Chiral perturbation theory

Gilberto Colangelo

UNIVERSITÄT BERN

Albert Einstein Center for Fundamental Physics

STRONGnet summer school, ZIF Bielefeld
14–25 June 2011

Lecture III: applications in phenomenology and lattice

$\pi\pi$ scattering beyond LO
- Experimental tests

The σ resonance

Finite volume effects
- Asymptotic formulae
- Resummation of higher exponentials
- The pion mass to two loops

Summary

$\pi\pi$ scattering at NLO

$$
\begin{aligned}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0+2a_2^2)\right. \\
&- \left.\frac{M_\pi^2}{672\pi^2F_\pi^2}(15\bar{\ell}_3-353)\right] = 0.16\cdot 1.25 = 0.20 \\
2a_0^0-5a_0^2 &= \frac{3M_\pi^2}{4\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{41M_\pi^2}{192\pi^2F_\pi^2}\right] = 0.624
\end{aligned}
$$

Gasser and Leutwyler (83)

Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

Gasser and Leutwyler (84)

Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

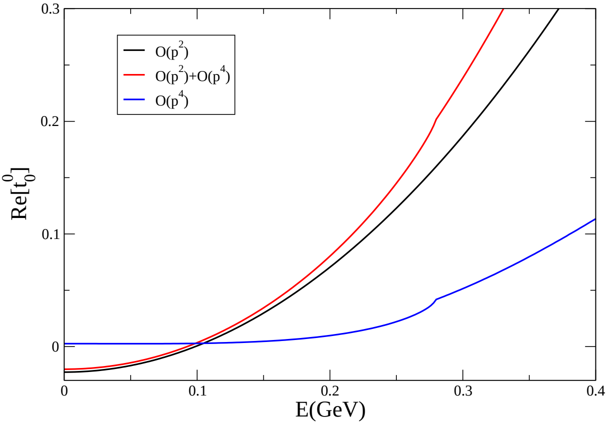
The reason for the rather large correction in a_0^0 is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{9}{2}\ell_\chi + \dots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{3}{2}\ell_\chi + \dots\right]$$

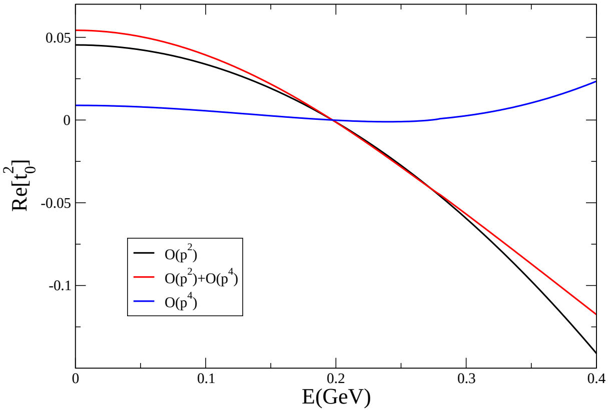
$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}\ln\frac{\mu^2}{M_\pi^2}$$

Gasser and Leutwyler (84)

Higher orders



Higher orders



Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

The reason for the rather large correction in a_0^0 is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{9}{2}\ell_\chi + \dots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{3}{2}\ell_\chi + \dots\right]$$

$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}\ln\frac{\mu^2}{M_\pi^2}$$

Gasser and Leutwyler (84)

How large are yet higher orders?
Is it at all possible to make a precise prediction?

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

Input: imaginary parts above 0.8 GeV
two subtraction constants, *e.g.* a_0^0 and a_0^2

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

Input: imaginary parts above 0.8 GeV
two subtraction constants, *e.g.* a_0^0 and a_0^2

Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV

Note: if a_0^0, a_0^2 are chosen within the universal band
the solution exists and is unique

Roy equations

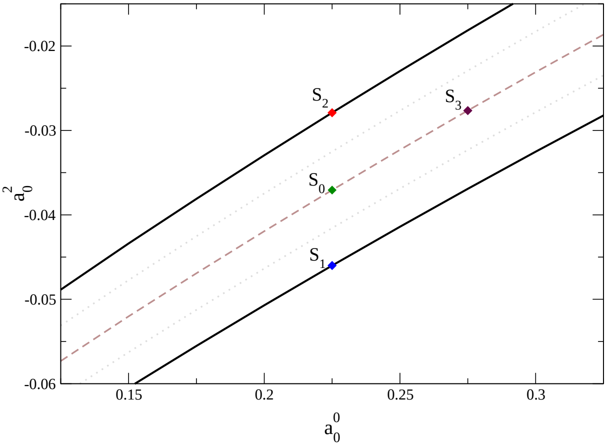
Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

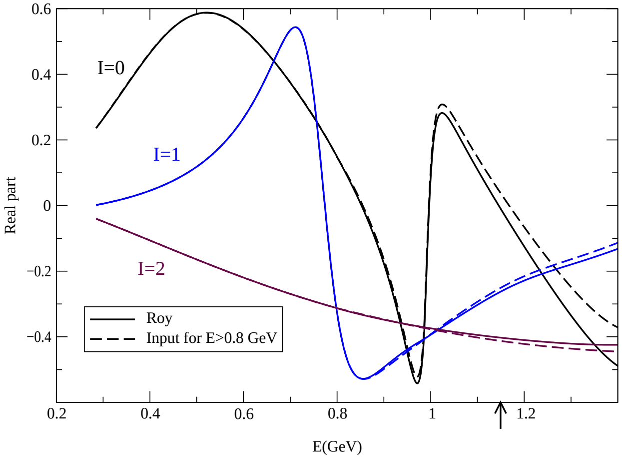
Input: imaginary parts above 0.8 GeV
two subtraction constants, *e.g.* a_0^0 and a_0^2
Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV
Note: if a_0^0, a_0^2 are chosen within the universal band
the solution exists and is unique

Numerical solutions of the Roy equations
Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)

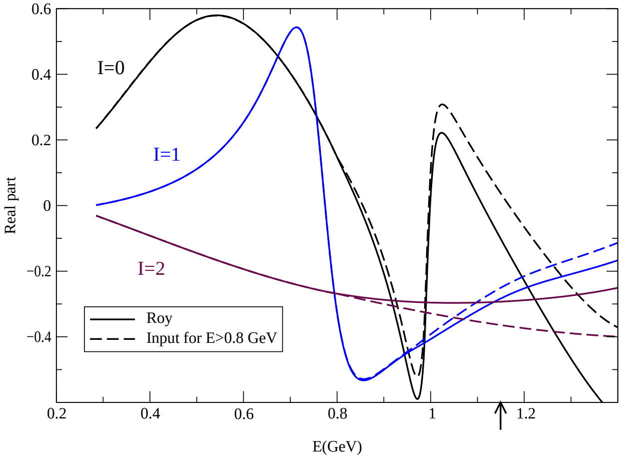
Numerical solutions



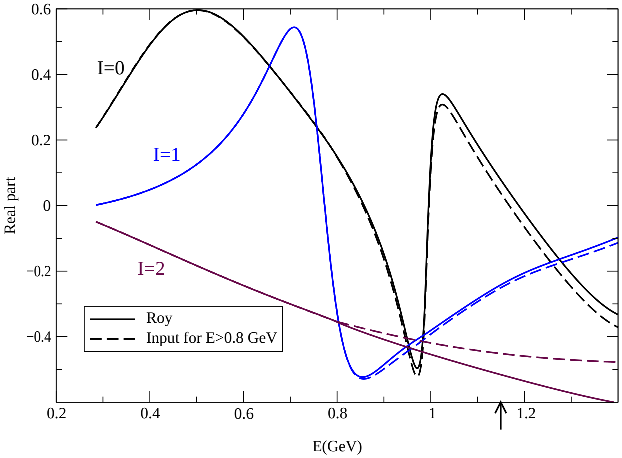
Numerical solutions



Numerical solutions



Numerical solutions



Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold (a_0^0, a_0^2) is not mandatory

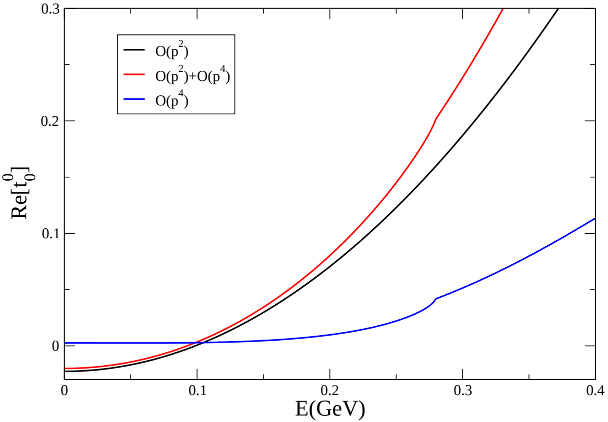
Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold (a_0^0, a_0^2) is not mandatory

The freedom in the choice of the subtraction point
can be exploited to use the chiral expansion
where it converges best, *i.e.* below threshold

Combining CHPT and dispersive methods



Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

$$\begin{aligned} a_0^0 &= \quad 0.159 \rightarrow \quad 0.200 \rightarrow \quad 0.216 \\ 10 \cdot a_0^2 &= -0.454 \rightarrow -0.445 \rightarrow -0.445 \\ & \qquad\quad p^2 \qquad\qquad p^4 \qquad\qquad p^6 \end{aligned}$$

GC, Gasser and Leutwyler (01)

Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

$$\begin{aligned} a_0^0 &= 0.159 \to 0.200 \to 0.216 \\ 10 \cdot a_0^2 &= -0.454 \to -0.445 \to -0.445 \\ & \quad\; p^2 \qquad\quad p^4 \qquad\quad p^6 \end{aligned}$$

CHPT below threshold + Roy

$$\begin{aligned} a_0^0 &= 0.197 \to 0.2195 \to 0.220 \\ 10 \cdot a_0^2 &= -0.402 \to -0.446 \to -0.444 \end{aligned}$$

GC, Gasser and Leutwyler (01)

Low-energy theorem for $\pi\pi$ scattering

$$\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) \equiv A(s,t,u) = \text{isospin invariant amplitude}$$

Low energy theorem: $$A(s,t,u) = \frac{s - M^2}{F^2} + \mathcal{O}(p^4)$$ Weinberg 1966

$$M^2 = B(m_u + m_d) \qquad M_\pi^2 = M^2 + O(m_q^2), \quad F_\pi = F + O(m_q)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0} = 3A(s,t,u) + A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection (I=0)

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \qquad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

Low-energy theorem for $\pi\pi$ scattering

$$\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) \equiv A(s,t,u) = \text{isospin invariant amplitude}$$

Low energy theorem: $$A(s,t,u) = \frac{s - M^2}{F^2} + \mathcal{O}(p^4)$$ Weinberg 1966

$$M^2 = B(m_u + m_d) \qquad M_\pi^2 = M^2 + O(m_q^2), \;\; F_\pi = F + O(m_q)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=2} = A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=2} = \frac{-s + 2M_\pi^2}{F_\pi^2}$$

S wave projection (I=2)

$$t_0^2(s) = \frac{2M_\pi^2 - s}{32\pi F_\pi^2} \qquad a_0^2 = t_0^2(4M_\pi^2) = \frac{-M_\pi^2}{16\pi F_\pi^2} = -0.045$$

Chiral predictions for a_0^0 and a_0^2

Quark mass dependence of M_π and F_π:

$$M_\pi^2 = M^2 \left(1 - \frac{M^2}{32\pi^2 F^2}\bar{\ell}_3 + O(p^4)\right)$$

$$M^2 \equiv -\frac{m_u + m_d}{F^2}\langle 0|\bar{q}q|0\rangle$$ Gell-Mann, Oakes, Renner (68)

$$F_\pi = F \left(1 + \frac{M^2}{16\pi^2 F^2}\bar{\ell}_4 + O(p^4)\right)$$

Phenomenological determinations (indirect):

$$\bar{\ell}_3 = 2.9 \pm 2.4$$ Gasser & Leutwyler (84)

$$\bar{\ell}_4 = 4.4 \pm 0.2$$ GC, Gasser & Leutwyler (01)

Lattice calculations determine these constants directly

Chiral predictions for a_0^0 and a_0^2

χPT calculations at NLO (Gasser & Leutwyler 84)

and at NNLO (Bijnens, GC, Ecker, Gasser & Sainio, 95)

Prediction obtained matching $O(p^6)$ χPT to Roy equations (disp. relation): GC, Gasser & Leutwyler (01)

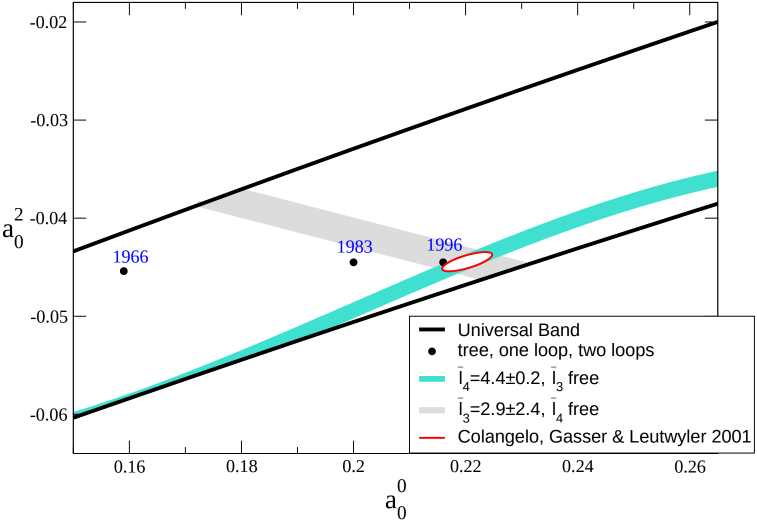
$$
\begin{aligned}
a_0^0 &= \quad 0.220 \pm 0.001 + 0.009\Delta\ell_4 - 0.002\Delta\ell_3 \\
10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.01\Delta\ell_4 - 0.004\Delta\ell_3
\end{aligned}
$$

where $\quad \bar{\ell}_4 = 4.4 + \Delta\ell_4 \qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3$

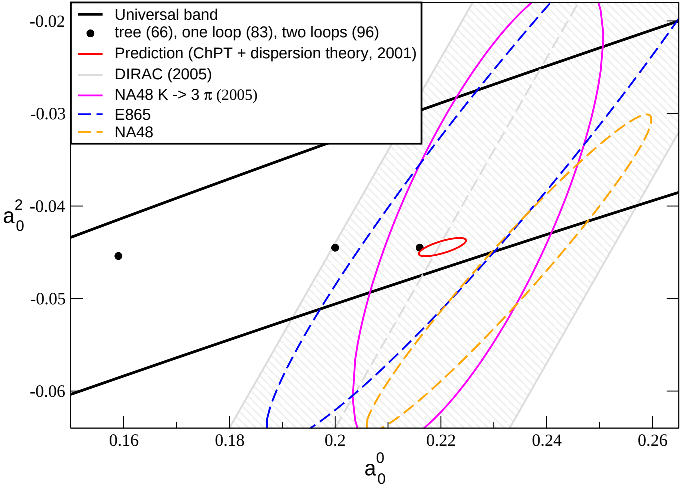
Adding errors in quadrature $[\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

$$
\begin{aligned}
a_0^0 &= \quad 0.220 \pm 0.005 \\
10 \cdot a_0^2 &= -0.444 \pm 0.01 \\
a_0^0 - a_0^2 &= \quad 0.265 \pm 0.004
\end{aligned}
$$

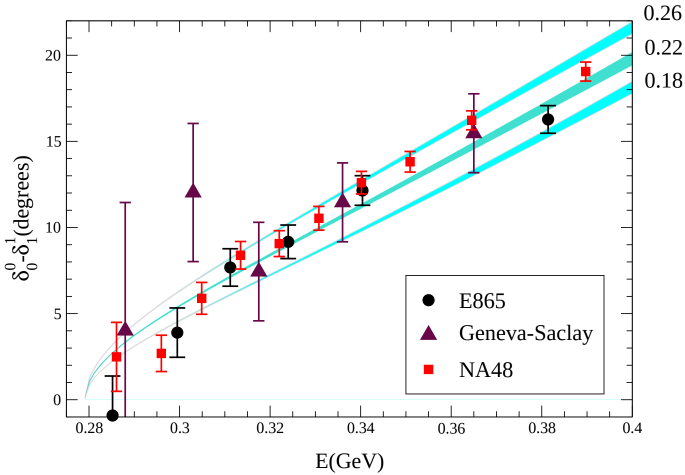
Chiral predictions for a_0^0 and a_0^2



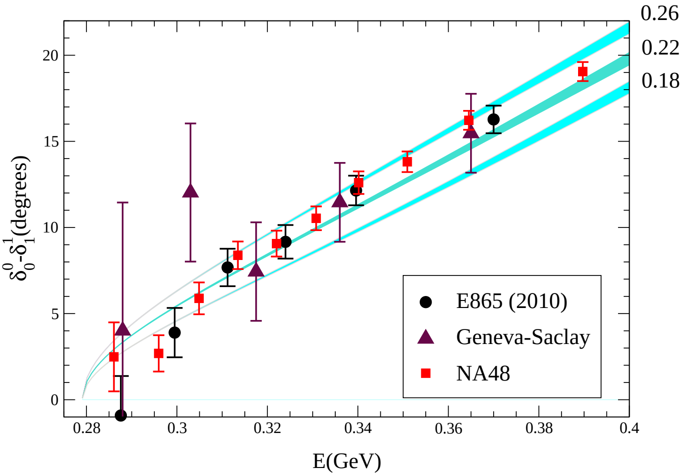
Experimental tests



Experimental tests

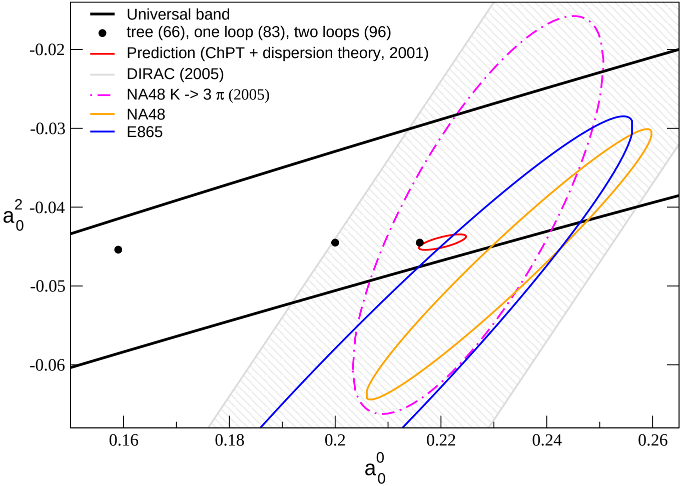


Experimental tests



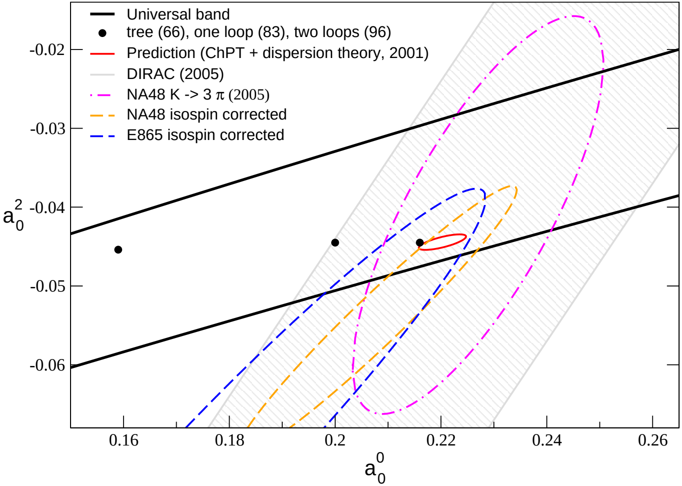
Recent update: E865 corrected their data

Experimental tests



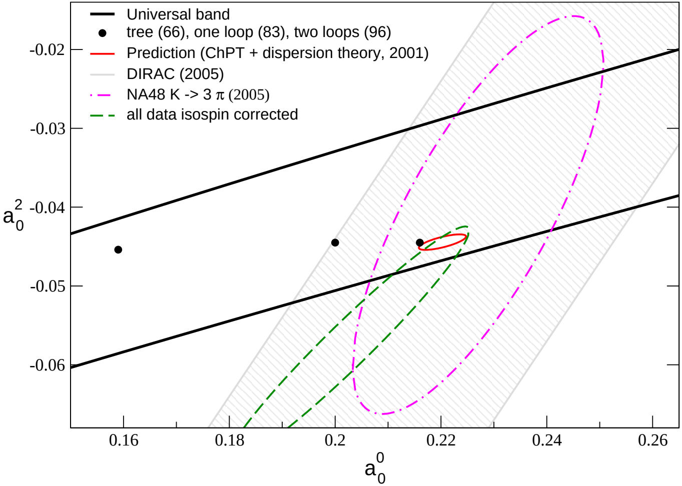
Recent update: E865 corrected their data

Experimental tests



isospin breaking corrections recently calculated for K_{e4} are essential at this level of precision GC, Gasser, Rusetsky (09)

Experimental tests



isospin breaking corrections recently calculated for K_{e4} are essential at this level of precision GC, Gasser, Rusetsky (09)

Experimental tests

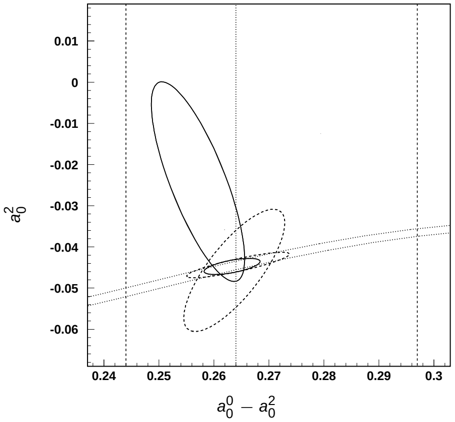
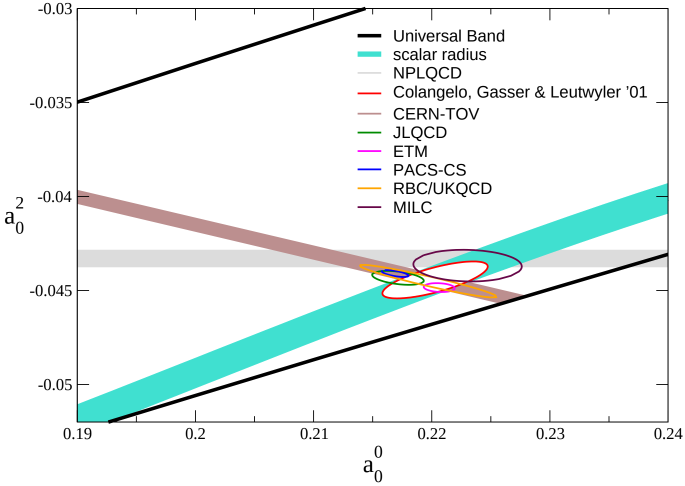


Figure from NA48/2 Eur.Phys.J.C64:589,2009

Lattice input for $\bar{\ell}_3$ and $\bar{\ell}_4$



Outline

The sigma resonance

The two S-wave scattering lengths are the *essential parameters* at low energy

For example, their knowledge fixes the σ pole position to a *remarkable* level of precision

$$M_\sigma = 441 \; ^{+16}_{-8} \; \text{MeV} \, , \quad \Gamma_\sigma = 544 \; ^{+18}_{-25} \; \text{MeV}$$

I. Caprini, GC, H. Leutwyler, PRL 06

The sigma resonance

: S. Eidelman *et al.* (Particle Data Group), Phys. Lett. B **592**, 1 (2004) and 2005 partial update for edition 2006 (URL: http://pdg.lbl.gov)

$f_0(600)$ or σ

$I^G(J^{PC}) = 0^+(0^{++})$

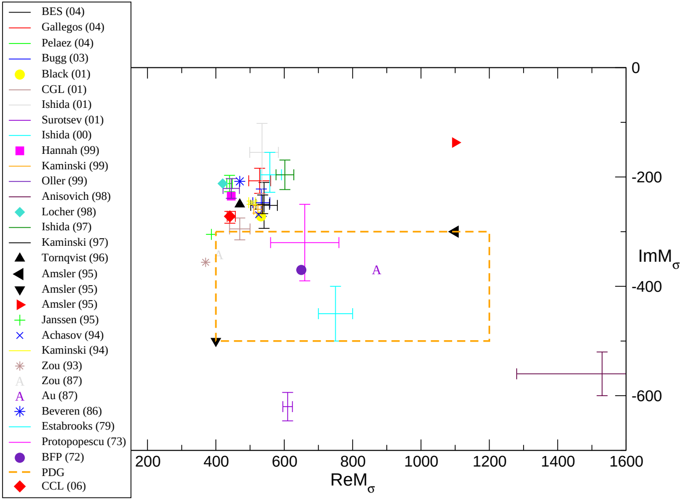
A REVIEW GOES HERE – Check our WWW List of Reviews

$f_0(600)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

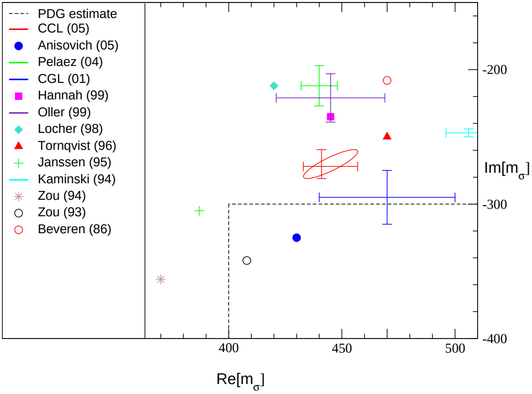
VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
(400–1200)−i(300–500) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(541\pm39)-i(252\pm42)$	[1] ABLIKIM	04A	BES2	$J/\psi \to \omega\pi^+\pi^-$
$(528\pm32)-i(207\pm23)$	[2] GALLEGOS	04	RVUE	Compilation
$(440\pm8)-i(212\pm15)$	[3] PELAEZ	04A	RVUE	$\pi\pi \to \pi\pi$
$(533\pm25)-i(247\pm25)$	[4] BUGG	03	RVUE	
$532 - i272$	BLACK	01	RVUE	$\pi^0\pi^0 \to \pi^0\pi^0$
$(470\pm30)-i(295\pm20)$	[5] COLANGELO	01	RVUE	$\pi\pi \to \pi\pi$
$(535^{+48}_{-36})-i(155^{+76}_{-53})$	[6] ISHIDA	01		$\Upsilon(3S) \to \Upsilon\pi\pi$
$610\pm14 - i620\pm26$	[7] SUROVTSEV	01	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$(558^{+34}_{-27})-i(196^{+32}_{-41})$	ISHIDA	00B		$p\overline{p} \to \pi^0\pi^0\pi^0$
$445 - i235$	HANNAH	99	RVUE	π scalar form factor
$(523\pm12)-i(259\pm7)$	KAMINSKI	99	RVUE	$\pi\pi \to \pi\pi, K\overline{K}, \sigma\sigma$
$442 - i\,227$	OLLER	99	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$469 - i203$	OLLER	99B	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$445 - i221$	OLLER	99C	RVUE	$\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$
$(1530^{+90}_{-250})-i(560\pm40)$	ANISOVICH	98B	RVUE	Compilation
$420 - i\,212$	LOCHER	98	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$(602\pm26)-i(196\pm27)$	[8] ISHIDA	97		$\pi\pi \to \pi\pi$
$(537\pm20)-i(250\pm17)$	[9] KAMINSKI	97B	RVUE	$\pi\pi \to \pi\pi, K\overline{K}, 4\pi$
$470 - i250$	[10,11] TORNQVIST	96	RVUE	$\pi\pi \to \pi\pi, K\overline{K}, K\pi, \eta\pi$
$\sim(1100 - i300)$	AMSLER	95B	CBAR	$\overline{p}p \to 3\pi^0$
$400 - i500$	[11,12] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$1100 - i137$	[11,13] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$387 - i305$	[11,14] JANSSEN	95	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$525 - i269$	[15] ACHASOV	94	RVUE	$\pi\pi \to \pi\pi$
$(506\pm10)-i(247\pm3)$	KAMINSKI	94	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$370 - i356$	[16] ZOU	94B	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$408 - i342$	[11,16] ZOU	93	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$870 - i370$	[11,17] AU	87	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$470 - i208$	[18] BEVEREN	86	RVUE	$\pi\pi \to \pi\pi, K\overline{K}, \eta\eta, \ldots$
$(750\pm50)-i(450\pm50)$	[19] ESTABROOKS	79	RVUE	$\pi\pi \to \pi\pi, K\overline{K}$
$(660\pm100)-i(320\pm70)$	PROTOPOP...	73	HBC	$\pi\pi \to \pi\pi, K\overline{K}$
$650 - i370$	[20] BASDEVANT	72	RVUE	$\pi\pi \to \pi\pi$

The sigma resonance



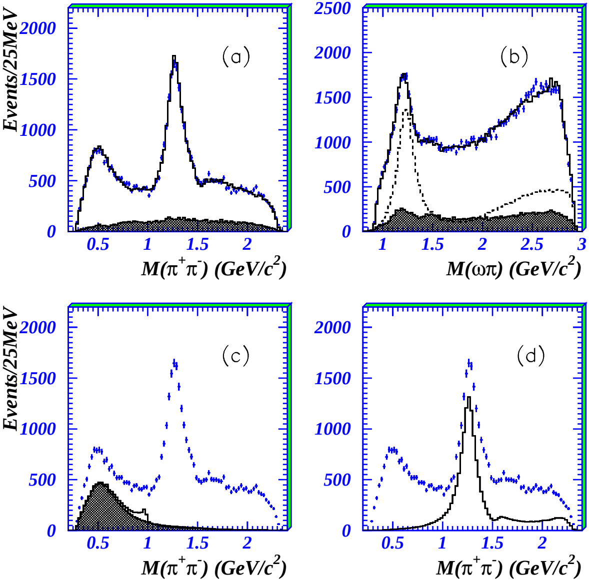
I. Caprini, GC, H. Leutwyler, PRL 06

The sigma resonance



I. Caprini, GC, H. Leutwyler, PRL 06

The σ in the data – BES (04), $J/\psi \to \omega\pi^+\pi^-$



How is the σ pole determined?

The relevant question is:

Where does the amplitude have a pole on the second Riemann sheet of the complex s plane?

The answer ought to be model- and parametrization-independent

How is the σ pole determined?

What is usually done is instead the following:

Fit the data with a parametrization, e.g.

$$
\begin{aligned}
f &= \frac{G_\sigma}{M^2 - s - iM\Gamma_{\rm tot}(s)} \\
\Gamma_{\rm tot}(s) &= g_1 \frac{\rho_{\pi\pi}(s)}{\rho_{\pi\pi}(M^2)} + g_2 \frac{\rho_{4\pi}(s)}{\rho_{4\pi}(M^2)}
\end{aligned}
$$

where $g_{1,2}$ can also be functions of s

How is the σ pole determined?

What is usually done is instead the following:

Fit the data with a parametrization, e.g.

$$f = \frac{G_\sigma}{M^2 - s - iM\Gamma_{\rm tot}(s)}$$

$$\Gamma_{\rm tot}(s) = g_1 \frac{\rho_{\pi\pi}(s)}{\rho_{\pi\pi}(M^2)} + g_2 \frac{\rho_{4\pi}(s)}{\rho_{4\pi}(M^2)}$$

where $g_{1,2}$ can also be functions of s

The fit to the data determines the σ parameters, M and $\Gamma_{\rm tot}$

How is the σ pole determined?

What is usually done is instead the following:
Fit the data with a parametrization, e.g.

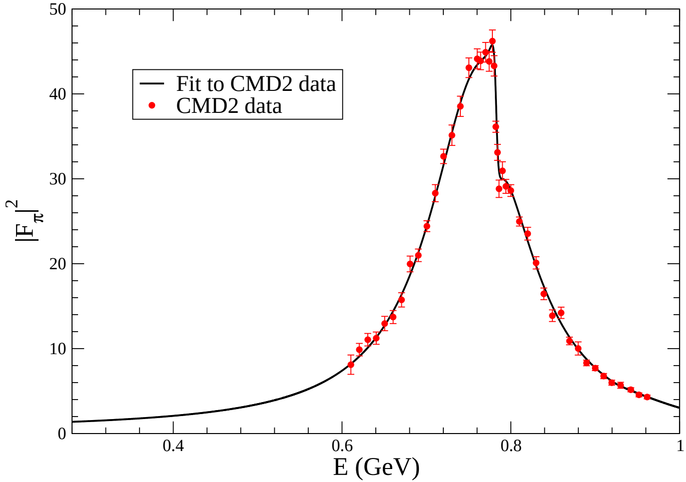
$$\begin{aligned} f &= \frac{G_\sigma}{M^2 - s - iM\Gamma_{\rm tot}(s)} \\ \Gamma_{\rm tot}(s) &= g_1 \frac{\rho_{\pi\pi}(s)}{\rho_{\pi\pi}(M^2)} + g_2 \frac{\rho_{4\pi}(s)}{\rho_{4\pi}(M^2)} \end{aligned}$$

where $g_{1,2}$ can also be functions of s

The fit to the data determines the σ parameters, M and $\Gamma_{\rm tot}$

The outcome is parametrization-dependent
Moreover, an obvious shortcoming of many of the parametrizations used to fit data is the neglect of the left-hand cut

Compare to the ρ in $e^+e^- \to \pi^+\pi^-$



Roy representation of t_0^0

Double-subtracted, crossing symmetric dispersion relation for t_0^0

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' \Big\{ K_0(s,s')\,\mathrm{Im}\, t_0^0(s') + K_1(s,s')\,\mathrm{Im}\, t_1^1(s') + K_2(s,s')\,\mathrm{Im}\, t_0^2(s') \Big\} + d_0^0(s)$$

$$a = a_0^0\,, \quad b = (2\,a_0^0 - 5\,a_0^2)/(12M_\pi^2)$$

$$K_0(s,s') = \frac{1}{\pi(s'-s)} + \frac{2\ln((s+s'-4M_\pi^2)/s')}{3\pi(s-4M_\pi^2)} - \frac{5s'+2s-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}$$

Roy representation of t_0^0

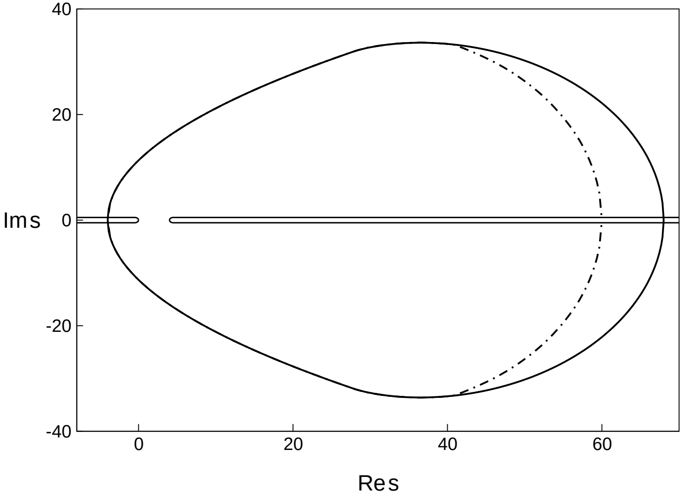
Double-subtracted, crossing symmetric dispersion relation for t_0^0

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' \Big\{ K_0(s,s')\, \mathrm{Im}\, t_0^0(s') + K_1(s,s')\, \mathrm{Im}\, t_1^1(s') + K_2(s,s')\, \mathrm{Im}\, t_0^2(s') \Big\} + d_0^0(s)$$

$$a = a_0^0 \, , \quad b = (2\, a_0^0 - 5\, a_0^2)/(12 M_\pi^2)$$

This representation allows one to evaluate t_0^0 in the complex plane – in its domain of validity on the first sheet.

Roy representation of t_0^0



Caprini, GC, Leutwyler (05)

Roy representation of t_0^0

Double-subtracted, crossing symmetric dispersion relation for t_0^0

$$
\begin{aligned}
t_0^0(s) &= a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' \left\{ K_0(s,s')\,\mathrm{Im}\, t_0^0(s') \right. \\
&\quad \left. + K_1(s,s')\,\mathrm{Im}\, t_1^1(s') + K_2(s,s')\,\mathrm{Im}\, t_0^2(s') \right\} + d_0^0(s)
\end{aligned}
$$

$$
a = a_0^0\,, \quad b = (2\,a_0^0 - 5\,a_0^2)/(12M_\pi^2)
$$

This representation allows one to evaluate t_0^0 in the complex plane – in its domain of validity on the first sheet.

Poles, however, are to be found on the second sheet

Roy representation of S_0^0

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M_\pi^2}{s} - 1}\,t_0^0(s)\;, \qquad 0 \le s \le 4M_\pi^2$$

Roy representation of S_0^0

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M_\pi^2}{s} - 1}t_0^0(s)\ , \qquad 0 \le s \le 4M_\pi^2$$

Unitarity implies that: $$S_0^{0\ I}(s + i\epsilon) = \left[S_0^{0\ I}(s - i\epsilon)\right]^{-1}$$

Roy representation of S_0^0

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M_\pi^2}{s} - 1}t_0^0(s) \; , \qquad 0 \le s \le 4M_\pi^2$$

Unitarity implies that: $S_0^{0\,I}(s+i\epsilon) = \left[S_0^{0\,I}(s-i\epsilon)\right]^{-1}$

The second sheet is reached by analytic continuation crossing the real axis from above: (for ϵ infinitesimally small)

$$S_0^{0\,II}(s-i\epsilon) = S_0^{0\,I}(s+i\epsilon) = \left[S_0^{0\,I}(s-i\epsilon)\right]^{-1}$$

Roy representation of S_0^0

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M_\pi^2}{s} - 1}\, t_0^0(s)\,, \qquad 0 \le s \le 4M_\pi^2$$

Unitarity implies that: $S_0^{0\,I}(s + i\epsilon) = \left[S_0^{0\,I}(s - i\epsilon)\right]^{-1}$

The second sheet is reached by analytic continuation crossing the real axis from above: (for ϵ infinitesimally small)

$$S_0^{0\,II}(s - i\epsilon) = S_0^{0\,I}(s + i\epsilon) = \left[S_0^{0\,I}(s - i\epsilon)\right]^{-1}$$

By analytic continuation, it is then true everywhere that

$$S_0^{0\,II}(s) = \left[S_0^{0\,I}(s)\right]^{-1}$$

Poles on the second sheet correspond to zeros on the first sheet!

Caprini, GC and Leutwyler, PRL (06)

Summary: method to determine the pole position

- Roy equations provide an explicit representation of t_0^0 on the first sheet, in terms of the imaginary parts of the partial waves on the real axis and two subtraction constants:

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' K_0(s,s')\, \mathrm{Im}\, t_0^0(s') + \ldots$$

Summary: method to determine the pole position

- Roy equations provide an explicit representation of t_0^0 on the first sheet, in terms of the imaginary parts of the partial waves on the real axis and two subtraction constants:

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' K_0(s,s')\, \mathrm{Im}\, t_0^0(s') + \ldots$$

- Unitarity implies that the S-matrix on the second sheet is equal to the inverse of the S-matrix on the first sheet

$$S_0^{0\,II}(s) = \left[S_0^{0\,I}(s)\right]^{-1}$$

Summary: method to determine the pole position

- Roy equations provide an explicit representation of t_0^0 on the first sheet, in terms of the imaginary parts of the partial waves on the real axis and two subtraction constants:

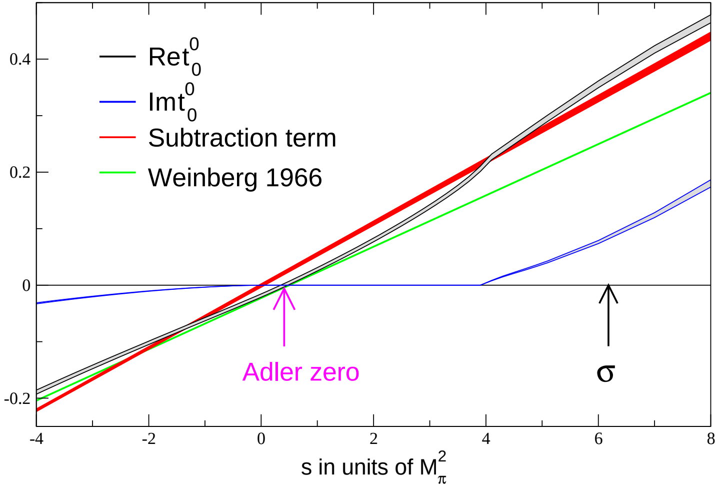
$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' K_0(s,s')\, \mathrm{Im}\, t_0^0(s') + \ldots$$

- Unitarity implies that the S-matrix on the second sheet is equal to the inverse of the S-matrix on the first sheet

$$S_0^{0\,II}(s) = \left[S_0^{0\,I}(s)\right]^{-1}$$

- Using as input the imaginary parts of the partial waves and the two S-wave scattering lengths one can determine the position of the poles of the S-matrix on the second sheet

Importance of the scattering lengths

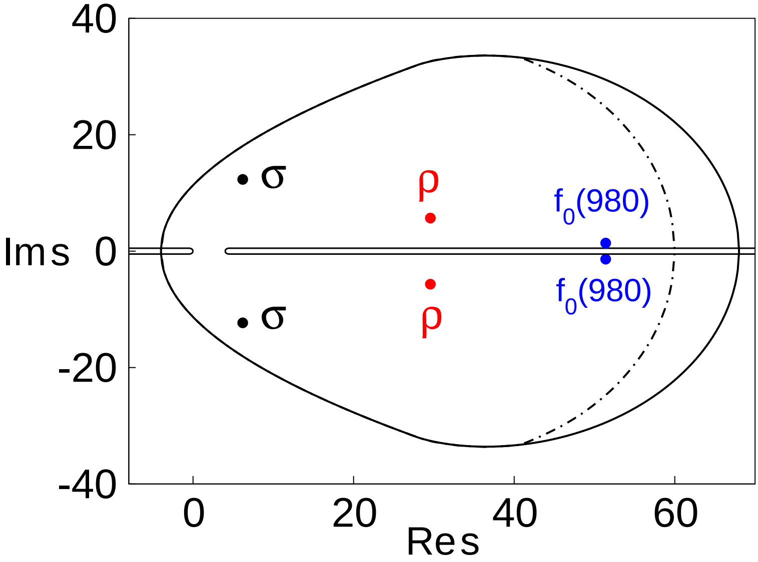


Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

Zeros of S_0^0 (and S_1^1)



Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

Error analysis: [at fixed a_0^0, a_0^2 and $\delta_A \equiv \delta_0^0(0.8\text{GeV})$]

$$m_\sigma \;=\; 441 \pm 4 - i(272 \pm 6)\ \text{MeV}$$

Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

Error analysis:

$$m_\sigma = 441 \pm 4 - i(272 \pm 6)\text{ MeV} + (-2.4 + i3.8)\Delta a_0^0$$

$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005}$$

Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

Error analysis:

$$\begin{aligned} m_\sigma &= 441 \pm 4 - i(272 \pm 6)\text{ MeV} + (-2.4 + i3.8)\Delta a_0^0 \\ &\quad +(0.8 - i4.0)\Delta a_0^2 \end{aligned}$$

$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005} \qquad \Delta a_0^2 = \frac{a_0^0 + 0.0444}{0.001}$$

Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

Error analysis:

$$\begin{aligned} m_\sigma &= 441 \pm 4 - i(272 \pm 6)\text{ MeV} + (-2.4 + i3.8)\Delta a_0^0 \\ &\quad +(0.8 - i4.0)\Delta a_0^2 + (5.3 + i3.3)\Delta\delta_A \end{aligned}$$

$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005} \qquad \Delta a_0^2 = \frac{a_0^0 + 0.0444}{0.001} \qquad \Delta\delta_A = \frac{\delta_A - 82.3}{3.4}$$

Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

Error analysis:

$$\begin{aligned} m_\sigma &= 441 \pm 4 - i(272 \pm 6)\text{ MeV} + (-2.4 + i3.8)\Delta a_0^0 \\ &\quad +(0.8 - i4.0)\Delta a_0^2 + (5.3 + i3.3)\Delta\delta_A \end{aligned}$$

$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005} \qquad \Delta a_0^2 = \frac{a_0^0 + 0.0444}{0.001} \qquad \Delta\delta_A = \frac{\delta_A - 82.3}{3.4}$$

$$M_\sigma = 441\,{}^{+16}_{-8}\text{ MeV}\,, \quad \Gamma_\sigma = 544\,{}^{+18}_{-25}\text{ MeV}$$

Caprini, GC and Leutwyler, PRL (06)

Comparison to lattice results?

- dispersion relations + chiral symmetry
  $\Rightarrow$ high precision determination of σ parameters
- this is the lowest resonance in QCD
- can the lattice provide a similar information?
- what method would allow one to calculate the σ parameters from first principles?
- see: lecture by C. Lang
  Meißner, Rusetsky et al. `1007.0860, 1010.6018`

Outline

Preliminary remarks

CHPT: expansion in m_{q_l}/Λ and p/Λ
In finite volume the momentum is quantized: $p = \frac{2\pi}{L} n$
Condition of applicability of CHPT:

$$m_{q_l} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1 \,, \quad L \gg 1\text{fm}$$

Two different physical regimes

$$LM_\pi \lesssim 1 \quad \Rightarrow \quad \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \quad \Rightarrow \quad p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

Preliminary remarks

CHPT: expansion in m_{q_l}/Λ and p/Λ
In finite volume the momentum is quantized: $p = \frac{2\pi}{L}n$
Condition of applicability of CHPT:

$$m_{q_l} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1 \, , \quad L \gg 1\mathrm{fm}$$

Two different physical regimes

$$LM_\pi \lesssim 1 \quad \Rightarrow \quad \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \quad \Rightarrow \quad p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

$$G_L(\vec{x},t) = \sum_{\vec{n}} G_\infty(\vec{x}+\vec{n}L,t)$$

p-regime

Examples:

Gasser and Leutwyler (88)

$$M_\pi(L) = M_\pi \left[1 + \frac{1}{2N_f}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)\right]$$
$$F_\pi(L) = F_\pi \left[1 - \frac{N_f}{2}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)\right]$$

with

$$\lambda = M_\pi L, \quad \xi = (M_\pi/4\pi F_\pi)^2$$

$$\tilde{g}_1(\lambda) = \left(\frac{4\pi}{M_\pi}\right)^2 \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)\big|_{t=\vec{x}=0} = \sum_{\vec{n}^2=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\lambda} K_1(|\vec{n}|\lambda)$$

p-regime

Examples:

Gasser and Leutwyler (88)

$$
\begin{aligned}
M_\pi(L) &= M_\pi \left[1 + \frac{1}{2N_f}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)\right] \\
F_\pi(L) &= F_\pi \left[1 - \frac{N_f}{2}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)\right]
\end{aligned}
$$

with

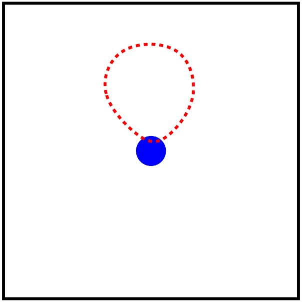
$$\lambda = M_\pi L, \quad \xi = (M_\pi/4\pi F_\pi)^2$$

$$\tilde{g}_1(\lambda) = \left(\frac{4\pi}{M_\pi}\right)^2 \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)\big|_{t=\vec{x}=0} = \sum_{\vec{n}^2=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\lambda} K_1(|\vec{n}|\lambda)$$

Asymptotic expansion of the Bessel function:

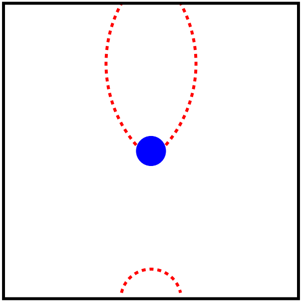
$$K_1(x) \simeq \sqrt{\frac{\pi}{2x}}e^{-x}$$

Masses in finite volume



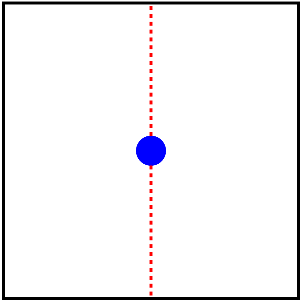
Loop-diagram

Masses in finite volume



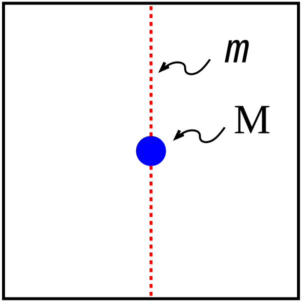
Loop diagram with
periodic
boundary conditions

Masses in finite volume



Loop diagram with
periodic
boundary conditions

Masses in finite volume



Loop diagram with periodic boundary conditions

This diagram exists only for $L \neq \infty$

Its effect is of the order $\exp\left[-mL\right]$

Lüscher's Formula

Lüscher has shown (86) that the leading exponential correction for $\lambda \gg 1$ has this general form:

$$M_{\pi,L} - M_\pi = C \int_{-\infty}^{\infty} dy \; e^{-\sqrt{M_\pi^2 + y^2} L} F(iy) + \ldots$$

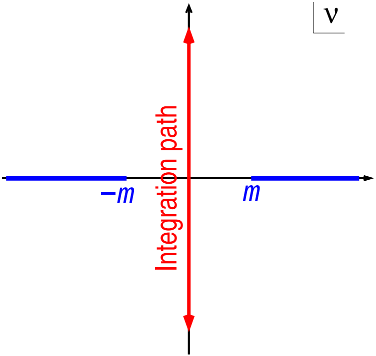
$F(\nu)$ is the forward $\pi\pi$ scattering amplitude

Lüscher's Formula

Lüscher has shown (86) that the leading exponential correction for $\lambda \gg 1$ has this general form:

$$M_{\pi,L} - M_\pi = C \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2 + y^2} L} F(iy) + \dots$$

$F(\nu)$ is the forward $\pi\pi$ scattering amplitude



Lüscher's Formula

Lüscher has shown (86) that the leading exponential correction for $\lambda \gg 1$ has this general form:

$$M_{\pi,L} - M_\pi = C \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \ldots$$

$F(\nu)$ is the forward $\pi\pi$ scattering amplitude

- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)

Lüscher's Formula

Lüscher has shown (86) that the leading exponential correction for $\lambda \gg 1$ has this general form:

$$M_{\pi,L} - M_\pi = C \int_{-\infty}^{\infty} dy\ e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \ldots$$

$F(\nu)$ is the forward $\pi\pi$ scattering amplitude

- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)
- The formula extends almost trivially to other particles: what matters for the behaviour of the corrections is not the mass of the particle itself, but the mass of the pion

Lüscher's Formula

Lüscher has shown (86) that the leading exponential correction for $\lambda \gg 1$ has this general form:

$$M_{\pi,L} - M_\pi = C \int_{-\infty}^{\infty} dy \; e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \ldots$$

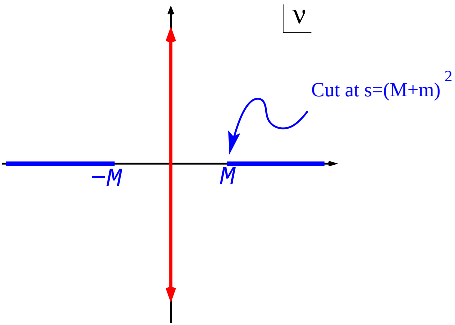
$F(\nu)$ is the forward $\pi\pi$ scattering amplitude

- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)
- The formula extends almost trivially to other particles: what matters for the behaviour of the corrections is not the mass of the particle itself, but the mass of the pion
- e.g. both the corrections for the pion as well as those for the proton mass depend exponentially on $M_\pi L$

Cuts and poles in the scattering amplitude

Any scattering amplitude must have a cut at

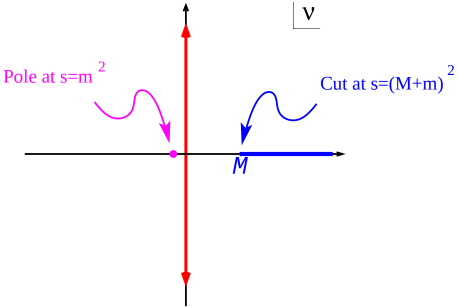
$s, u = (M + m)^2 \Rightarrow \nu = \frac{s-u}{4m} = \pm M_\pi$



Cuts and poles in the scattering amplitude

In addition it may have poles,

e.g. at $s, u = m^2 \Rightarrow \nu = \frac{s-u}{4m} = \mp \frac{M_\pi^2}{2m}$



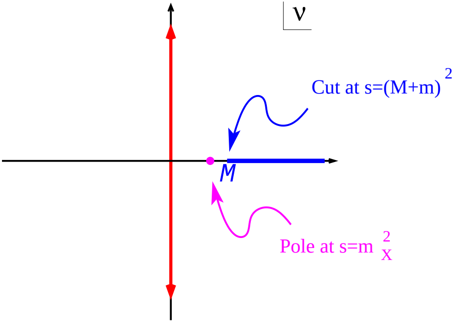
Poles on the lhs of the imaginary axis generate an extra term in the Lüscher's formula

GC, Fuhrer, Lanz, 09

Cuts and poles in the scattering amplitude

In addition it may have poles,

or at $s, u = m_X^2 \Rightarrow \nu = \frac{s-u}{4m} = \mp \frac{M_\pi^2}{2m} + \Delta m \left(1 + \frac{\Delta m}{2m}\right)$ $\Delta m = m_X - m$



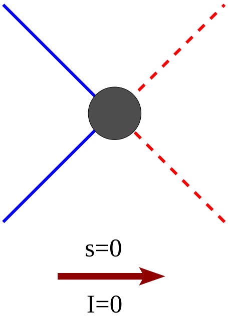
Poles on the rhs of the imaginary axis do not generate an extra term in the Lüscher's formula, cf. Arndt and Lin (04)

GC, Fuhrer, Lanz, 09

Corrections for M_π

$\pi\pi$ scattering amplitude with forward kinematics

$$F_{\pi\pi}(\nu) = T^{I=0}\left[0, 2M_\pi(M_\pi + \nu), 2M_\pi(M_\pi - \nu)\right] = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)$$



Lüscher's formula or CHPT?

$$\Delta M^L_{\pi\ \text{Lüscher}} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\ F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M^L_{\pi\ \text{CHPT}} = \frac{1}{4}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)$$

$$F(\nu) = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)\ , \quad \tilde{g}_1(\lambda) = \sum_{\vec{n}^2=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\lambda} K_1(|\vec{n}|\lambda)$$

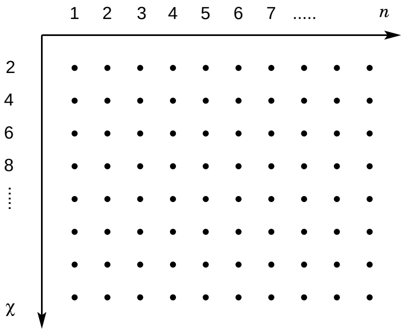
The leading exponential term at leading order in the chiral expansion is the same in both formulae

Each formula gives the leading term in two different expansions

Lüscher's formula or CHPT?

$$\Delta M^L_{\pi\ \text{Lüscher}} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

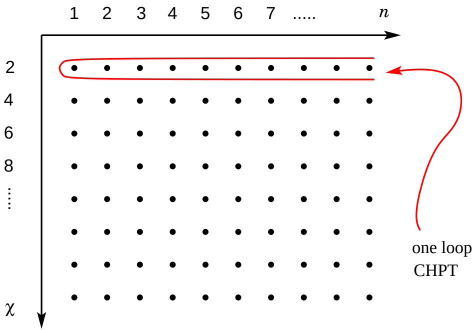
$$\Delta M^L_{\pi\ \text{CHPT}} = \frac{1}{4}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)$$



Lüscher's formula or CHPT?

$$\Delta M_{\pi\ \text{Lüscher}}^{L} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

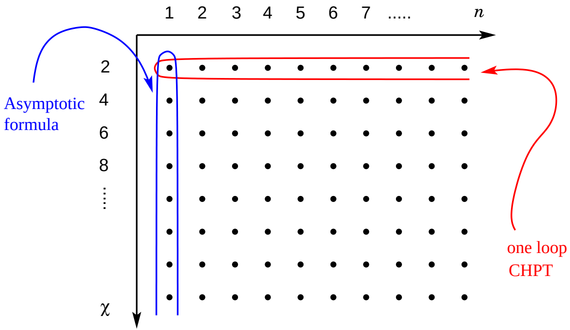
$$\Delta M_{\pi\ \text{CHPT}}^{L} = \frac{1}{4}\xi\,\tilde{g}_1(\lambda) + O(\xi^2)$$



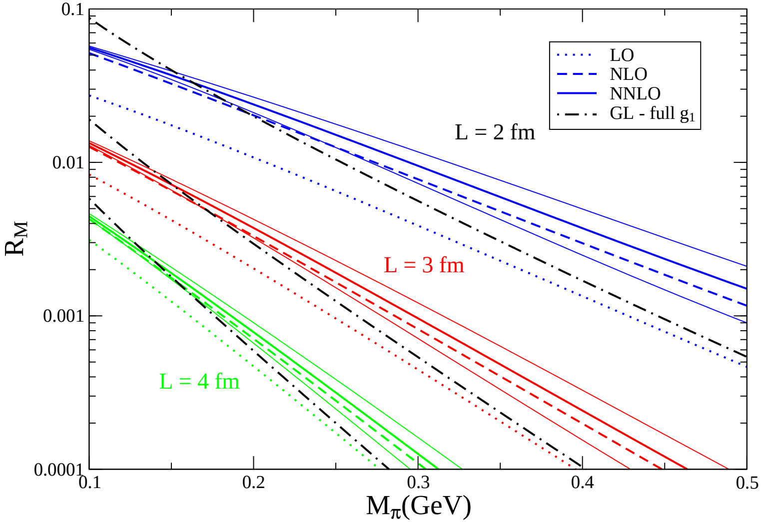
Lüscher's formula or CHPT?

$$\Delta M_{\pi\ \text{Lüscher}}^{L} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M_{\pi\ \text{CHPT}}^{L} = \frac{1}{4}\xi\, \tilde{g}_1(\lambda) + O(\xi^2)$$



Lüscher's formula vs CHPT: numerics



$R_M = M_{\pi L}/M_\pi - 1$

GC and S. Dürr 03

Extension of the Lüscher's Formula

One-loop CHPT corrections

$$\tilde{g}_1(\lambda) = \left(\frac{4\pi}{M_\pi}\right)^2 \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L, t)\big|_{t=\vec{x}=0} = \sum_{\vec{n}^2=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda} K_1\left(|\vec{n}|\,\lambda\right)$$

Extension of the Lüscher's Formula

One-loop CHPT corrections

$$\tilde{g}_1(\lambda) = \left(\frac{4\pi}{M_\pi}\right)^2 \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)\big|_{t=\vec{x}=0} = \sum_{\vec{n}^2=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda} K_1\left(|\vec{n}|\,\lambda\right)$$

Analogously one can extend the Lüscher's Formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{\vec{n}^2=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

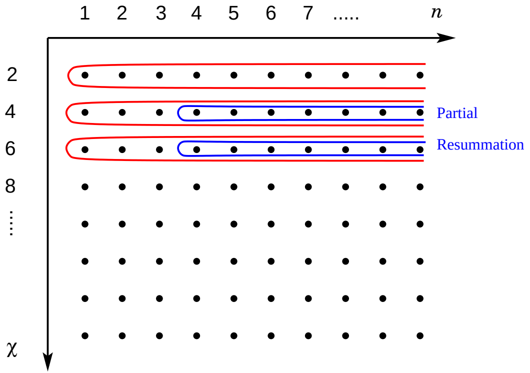
The extension does not give all exponentially subleading terms!

but pushes the accuracy of the formula toward $O\left(e^{-2\lambda}\right)$

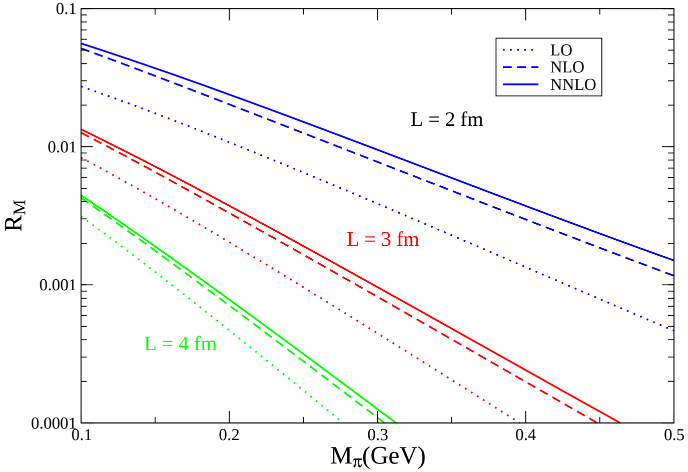
GC S. Dürr and C. Haefeli (05)

Extension of the Lüscher's Formula

$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda}\sum_{\vec{n}^2=1}^{\infty}\frac{m(|\vec{n}|)}{|\vec{n}|}\int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

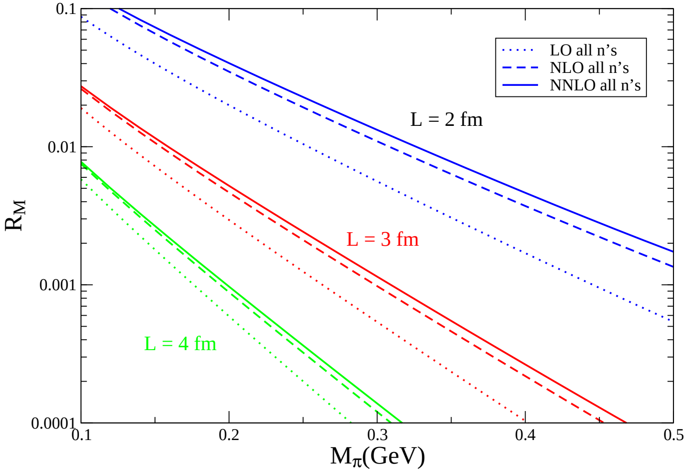


Nonleading exp. terms in $M_{\pi,L}$



GC S. Dürr and C. Haefeli (05)

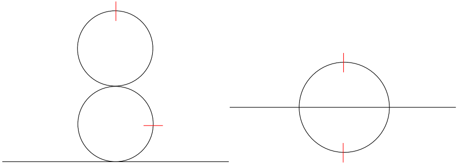
Nonleading exp. terms in $M_{\pi,L}$



GC S. Dürr and C. Haefeli (05)

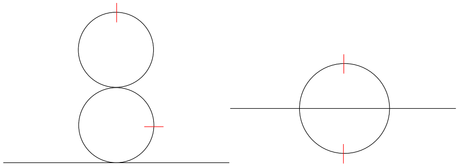
Two loop effects beyond Lüscher

Diagrams with two propagators in finite volume are not included in Lüscher's formula

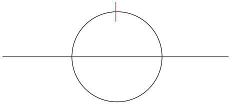


Two loop effects beyond Lüscher

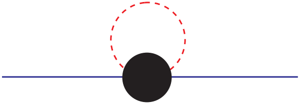
Diagrams with two propagators in finite volume are not included in Lüscher's formula



However, also diagrams with one propagator in finite volume are not fully included



Two loop effects beyond Lüscher

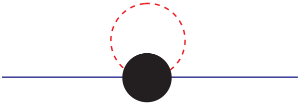


$$= \int d^4\ell \, \Gamma(p,\ell,-\ell,-p) G_L(\ell)$$

$$M_L - M_\infty = \sum_{\vec{n}\neq\vec{0}} \int d^4\ell \, \Gamma(p,\ell,-\ell,p) G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L}$$

$$G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}$$

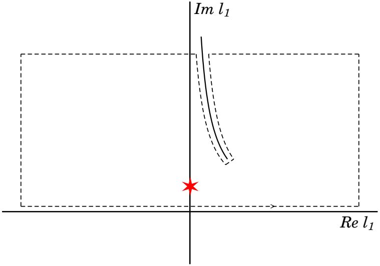
Two loop effects beyond Lüscher



$$= \int d^4\ell \, \Gamma(p,\ell,-\ell,-p) G_L(\ell)$$

$$M_L - M_\infty = \sum_{\vec{n}\neq\vec{0}} \int d^4\ell \, \Gamma(p,\ell,-\ell,p) G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L}$$

$$G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}$$

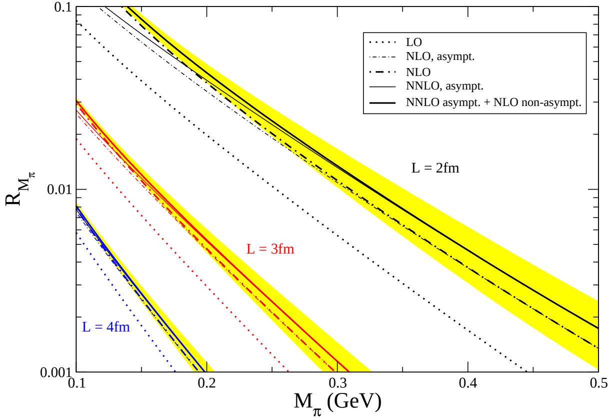


The pion mass to two loops

$$\Delta M_\pi = \text{Lüscher resummed} + \text{cut} + \text{2 FV-propagators}$$

The pion mass to two loops

$$\Delta M_\pi = \text{Lüscher resummed} + \text{cut} + \text{2 FV-propagators}$$



GC and C. Haefeli (05)

The pion mass to two loops

$$\Delta M_\pi = \text{Lüscher resummed} + \text{cut} + \text{2 FV-propagators}$$

The contributions beyond the resummed Lüscher formula are negligibly small

Outline

Summary

As an illustration of the effective field theory method I have discussed a few applications:

- the analysis of the $\pi\pi$ scattering amplitude
- the determination of the σ resonance parameters
- the calculation of finite volume effects