f has to obey the composition law

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi})$$

$\vec{f}(g, 0)$ = image of the origin : the elements which leave the origin invariant form a subgroup – the conserved subgroup H

$\vec{f}(gh, 0)$ coincides with $\vec{f}(g, 0)$ for each $g \in G$ and $h \in H \Rightarrow$ the function $\vec{f}$ maps elements of G/H onto the space of pion fields

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The mapping is invertible: $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies $g_1 g_2^{-1} \in H$
$\Rightarrow$ pions can be identified with elements of G/H

Action of G on G/H

Two elements of G, $g_{1,2}$ are identified with the same element of G/H if

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The transformation properties of the coordinates of G/H under the action of G are nonlinear (h is in general a nonlinear function of q_1 and g)

The space G/H for QCD

The choice of a representative element inside each equivalence class is arbitrary. For example

$$g = (g_L, g_R) = (1, g_R g_L^{-1}) \cdot (g_L, g_L) =: q \cdot h$$

but also $$g = (g_L, g_R) = (g_L g_R^{-1}, 1) \cdot (g_R, g_R) =: q' \cdot h'$$

where $$q, q' \in G/H \quad \text{and} \quad h, h' \in H$$

Action of G on G/H

$$\begin{aligned}(V_L, V_R) \cdot (1, g_R g_L^{-1}) &= (V_L, V_R g_R g_L^{-1}) \\ &= (1, V_R g_R g_L^{-1} V_L^{-1}) \cdot (V_L, V_L)\end{aligned}$$

The space G/H for QCD

In the literature the pion fields are usually collected in a matrix–valued field U, which transforms like

$$U \xrightarrow{G} U' = V_R U V_L^{-1}$$

U is nothing but a shorthand notation for $(1, g_R g_L^{-1})$, or its non-trivial part $g_R g_L^{-1}$

As a matrix U is a member of $SU(3)$, and therefore it can be written as

$$U = e^{i\phi^a \lambda_a}$$

where ϕ^a are the eight pion fields

Construction of the effective Lagrangian

In order to reproduce the low–energy structure of QCD we construct an effective Lagrangian which:

- contains the pion fields as the only degrees of freedom
- is invariant under G
- and expand it in powers of momenta

$$\begin{aligned}\mathcal{L}_{eff} &= f_1(U) + f_2(U)\langle U^+ \Box U\rangle \\ &+ f_3(U)\langle \partial_\mu U^+ \partial^\mu U\rangle + O(p^4)\end{aligned}$$

The invariance under transformations $U \xrightarrow{G} U' = V_R U V_L^{-1}$ implies that $f_{1,2,3}(U)$ do not depend on U
$\Rightarrow f_1$ is an irrelevant constant and can simply be dropped

Construction of the effective Lagrangian

Using partial integration we end up with

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \qquad \mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^+ \partial^\mu U \rangle$$

where we have fixed the constant in front of the trace by looking at the Noether currents of the G symmetry:

$$V_i^\mu = i\frac{F^2}{4}\langle \lambda_i [\partial^\mu U, U^+] \rangle \qquad A_i^\mu = i\frac{F^2}{4}\langle \lambda_i \{\partial^\mu U, U^+\} \rangle$$

and comparing the result of the matrix element with the definition

$$\langle 0 | A_i^\mu | \pi^k(p) \rangle = i p^\mu \delta_{ik} F$$

Some technical details

The matrix field U is an exponential of the pion fields π. If we want fields π of canonical dimension, we have to introduce a dimensional constant in the definition of U:

$$U = \exp\left\{\frac{i}{F'}\pi^k\lambda_k\right\}$$

The requirement that the kinetic term of the pion fields is standard:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\pi^i\partial^\mu\pi^i \qquad \text{implies:} \qquad F = F'$$

The Lagrangian contains only one coupling constant which is the pion decay constant

The first prediction: $\pi\pi$ scattering

Isospin invariant amplitude:

$$M(\pi^a\pi^b \to \pi^c\pi^d) = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,u,s) + \delta_{ad}\delta_{bc}A(u,s,t)$$

Using the effective Lagrangian above

$$A(s,t,u) = \frac{s}{F^2}$$

Exercise: calculate it!

χPT and explicit symmetry breaking?

- The effective Lagrangian was constructed in order to systematically account for symmetry relations.
  If the symmetry is explicitly broken can we still use it?
- If the symmetry breaking is weak we can make a perturbative expansion: matrix elements of the symmetry breaking Lagrangian (or of powers thereof) will appear
- Once we know the transformation properties of the symmetry breaking term, we can use symmetry to constrain its matrix elements
- The effective Lagrangian is still the appropriate tool to be used if we want to derive systematically all symmetry relations

Effective Lagrangian with ESB

$$\mathcal{L}^{\text{QCD}} = \mathcal{L}_0^{\text{QCD}} - \bar{q}\mathcal{M}q$$

The symmetry breaking term

$$\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \text{h.c.}$$

becomes also chiral invariant if we impose that the quark mass matrix $\mathcal{M}$ transforms according to

$$\mathcal{M} \to \mathcal{M}' = V_R\mathcal{M}V_L^+$$

We can now proceed to construct a chiral invariant effective Lagrangian that includes explicitly the matrix $\mathcal{M}$:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

Effective Lagrangian with ESB

To first order in $\mathcal{M}$ there is only one chiral invariant term which one can construct:

$$\mathcal{L}_{\mathcal{M}}^{(1)} = \frac{F^2}{2}\left[B\langle \mathcal{M} U^+\rangle + B^*\langle \mathcal{M}^+ U\rangle\right]$$

Strong interactions respect parity $\Rightarrow$ B must be real:

$$\mathcal{L}_{\mathcal{M}}^{(1)} = \frac{F^2 B}{2}\langle \mathcal{M}\left(U + U^+\right)\rangle$$

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Before using this Lagrangian, pin down the constant B:

$$B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle \qquad\qquad M_\pi^2 = 2B\hat{m}$$

Leading order effective Lagrangian

The complete leading order effective Lagrangian of QCD reads:

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle\partial_\mu U^+\partial^\mu U\rangle + \langle 2B\mathcal{M}\left(U+U^+\right)\rangle\right]$$

F is the pion decay constant in the chiral limit

B is related to the $\bar{q}q$–condensate and to the pion mass

$$M_\pi^2 = 2B\hat{m} + O(\hat{m}^2)$$

$\pi\pi$ scattering to leading order

In the presence of quark masses the $\pi\pi$ scattering amplitude becomes

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}$$ Weinberg (66)

The two S–wave scattering lengths read

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045$$

External fields

QCD coupled to external fields ($\mathcal{M} \to s$):

$$\mathcal{L} = \mathcal{L}_{\mathrm{QCD}}^{(0)} + \bar{q}\gamma^{\mu}(v_{\mu} + \gamma_5 a_{\mu})q - \bar{q}(s - i\gamma_5 p)q$$

Generating functional of Green functions of quark bilinears

$$\langle 0|Te^{\mathrm{i}\int d^4x\mathcal{L}}|0\rangle = e^{\mathrm{i}Z[v,a,s,p]}$$

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Generating functional of Green functions of quark bilinears

$$\langle 0|T e^{\mathrm{i}\int d^4x\mathcal{L}}|0\rangle = e^{\mathrm{i}Z[v,a,s,p]} = \mathcal{N}^{-1}\int [dU] e^{\mathrm{i}\int d^4x\mathcal{L}_{\rm eff}}$$

External fields in $\mathcal{L}_{\rm eff} = \mathcal{L}_2(U,v,a,s,p) + \mathcal{L}_4(U,v,a,s,p) + \ldots$

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle D_\mu U^\dagger D^\mu U\rangle + \langle U\chi^\dagger + \chi U^\dagger\rangle\right]$$

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s+ip) \quad (r_\mu, l_\mu) = v_\mu \pm a_\mu$$

Gasser, Leutwyler (84)

The chiral Lagrangian to higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

$\mathcal{L}_2$	contains	$(2,2)$	constants	
$\mathcal{L}_4$	contains	$(7,10)$	constants	Gasser, Leutwyler (84)
$\mathcal{L}_6$	contains	$(53,90)$	constants	Bijnens, GC, Ecker (99)

The number in parentheses are for an $SU(N)$ theory with $N=(2,3)$

The $\mathcal{L}_4$ Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 \quad = \quad & L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
& + L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
& + L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
& + L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
& - iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\
& + L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_\mu U \quad &= \quad \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s + ip) \\
F_R^{\mu\nu} \quad &= \quad \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
r_\mu \quad &= \quad v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu
\end{aligned}
$$

Summary

- I have discussed Goldstone's theorem and some of its physical implications at low energy
- The effective Lagrangian for Goldstone bosons is a tool to derive systematically the consequences of the symmetry on their interactions – I have discussed the principles that allow one to construct it
- The effective Lagrangian is useful also in the presence of a (small) explicit symmetry breaking – I have shown how to construct it even in this case
- I have emphasized the importance of the external fields in formulating the effective Lagrangian method