Chiral perturbation theory

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Lecture II: Higher orders, applications

Why go beyond $O(p^2)$? Why loops?

$$\langle 0|T e^{i\int d^4x \mathcal{L}}|0\rangle = e^{iZ[v,a,s,p]} = \mathcal{N}^{-1}\int [dU] e^{i\int d^4x \mathcal{L}_{\rm eff}}$$

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- Why not? Chiral Symmetry forbids $O(p^0)$ interactions between pions, but allows all higher orders
- Unitarity requires that if an amplitude at order p^2 is purely real, at order p^4 its imaginary part is nonzero.
  Take the $\pi\pi$ scattering amplitude. The elastic unitarity relation for the partial waves t^I_ℓ of isospin I and angular momentum ℓ reads:

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- *The correct imaginary parts are generated automatically by loops*
- *The divergences occuring in the loops can be disposed of just like in a renormalizable field theory*

Effective quantum field theory

The method of effective quantum field theory provides a rigorous framework to compute Green functions that respect:

symmetry, analyticity, unitarity

The method yields a systematic expansion of the Green functions in powers of momenta and quark masses

In the following I will discuss in detail how this works when you consider loops:

- I will consider the finite, analytically nontrivial part of the loops and discuss in detail its physical meaning
- I will consider the divergent part of the loops and discuss how the renormalization program works

Scalar form factor of the pion

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u + \bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \ , \quad t = (p_1+p_2)^2 \ ,$$

At tree level:

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + O(p^4) \ ,$$

in agreement with the Feynman–Hellman theorem:
the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

$$\hat{m}\frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi|\hat{m}\bar{q}q|\pi\rangle = \Gamma(0) \ .$$

This matrix element is relevant for the decay $h \to \pi\pi$, which, for a light Higgs would have been the main decay mode

Donoghue, Gasser & Leutwyler (90)

Dispersion relation for $\Gamma(t)$

For $t \geq 4M_\pi^2$ $\mathrm{Im}\,\Gamma(t) \neq 0$. $\Gamma(t)$ is analytic everywhere else in the complex t plane, and obeys the following dispersion relation: $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\mathrm{Im}\,\bar{\Gamma}(t')}{t' - t}$$

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Unitarity implies $[\sigma(t) = \sqrt{1 - 4M_\pi^2/t}]$

$$\mathrm{Im}\,\bar{\Gamma}(t) = \sigma(t)\bar{\Gamma}(t)t_0^{0*}(t) = \bar{\Gamma}(t)e^{-i\delta_0^0} \sin\delta_0^0 = |\bar{\Gamma}(t)| \sin\delta_0^0$$

where t_0^0 is the S–wave, $I = 0$ $\pi\pi$ scattering amplitude

Strictly speaking, the above unitarity relation is valid only for $t \leq 16M_\pi^2$. To a good approximation, however, it holds up to the $K\bar{K}$ threshold

Dispersion relation and chiral counting

$$
\begin{aligned}
\bar{\Gamma}(t) &= 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{|\bar{\Gamma}(t')|\sin\delta_0^0(t')}{t'-t} \\
b &\sim O(1)\left(1 + O(M_\pi^2)\right) \\
\delta_0^0 &\sim O(p^2)\left(1 + O(p^2)\right)
\end{aligned}
$$

There are two $O(p^2)$ correction to $\bar{\Gamma}$:

1. $O(1)$ contribution to b;
2. the dispersive integral containing the $O(p^2)$ phase δ_0^0.

Notice that the latter is fixed by unitarity and analyticity

Are these respected by the one loop calculation?

Dispersion relation and one–loop CHPT

The full one–loop expression of $\bar{\Gamma}(t)$ reads as follows:

$$\bar{\Gamma}(t) = 1 + \frac{t}{16\pi^2 F_\pi^2}(\bar{\ell}_4 - 1) + \frac{2t - M_\pi^2}{2F_\pi^2}\bar{J}(t)$$

$$\bar{J}(t) = \frac{1}{16\pi^2}\left[\sigma(t)\ln\frac{\sigma(t)-1}{\sigma(t)+1} + 2\right]$$

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To prove that unitarity and analyticity are respected at this order is sufficient to add:

$$\delta_0^0(t) = \sigma(t)\frac{2t - M_\pi^2}{32\pi F_\pi^2} + O(p^4) \qquad \bar{J}(t) = \frac{t}{16\pi^2}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\sigma(t')}{t'-t}$$

Can you prove it?

Hints:

- Subtract $\bar{J}(t)$ once more

$$\bar{J}(t) = \frac{t}{96\pi^2 M_\pi^2} + \frac{t^2}{16\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t}$$

- Trick to pull out a linear term from the dispersive integral:

$$\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{t'\sigma(t')}{t' - t} = t \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t} + \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \sigma(t')$$

High-energy contributions

The dispersive integral goes up to $s' = \infty$, but the integrand is correct only at low energy!

$$
\begin{aligned}
\bar{\Gamma}(t)_{h.e.} &= \frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{dt'}{t'^2}\frac{|\bar{\Gamma}(t')|\sin\delta_0^0(t')}{t'-t} \\
&\sim \frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{dt'}{t'^2}|\bar{\Gamma}(t')|\sin\delta_0^0(t')\frac{1}{t'}\left(1+\frac{t}{t'}+\ldots\right) \\
&\sim ct^2+\mathcal{O}(t^3)
\end{aligned}
$$

The contributions from the high-energy region of the dispersive integral are formally of higher order – introducing a cut-off to remove them would only make the formulae more cumbersome

Renormalization at one loop

$$\int \frac{d^4l}{(2\pi)^4} \frac{\{p^2, p\cdot l, l^2\}}{(l^2-M^2)((p-l)^2-M^2)}\ , \qquad p = p_1 + p_2$$

$$\sim \underbrace{\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2-M^2)}}_{T(M^2)} + p^2 \underbrace{\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2-M^2)((p-l)^2-M^2)}}_{J(p^2)}$$

$$T(M^2) = a + bM^2 + \bar{T}(M^2) \qquad J(t) = J(0) + \bar{J}(t)$$

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$$T(M^2) = a + bM^2 + \bar{T}(M^2) \qquad J(t) = J(0) + \bar{J}(t)$$

$\bar{T}(M^2)$ and $\bar{J}(t)$ are finite

$$\Gamma(t) \sim M^2 \left[1 + \underbrace{bM^2 + tJ(0)}_{\text{divergent part}} + \bar{T}(M^2) + \bar{J}(t) \right]$$

Counterterms

$$\mathcal{L}_2 \quad \Rightarrow \quad \Gamma^{(2)}(t) \sim M^2$$

$$\mathcal{L}_4 \quad \Rightarrow \quad \Gamma^{(4)}(t) \sim \ell_3 M^4 + \ell_4 M^2 t$$

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To remove the divergences one only needs to properly define the couplings ($\ell_{3,4}$) in the lagrangian at order $O(p^4)$

Quote from Weinberg's book on QFT, vol. I: "(...) as long as we include every one of the infinite number of interactions allowed by symmetries, the so–called non–renormalizable theories are actually just as renormalizable as renormalizable theories."

Chiral logarithms

Scalar radius of the pion

$$\begin{aligned}\Gamma(t) &= \Gamma(0)\left[1 + \frac{1}{6}\langle r^2\rangle_S^\pi t + O(t^2)\right] \\ \langle r^2\rangle_S^\pi \sim J(0) &= \int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2-M^2)^2} \sim \ln\frac{M^2}{\Lambda^2}\end{aligned}$$

The integral is UV divergent, but also IR divergent if $M \to 0$:

$$\lim_{M^2\to 0}\langle r^2\rangle_S^\pi \sim \ln M^2 \quad ,$$

The extension of the cloud of pions surrounding a pion (or any other hadron) goes to infinity if pions become massless (Li and Pagels ’72)

Chiral symmetry and renormalization

To remove the divergent part in $\Gamma(t)$ we have to fix the divergent part of chiral–invariant operator of order $O(p^4)$

e.g. $\langle \partial_\mu U^\dagger \partial^\mu U \rangle \langle B\mathcal{M}(U + U^\dagger) \rangle \sim \ldots + M^2 \phi^2 \partial_\mu \phi^4 \partial^\mu \phi^6 + \ldots$

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Chiral symmetry implies that after calculating the divergent part of $\Gamma(s)$ I also <u>know</u> the divergent part of the $6\pi \to 6\pi$ scattering amplitude

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1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?

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1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?
2. Is there a tool that allows one to calculate the divergences keeping chiral invariance explicit in every step of the calculation?

Leutwyler's theorem

What is the most general way of constructing a chiral-invariant generating functional out of a path integral over the Goldstone boson degrees of freedom?

$$Z[v',a',s',p'] = Z[v,a,s,p] \Leftrightarrow \mathcal{L}_{\rm eff}[v',a',s',p'] = \mathcal{L}_{\rm eff}[v,a,s,p]?$$

For Lorentz–invariant theories in 4 dimensions, a path integral constructed with gauge–invariant lagrangians is a necessary and sufficient condition to obtain a gauge–invariant generating functional

The theorem also includes the case in which the symmetry is anomalous and the case in which the symmetry is explicitly broken

Leutwyler (94), d'Hoker & Weinberg (94)

Chiral invariant renormalization

- Gasser & Leutwyler (84) have shown that, using the background field method and heat kernel techniques, the calculation of the divergences at one loop – and the corresponding renormalization – can be performed in an explicitly chiral invariant manner
- The method has been extended and applied to two loops (Bijnens, GC & Ecker 98). After a long and tedious calculation, the divergent parts of all the counterterms at $\mathcal{O}(p^6)$ has been provided
- The renormalization of CHPT up to two loops has been performed explicitly: the calculation of any amplitude at two loops can be immediately checked by comparing the divergent part of Feynman diagrams to the divergent parts of the relevant counterterms

Chiral perturbation theory

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  symmetry, analyticity, unitarity
- The method yields a systematic expansion of the Green functions in powers of momenta and quark masses
- The method has been rigorously established and can be formulated as a set of calculational rules:

LO	tree level diagrams with $\mathcal{L}_2$
NLO	tree level diagrams with $\mathcal{L}_4$
	1-loop diagrams with $\mathcal{L}_2$
NNLO	tree level diagrams with $\mathcal{L}_6$
	2-loop diagrams with $\mathcal{L}_2$
	1-loop diagrams with one vertex from $\mathcal{L}_4$

$\pi\pi$ scattering at NLO

$$
\begin{aligned}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0 + 2a_2^2)\right. \\
&- \left.\frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{\ell}_3 - 353)\right] = 0.16 \cdot 1.25 = 0.20 \\
2a_0^0 - 5a_0^2 &= \frac{3M_\pi^2}{4\pi F_\pi^2}\left[1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{41M_\pi^2}{192\pi^2 F_\pi^2}\right] = 0.624
\end{aligned}
$$

Gasser and Leutwyler (83)

Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

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The reason for the rather large correction in a_0^0 is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{9}{2}\ell_\chi + \ldots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{3}{2}\ell_\chi + \ldots\right]$$

$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}\ln\frac{\mu^2}{M_\pi^2}$$

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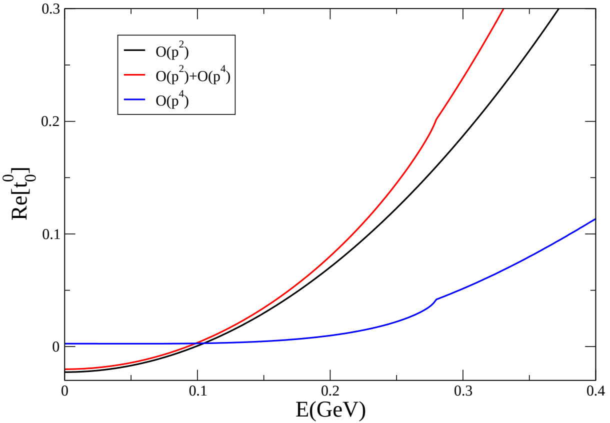

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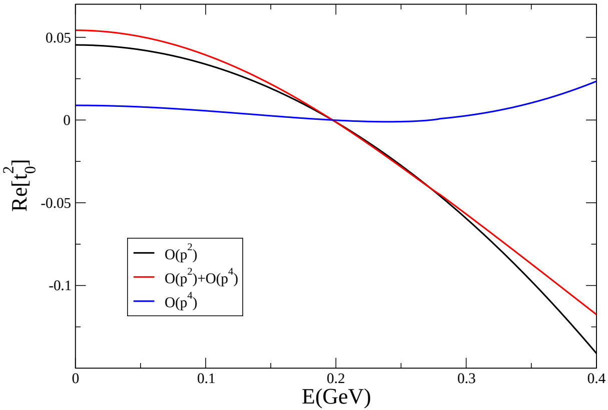

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How large are yet higher orders?
Is it at all possible to make a precise prediction?

Roy equations

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Input: imaginary parts above 0.8 GeV
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Note: if a_0^0, a_0^2 are chosen within the universal band
the solution exists and is unique

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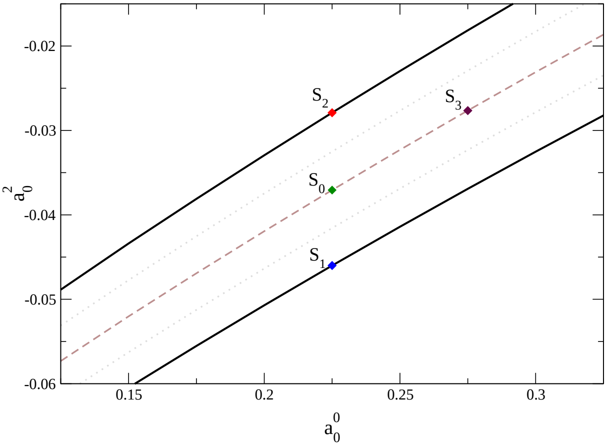
Numerical solutions of the Roy equations
Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)

Numerical solutions



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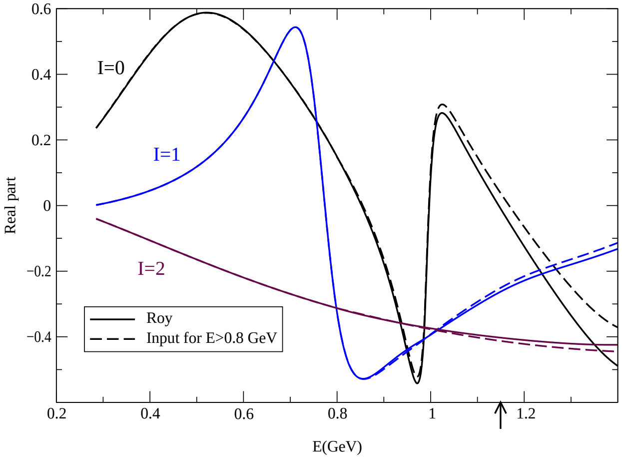

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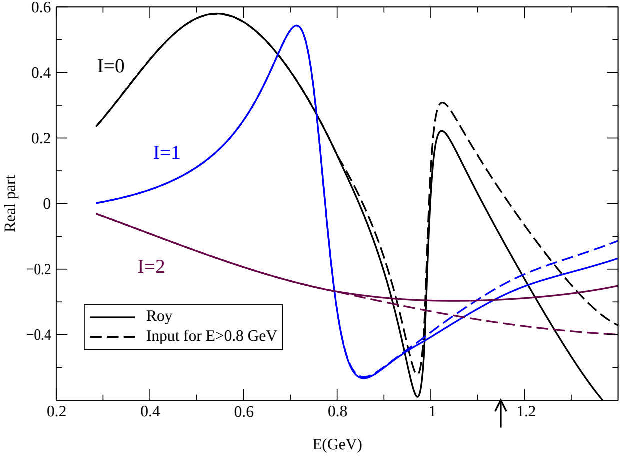

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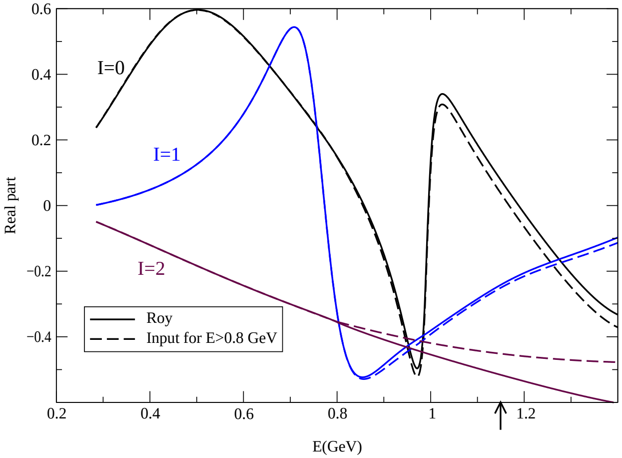

Combining CHPT and dispersive methods

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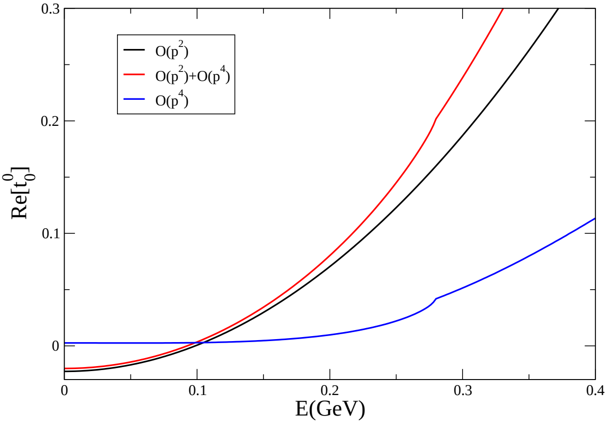
In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold (a_0^0, a_0^2) is not mandatory

The freedom in the choice of the subtraction point can be exploited to use the chiral expansion where it converges best, *i.e.* below threshold

Combining CHPT and dispersive methods



Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

$$
\begin{array}{rcccc}
a_0^0 & = & 0.159 \to & 0.200 \to & 0.216 \\
10 \cdot a_0^2 & = & -0.454 \to & -0.445 \to & -0.445 \\
 & & p^2 & p^4 & p^6
\end{array}
$$

GC, Gasser and Leutwyler (01)

Combining CHPT and dispersive methods

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CHPT below threshold + Roy

$$\begin{array}{rcccc} a_0^0 & = & 0.197 \to & 0.2195 \to & 0.220 \\ 10 \cdot a_0^2 & = & -0.402 \to & -0.446 & \to -0.444 \end{array}$$

GC, Gasser and Leutwyler (01)

Low-energy theorem for $\pi\pi$ scattering

$$\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) \equiv A(s,t,u) = \text{isospin invariant amplitude}$$

Low energy theorem: $$A(s,t,u) = \frac{s - M^2}{F^2} + \mathcal{O}(p^4)$$ Weinberg 1966

$$M^2 = B(m_u + m_d) \qquad M_\pi^2 = M^2 + O(m_q^2), \quad F_\pi = F + O(m_q)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0} = 3A(s,t,u) + A(t,s,u) + A(u,t,s) \ \Rightarrow \ T^{I=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection (I=0)

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \qquad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

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All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=2} = A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=2} = \frac{-s + 2M_\pi^2}{F_\pi^2}$$

S wave projection (I=2)

$$t_0^2(s) = \frac{2M_\pi^2 - s}{32\pi F_\pi^2} \qquad a_0^2 = t_0^2(4M_\pi^2) = \frac{-M_\pi^2}{16\pi F_\pi^2} = -0.045$$

Chiral predictions for a_0^0 and a_0^2

Quark mass dependence of M_π and F_π:

$$M_\pi^2 = M^2 \left(1 - \frac{M^2}{32\pi^2 F^2}\bar{\ell}_3 + O(p^4)\right)$$

$$M^2 \equiv -\frac{m_u + m_d}{F^2}\langle 0|\bar{q}q|0\rangle$$

Gell-Mann, Oakes, Renner (68)

$$F_\pi = F \left(1 + \frac{M^2}{16\pi^2 F^2}\bar{\ell}_4 + O(p^4)\right)$$

Phenomenological determinations (indirect):

$$\bar{\ell}_3 = 2.9 \pm 2.4$$

Gasser & Leutwyler (84)

$$\bar{\ell}_4 = 4.4 \pm 0.2$$

GC, Gasser & Leutwyler (01)

Lattice calculations determine these constants directly

Chiral predictions for a_0^0 and a_0^2

χPT calculations at NLO (Gasser & Leutwyler 84)

and at NNLO (Bijnens, GC, Ecker, Gasser & Sainio, 95)

Prediction obtained matching $O(p^6)$ χPT to Roy equations (disp. relation): GC, Gasser & Leutwyler (01)

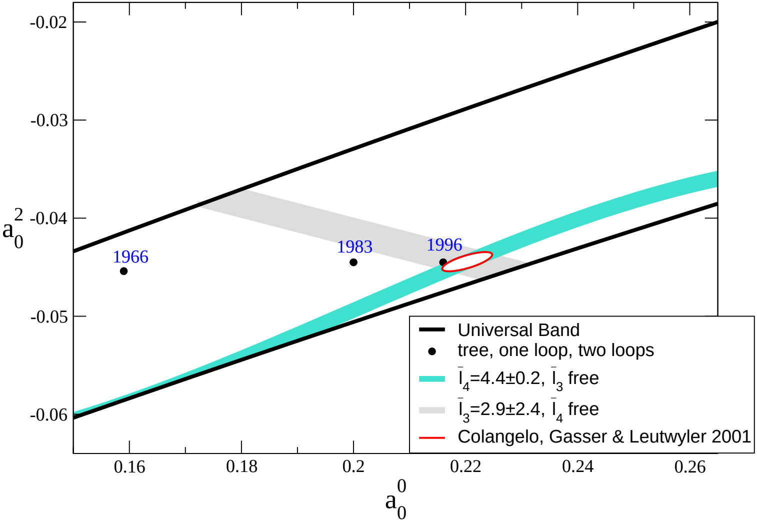
$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 + 0.009\Delta\ell_4 - 0.002\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.01\Delta\ell_4 - 0.004\Delta\ell_3 \end{aligned}$$

where $\bar{\ell}_4 = 4.4 + \Delta\ell_4 \qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3$

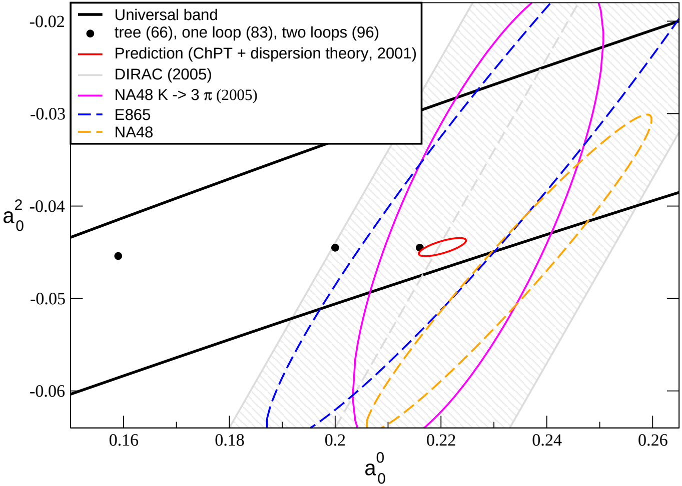
Adding errors in quadrature $[\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.005 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.01 \\ a_0^0 - a_0^2 &= 0.265 \pm 0.004 \end{aligned}$$

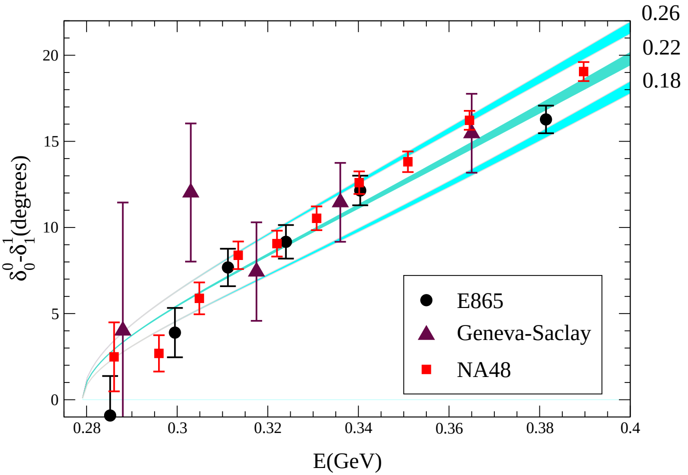
Chiral predictions for a_0^0 and a_0^2



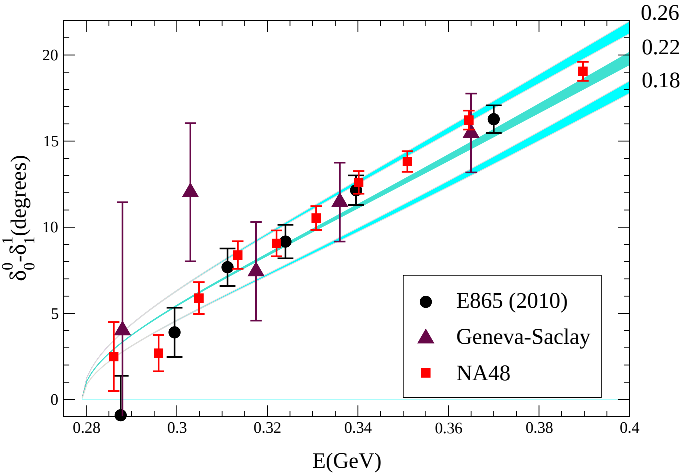
Experimental tests



Experimental tests

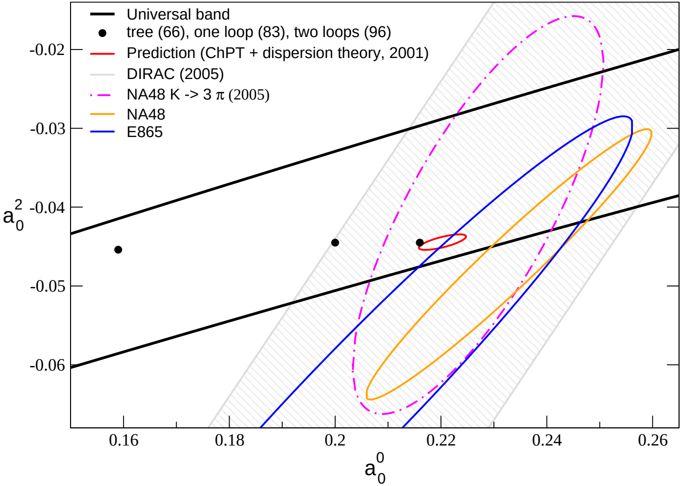


Experimental tests



Recent update: E865 corrected their data

Experimental tests



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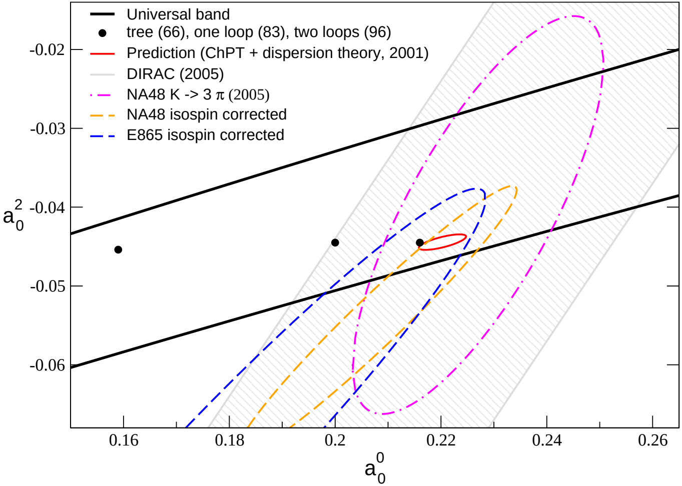
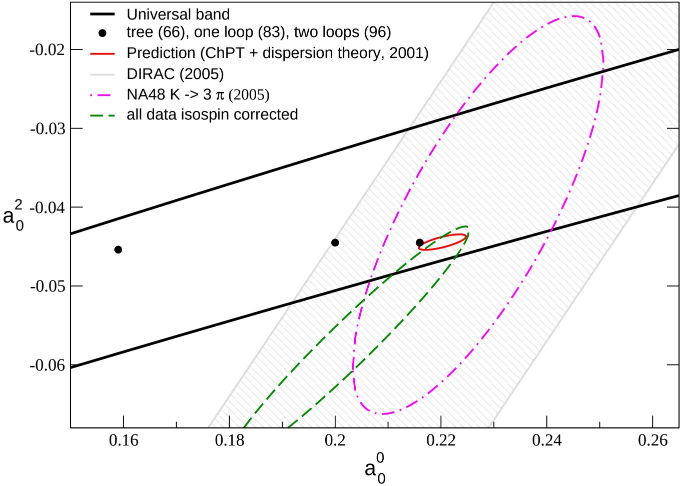


isospin breaking corrections recently calculated for K_{e4} are essential at this level of precision GC, Gasser, Rusetsky (09)

Experimental tests



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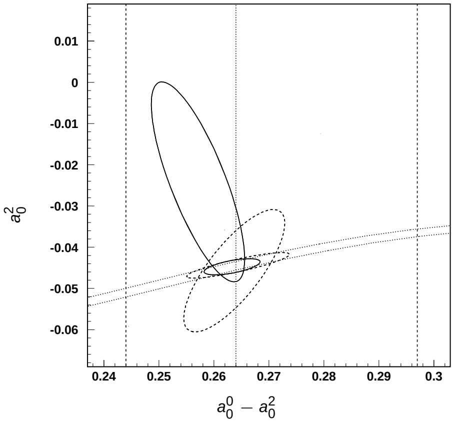


Figure from NA48/2 Eur.Phys.J.C64:589,2009

Summary

- The finite, analytically nontrivial part of the one loop integrals automatically generates the correct imaginary parts, as required by unitarity.
- Effective quantum field theory is a systematic method to generate a perturbative solution of dispersion relations
- The UV divergences encountered in loop integrals can be removed according to standard renormalization methods
- Some loop integrals have also an IR singular behaviour which has a very clear physical meaning, and again shows the necessity of taking loop effects into account
- Leutwyler's theorem: doing a path integral over an effective Lagrangian is the most general way to construct an invariant generating functional
- As an example of the accuracy one can reach with this method I have discussed:
  - the calculation of $\pi\pi$ S-wave scattering lengths

Experiments on $\pi\pi$ scattering

$\pi\pi$ scattering at low energy can be measured in

- pionium decay
  the decay width is proportional to $(a_0 - a_2)^2$ DIRAC
- $K^\pm \to \pi^0\pi^0\pi^\pm$
  a cusp at $M_{\pi^0\pi^0} = 2M_{\pi^+}$ is proportional to $(a_0 - a_2)^2$ NA48
- K_{e4} decays
  $\delta_0^0 - \delta_1^1$ between threshold and $s \sim M_K^2$ can be extracted by measuring certain angular distributions E865, NA48

All measurements have reached a remarkably high accuracy
$\Rightarrow$ necessary to take isospin breaking corrections into account

Pionium lifetime measurement

Master formula:

Deser, Goldberger, Baumann, Thirring 54

Gall, Gasser, Lyubovitskij, Rusetsky 99, 01

cf. also Sazdjian, and Gashi, Oades, Rasche, Woolcock

$$\Gamma_{2\pi^0} = \frac{2}{9}\alpha^3 p^\star (a_0^0 - a_0^2)^2 (1 + \delta_1) + o(\delta^{9/2})$$

with $p^\star$ the modulus of the π^0 3-momentum and

$\delta_1 = (5.8 \pm 1.2) \times 10^{-2}$ isospin breaking corrections

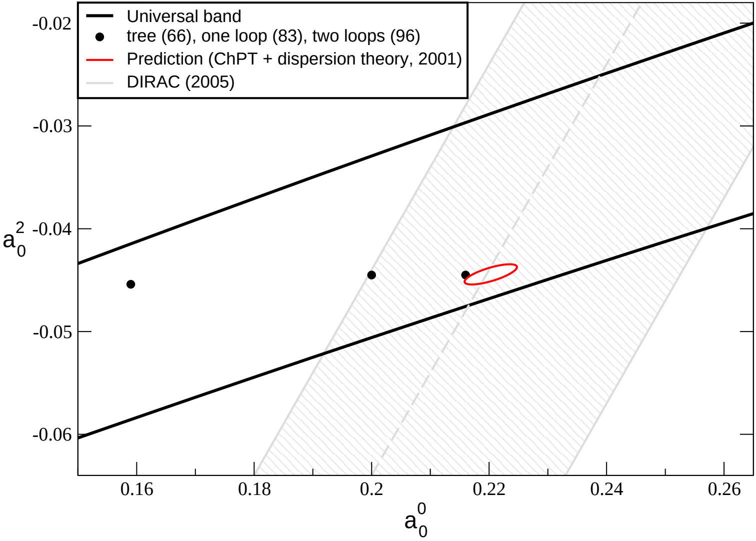
DIRAC (05):

$$\tau = 2.91 \,{}^{+0.45}_{-0.38}\,(\mathrm{stat})\,{}^{+0.19}_{-0.49}\,(\mathrm{syst}) \times 10^{-15}\mathrm{s}$$

which translates to

$$a_0^0 - a_0^2 = 0.264 \,{}^{+0.033}_{-0.020}$$

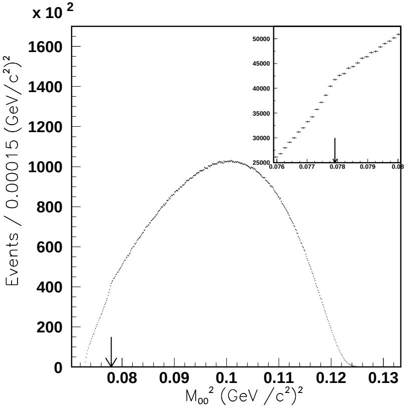
Pionium lifetime measurement



Cusp in $K^{\pm} \to \pi^0\pi^0\pi^{\pm}$

NA48: high-statistics measurement of $K^{\pm} \to \pi^0\pi^0\pi^{\pm}$.

Cusp in the $\pi^0\pi^0$ spectrum @ $\pi^+\pi^-$ threshold clearly observed



Cusp in $K^{\pm} \to \pi^0\pi^0\pi^{\pm}$

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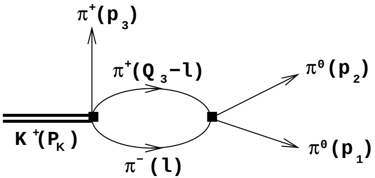
Cusp in the $\pi^0\pi^0$ spectrum @ $\pi^+\pi^-$ threshold clearly observed

Theoretical interpretation: Cabibbo 04

Early history: Wigner (48), Budini, Fonda (61),... , Bernstein et al. (97), Meißner et al. (97)

Two-loop treatment: Cabibbo, Isidori 05, Gamiz, Prades, Scimemi 06, GC, Gasser, Kubis, Rusetsky 06

a $\pi^+\pi^-$ intermediate state is responsible for the cusp



Cusp in $K^\pm \to \pi^0\pi^0\pi^\pm$: results

NA48 published the first results of their analysis in 2006:

$$\begin{aligned} a_0^0 - a_0^2 &= 0.268 \pm 0.010(\text{stat}) \pm 0.004(\text{syst}) \pm 0.013(\text{ext}) \\ a_0^2 &= -0.041 \pm 0.022(\text{stat}) \pm 0.014(\text{syst}) \end{aligned}$$

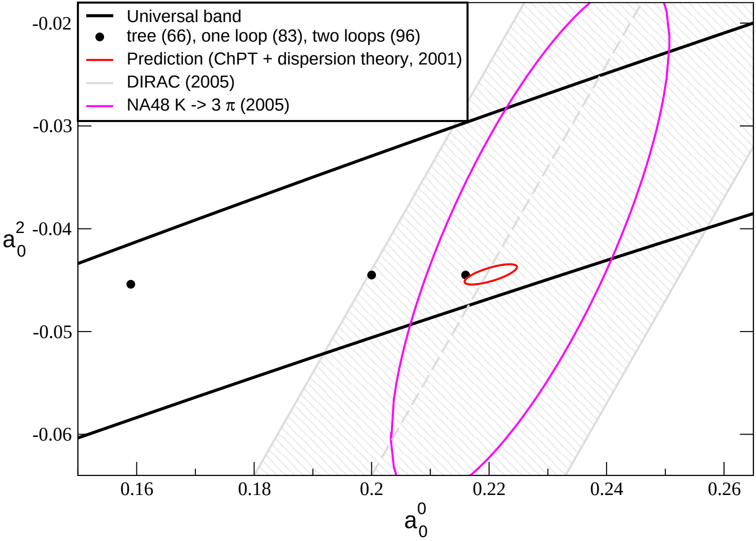
Correlation coeff. $= -0.858$

Constraining a_0^0 and a_0^2 to respect the low-energy theorem which relates them to the scalar radius they get

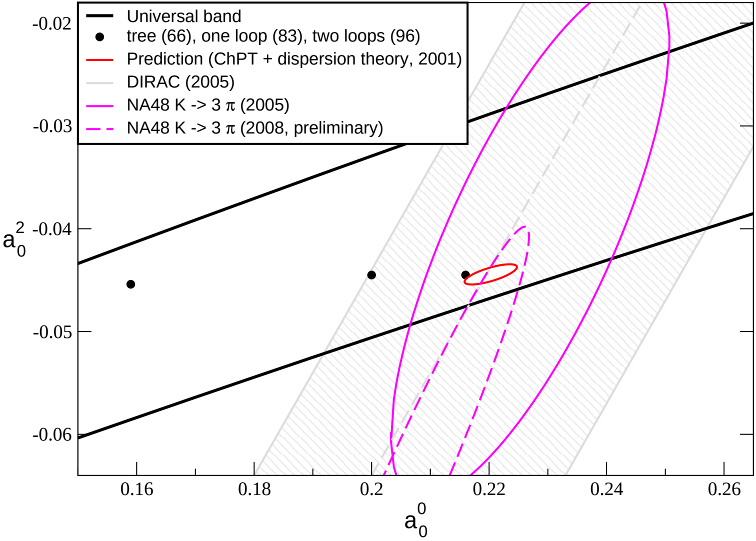
$$a_0^0 = 0.220 \pm 0.006(\text{stat}) \pm 0.004(\text{syst}) \pm 0.011(\text{ext})$$

More statistics already analyzed – results not yet published

Cusp in $K^\pm \to \pi^0\pi^0\pi^\pm$: results



Cusp in $K^\pm \to \pi^0\pi^0\pi^\pm$: results



Extracting the $\pi\pi$ phases from K_{e4}

Isospin limit

$$\begin{aligned} V_\mu - A_\mu &\equiv \langle \pi(p_1)\pi(p_2)|\bar{s}\gamma_\mu(1-\gamma_5)u|K(p)\rangle \\ A_\mu &= \frac{-i}{M_K}\left[(p_1+p_2)_\mu F + (p_1-p_2)_\mu G + L_\mu R\right] \end{aligned}$$

$$F = f_s e^{i\delta_0^0} + f_p e^{i\delta_1^1}\cos\theta + D\text{-wave} \qquad G = g_p e^{i\delta_1^1} + D\text{-wave}$$

the phases δ_ℓ^I are those of $\pi\pi$ scattering (Watson's theorem)

Measuring certain angular distributions one can extract very cleanly the phase difference:

$$\delta_0^0(s) - \delta_1^1(s) \quad 2M_\pi < \sqrt{s} < M_K - m_l$$

Extracting the $\pi\pi$ phases from K_{e4}

Isospin breaking effects

radiative corrections and $m_u - m_d \neq 0$ break isospin

NA48 corrects two effects

- the effect of the Coulomb attraction between two slow pions is removed by applying the Gamow factor
- the effect of real emission of photons is taken into account with the help of the program PHOTOS Was et al.

The mass splitting between the charged and neutral pions and the difference $m_d - m_u$, however, are not corrected for

We (GC, Gasser and Rusetsky) proceed by assuming that

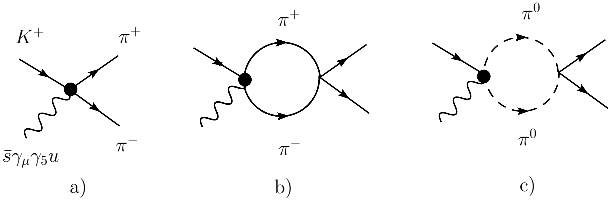
Full isospin breaking eff. = Coulomb factor × PHOTOS × mass effects

and discuss how the latter affect the measured phases

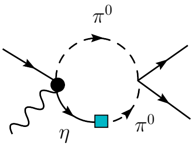
Isospin breaking effects in K_{e4}

GC, Gasser Rusetsky

Tree and one-loop diagrams in K_{e4}:



The different thresholds between b) and c) affect the phases



The $\pi^0 - \eta$ mixing $\propto (m_u - m_d)$ also modifies the phases

Isospin breaking effects in K_{e4}

GC, Gasser Rusetsky

Both effects add up to [$\delta =$measured phase]

$$\delta = \frac{1}{32\pi F^2}\left\{(4\Delta_\pi + s)\sigma + (s - M_{\pi^0}^2)\left(1 + \frac{3}{2R}\right)\sigma_0\right\} - \delta_1^1 + O(p^4)$$

where $\Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2, \quad \sigma = \sqrt{1 - \frac{4M_\pi^2}{s}}, \quad R = \frac{m_s - \hat{m}}{m_d - m_u}$

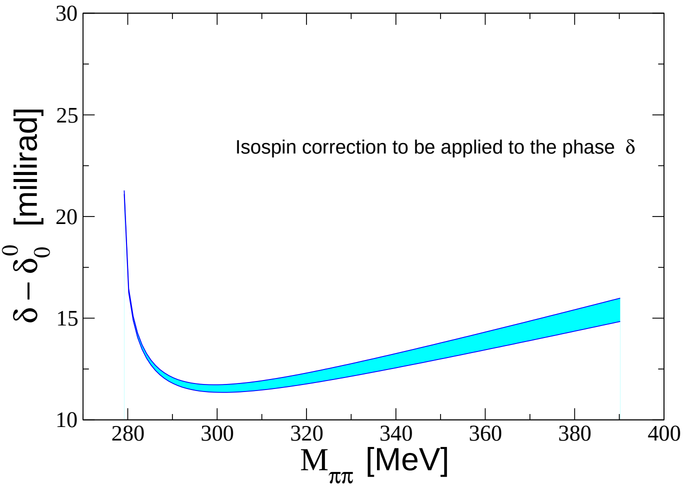
For the numerical analysis we use $R = 37 \pm 4$

Related work also by

- Cuplov and Nehme, Nehme [CHPT calculation at $O(p^4)$ including isospin breking effects]
- Gevorkyan et al. 07 [Coulomb and real photon corrections $\Leftrightarrow$ double counting problems – approach not sound]
- Descotes-Genon and Knecht (in preparation)

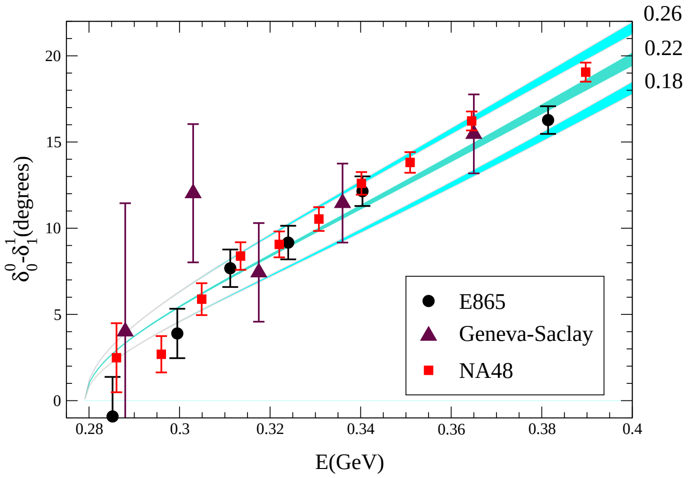
Isospin breaking effects in K_{e4}

GC, Gasser Rusetsky



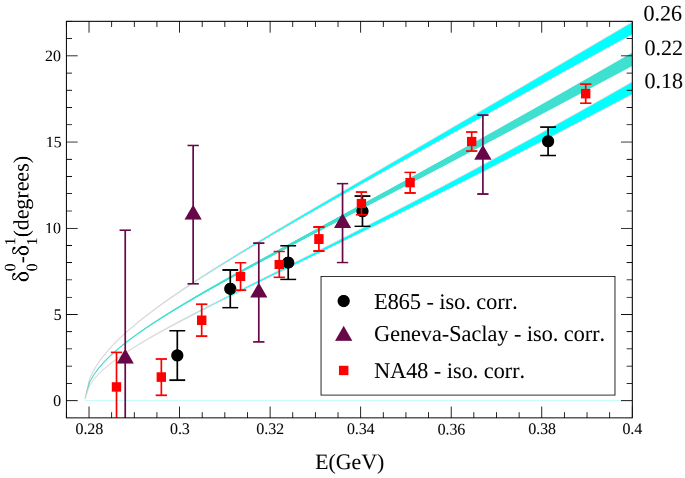
Isospin breaking effects in K_{e4}

GC, Gasser Rusetsky

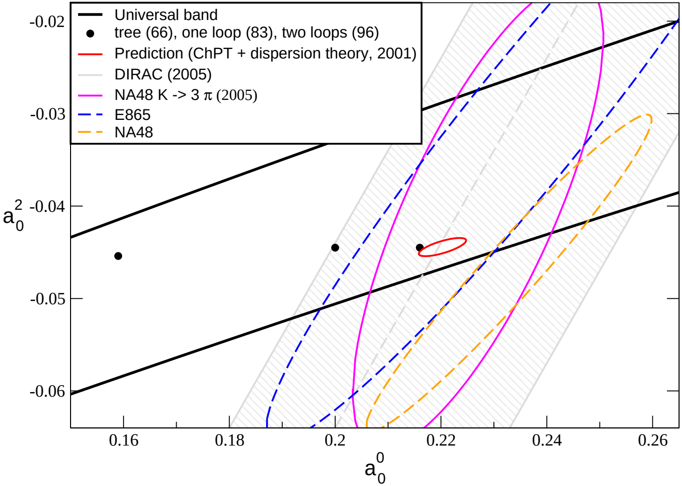


Isospin breaking effects in K_{e4}

GC, Gasser Rusetsky



Effect of isospin breaking on the scattering lengths



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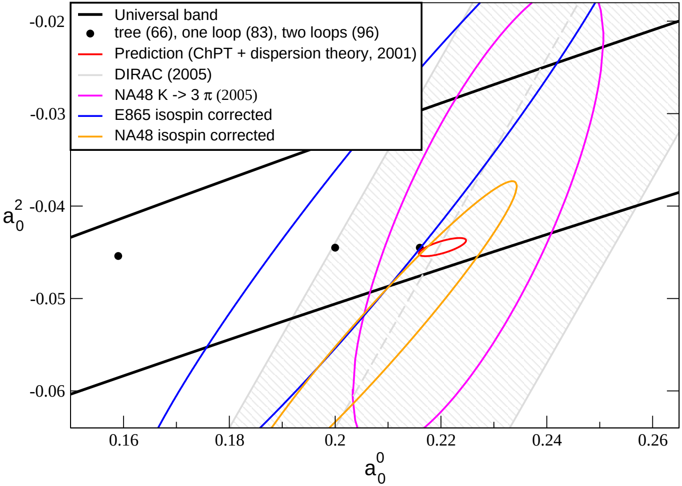

GC, Gasser Rusetsky

Effect of isospin breaking on the scattering lengths

Fits assuming the LET ($a_0^2(a_0^0)$)

Before...

$$a_0^0 = \begin{cases} 0.243 \pm 0.037 & \chi^2 = 2.2 \quad \text{Geneva-Saclay} \quad [\ 5 \text{ data}] \\ 0.218 \pm 0.013 & \chi^2 = 5.7 \quad \text{E865} \quad [\ 6 \text{ data}] \\ 0.245 \pm 0.007 & \chi^2 = 9.6 \quad \text{NA48} \quad [10 \text{ data}] \end{cases}$$

and after applying isospin breaking corrections

$$a_0^0 = \begin{cases} 0.222 \pm 0.040 & \chi^2 = 2.1 \quad \text{Geneva-Saclay} \\ 0.195 \pm 0.013 & \chi^2 = 6.6 \quad \text{E865} \\ 0.223 \pm 0.007 & \chi^2 = 11.5 \quad \text{NA48} \end{cases}$$

Averaging the latter three independent determinations yields

$$a_0^0 = 0.217 \pm 0.008_{\rm exp} \pm 0.006_{\rm th} \qquad [S = 1.3]$$

GC, Gasser Rusetsky, 08

new results with more statistics presented at conferences but not yet published