Chiral perturbation theory

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Lecture III: applications in phenomenology and lattice

ππ scattering beyond LO
  Experimental tests

The σ resonance

Finite volume effects
  Asymptotic formulae
  Resummation of higher exponentials
  The pion mass to two loops

Summary
\( \pi \pi \) scattering at NLO

\[
a_0^0 = \frac{7 M^2}{32 \pi F^2} \left[ 1 + \frac{M^2}{3} \langle r^2 \rangle \pi S + \frac{200 \pi F^2 M^2}{7} (a_2^0 + 2a_2^2) \right]
- \frac{M^2}{672 \pi^2 F^2} (15\ell_3 - 353) = 0.16 \cdot 1.25 = 0.20
\]

\[
2a_0^0 - 5a_2^0 = \frac{3 M^2}{4 \pi F^2} \left[ 1 + \frac{M^2}{3} \langle r^2 \rangle \pi S + \frac{41 M^2}{192 \pi^2 F^2} \right] = 0.624
\]

Gasser and Leutwyler (83)
Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)\neq 1$ GeV $\Rightarrow$ expected to be a few percent

\[ a_0^0 = 0.200 + \mathcal{O}(p^6) \quad a_0^2 = -0.0445 + \mathcal{O}(p^6) \]

Gasser and Leutwyler (84)
Higher orders

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$
\Lambda \sim 1 \text{ GeV} \implies \text{expected to be a few percent}
$

\begin{align*}
a_0^0 &= 0.200 + \mathcal{O}(p^6) \quad a_0^2 = -0.0445 + \mathcal{O}(p^6)
\end{align*}

The reason for the rather large correction in $a_0^0$ is a chiral log

\begin{align*}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2} \left[ 1 + \frac{9}{2} \ell_\chi + \ldots \right] \\
a_0^2 &= -\frac{M_\pi^2}{16\pi F_\pi^2} \left[ 1 - \frac{3}{2} \ell_\chi + \ldots \right]
\end{align*}

\begin{align*}
\ell_\chi &= \frac{M_\pi^2}{16\pi^2 F_\pi^2} \ln \frac{\mu^2}{M_\pi^2}
\end{align*}

Gasser and Leutwyler (84)
Higher orders

\[ O(p^2) \]
\[ O(p^2) + O(p^4) \]
\[ O(p^4) \]
Higher orders

\[ \text{Re}[t_0^2] = O(p^2) + O(p^4) \]
Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
\[ \Lambda \sim 1 \text{ GeV} \Rightarrow \text{expected to be a few percent} \]

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Gasser and Leutwyler (84)

How large are yet higher orders?
Is it at all possible to make a precise prediction?
Roy equations

Unitarity effects can be calculated exactly using dispersive methods
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Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations
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Input: imaginary parts above 0.8 GeV
    two subtraction constants, e.g. $a_0^0$ and $a_0^2$
Roy equations

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Unitarity, analyticity and crossing symmetry \( \equiv \) Roy equations

**Input:** imaginary parts above 0.8 GeV
two subtraction constants, e.g. \( a_0^0 \) and \( a_0^2 \)

**Output:** the full \( \pi\pi \) scattering amplitude below 0.8 GeV

**Note:** if \( a_0^0, a_0^2 \) are chosen within the universal band
the solution exists and is unique
Roy equations

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Unitarity, analyticity and crossing symmetry \( \equiv \) Roy equations

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**Numerical solutions of the Roy equations**
- Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
- Ananthanarayan, GC, Gasser and Leutwyler (00)
- Descotes-Genon, Fuchs, Girlanda and Stern (01)
Numerical solutions
Numerical solutions
Numerical solutions
Numerical solutions

The graph shows the real part of the scattering amplitude for different values of the angular momentum quantum number I, labeled I=0, I=1, and I=2. The graph is plotted against E (GeV) for energy values up to 1.2 GeV.

- The solid black line represents the Roy solution.
- The dashed black line represents the input for E > 0.8 GeV.

The graph indicates the behavior of the scattering amplitude under finite volume effects.
Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted
Combining CHPT and dispersive methods

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Subtracting the amplitude at threshold \((a_0^0, a_0^2)\) is not mandatory.
Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold \(a_0^0, a_0^2\) is not mandatory

The freedom in the choice of the subtraction point can be exploited to use the chiral expansion where it converges best, i.e. below threshold
Combining CHPT and dispersive methods

\[ \text{Re}[t_0]\]

- \(O(p^2)\)
- \(O(p^2) + O(p^4)\)
- \(O(p^4)\)

\(E(\text{GeV})\)
Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

\[
\begin{align*}
  a_0^0 &= 0.159 \rightarrow 0.200 \rightarrow 0.216 \\
  10 \cdot a_0^2 &= -0.454 \rightarrow -0.445 \rightarrow -0.445 \\
  p^2 & \quad p^4 & \quad p^6
\end{align*}
\]

GC, Gasser and Leutwyler (01)
Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

\[
a_0^0 = 0.159 \rightarrow 0.200 \rightarrow 0.216
\]

\[
10 \cdot a_0^2 = -0.454 \rightarrow -0.445 \rightarrow -0.445
\]

\[
p^2 \quad p^4 \quad p^6
\]

CHPT below threshold + Roy

\[
a_0^0 = 0.197 \rightarrow 0.2195 \rightarrow 0.220
\]

\[
10 \cdot a_0^2 = -0.402 \rightarrow -0.446 \rightarrow -0.444
\]

GC, Gasser and Leutwyler (01)
Low-energy theorem for $\pi\pi$ scattering

$\mathcal{M}(\pi^0\pi^0 \rightarrow \pi^+\pi^-) \equiv A(s, t, u) = \text{isospin invariant amplitude}$

Low energy theorem: $A(s, t, u) = \frac{s - M^2}{F^2} + O(p^4)$  \text{ Weinberg 1966}

$M^2 = B(m_u + m_d)$ \quad $M^2_{\pi} = M^2 + O(m_q^2)$, $F_{\pi} = F + O(m_q)$

All physical amplitudes can be expressed in terms of $A(s, t, u)$

$T^{I=0} = 3A(s, t, u) + A(t, s, u) + A(u, t, s) \Rightarrow T^{I=0} = \frac{2s - M^2_{\pi}}{F^2_{\pi}}$

S wave projection \quad (I=0)

$t_0^0(s) = \frac{2s - M^2_{\pi}}{32\pi F^2_{\pi}}$ \quad $a_0^0 = t_0^0(4M^2_{\pi}) = \frac{7M^2_{\pi}}{32\pi F^2_{\pi}} = 0.16$
Low-energy theorem for \( \pi \pi \) scattering

\[
\mathcal{M}(\pi^0 \pi^0 \to \pi^+ \pi^-) \equiv A(s, t, u) = \text{isospin invariant amplitude}
\]

Low energy theorem: 
\[
A(s, t, u) = \frac{s - M^2}{F^2} + O(p^4) \quad \text{Weinberg 1966}
\]

\[
M^2 = B(m_u + m_d) \quad M^2_\pi = M^2 + O(m_q^2), \quad F_\pi = F + O(m_q)
\]

All physical amplitudes can be expressed in terms of \( A(s, t, u) \)

\[
T^{I=2} = A(t, s, u) + A(u, t, s) \quad \Rightarrow \quad T^{I=2} = -s + \frac{2M^2_\pi}{F^2_\pi}
\]

S wave projection \((I=2)\)

\[
t_0^2(s) = \frac{2M^2_\pi - s}{32\pi F^2_\pi} \quad a_0^2 = t_0^2(4M^2_\pi) = \frac{-M^2_\pi}{16\pi F^2_\pi} = -0.045
\]
Chiral predictions for $a_0^0$ and $a_0^2$

Quark mass dependence of $M_\pi$ and $F_\pi$:

$$M_\pi^2 = M^2 \left( 1 - \frac{M^2}{32\pi^2 F^2} \, \bar{\ell}_3 + O(p^4) \right)$$

$$M^2 \equiv -\frac{m_u + m_d}{F^2} \langle 0 | \bar{q} q | 0 \rangle$$

Gell-Mann, Oakes, Renner (68)

$$F_\pi = F \left( 1 + \frac{M^2}{16\pi^2 F^2} \, \bar{\ell}_4 + O(p^4) \right)$$

Phenomenological determinations (indirect):

$$\bar{\ell}_3 = 2.9 \pm 2.4$$

Gasser & Leutwyler (84)

$$\bar{\ell}_4 = 4.4 \pm 0.2$$

GC, Gasser & Leutwyler (01)

Lattice calculations determine these constants directly
Chiral predictions for $a_0^0$ and $a_0^2$

$\chi$PT calculations at NLO
and at NNLO

Prediction obtained matching $O(p^6)$ $\chi$PT to Roy equations (disp. relation):

\[
\begin{align*}
a_0^0 & = 0.220 \pm 0.001 + 0.009\Delta \ell_4 - 0.002\Delta \ell_3 \\
10 \cdot a_0^2 & = -0.444 \pm 0.003 - 0.01\Delta \ell_4 - 0.004\Delta \ell_3
\end{align*}
\]

where \( \bar{\ell}_4 = 4.4 + \Delta \ell_4 \)
\( \bar{\ell}_3 = 2.9 + \Delta \ell_3 \)

Adding errors in quadrature \([\Delta \ell_4 = 0.2, \Delta \ell_3 = 2.4]\)

\[
\begin{align*}
a_0^0 & = 0.220 \pm 0.005 \\
10 \cdot a_0^2 & = -0.444 \pm 0.01 \\
a_0^0 - a_0^2 & = 0.265 \pm 0.004
\end{align*}
\]
Chiral predictions for $a_0^0$ and $a_0^2$

![Graph showing chiral predictions for $a_0^0$ and $a_0^2$.](image)

- **Universal Band**
- **Tree, one loop, two loops**
- **$\bar{l}_4 = 4.4 \pm 0.2$, $\bar{l}_3$ free**
- **$\bar{l}_3 = 2.9 \pm 2.4$, $\bar{l}_4$ free**

Colangelo, Gasser & Leutwyler 2001
Experimental tests

Experimental tests

- Universal band
- Tree (66), one loop (83), two loops (96)
- Prediction (ChPT + dispersion theory, 2001)
- DIRAC (2005)
- NA48 K -> 3 \pi (2005)
- E865
- NA48
Experimental tests

![Graph showing experimental tests data]

- **E865**
- **Geneva-Saclay**
- **NA48**
Recent update: E865 corrected their data
Experimental tests

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Experimental tests

isospin breaking corrections recently calculated for $K_{e4}$ are essential at this level of precision

GC, Gasser, Rusetsky (09)
Experimental tests

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GC, Gasser, Rusetsky (09)
Experimental tests

Figure from NA48/2 Eur.Phys.J.C64:589,2009
Lattice input for $\bar{\ell}_3$ and $\bar{\ell}_4$
Outline

\( \pi \pi \) scattering beyond LO

Experimental tests

The \( \sigma \) resonance

Finite volume effects

Asymptotic formulae

Resummation of higher exponentials

The pion mass to two loops

Summary
The sigma resonance

The two S-wave scattering lengths are the essential parameters at low energy.

For example, their knowledge fixes the $\sigma$ pole position to a remarkable level of precision:

$$M_\sigma = 441^{+16}_{-8} \text{ MeV}, \quad \Gamma_\sigma = 544^{+18}_{-25} \text{ MeV}$$

I. Caprini, GC, H. Leutwyler, PRL 06
\[ iG(JPC) = 0^+(0^+ +) \]

A REVIEW GOES HERE – Check our WWW List of Reviews

\[ f_0(600) \text{ T-MATRIX POLE } \sqrt{s} \]

Note that \( \Gamma \approx 2 \Im(\sqrt{s}_{\text{pole}}) \).

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<td>660 ± 100 – i(320 ± 70)</td>
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\( \pi \pi \text{ scattering} \)

\( \sigma \text{ finite volume effects} \)

\( \eta \eta \text{ summary} \)
The sigma resonance

![Graph showing the sigma resonance with various data points and error bars.](image)
The sigma resonance

![Graph showing the sigma resonance with data points from various authors and their estimates.](image)

I. Caprini, GC, H. Leutwyler, PRL 06
The $\sigma$ in the data – BES (04), $J/\psi \rightarrow \omega \pi^+ \pi^-$
How is the $\sigma$ pole determined?

The relevant question is:

Where does the amplitude have a pole on the second Riemann sheet of the complex $s$ plane?

The answer ought to be model- and parametrization-independent.
How is the $\sigma$ pole determined?

What is usually done is instead the following:
Fit the data with a parametrization, e.g.

$$f = \frac{G_\sigma}{M^2 - s - iM \Gamma_{\text{tot}}(s)}$$

$$\Gamma_{\text{tot}}(s) = g_1 \frac{\rho_{\pi\pi}(s)}{\rho_{\pi\pi}(M^2)} + g_2 \frac{\rho_{4\pi}(s)}{\rho_{4\pi}(M^2)}$$

where $g_{1,2}$ can also be functions of $s$
How is the $\sigma$ pole determined?

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The fit to the data determines the $\sigma$ parameters, $M$ and $\Gamma_{\text{tot}}$
How is the $\sigma$ pole determined?

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$$ \Gamma_{\text{tot}}(s) = g_1 \frac{\rho_{\pi\pi}(s)}{\rho_{\pi\pi}(M^2)} + g_2 \frac{\rho_{4\pi}(s)}{\rho_{4\pi}(M^2)} $$

where $g_{1,2}$ can also be functions of $s$

The fit to the data determines the $\sigma$ parameters, $M$ and $\Gamma_{\text{tot}}$

The outcome is parametrization-dependent

Moreover, an obvious shortcoming of many of the parametrizations used to fit data is the neglect of the left-hand cut
Compare to the $\rho$ in $e^+ e^- \rightarrow \pi^+ \pi^-$
Roy representation of $t_0^0$

Double-subtracted, crossing symmetric dispersion relation for $t_0^0$

$$t_0^0(s) = a + (s - 4M_{\pi}^2) b + \int_{4M_{\pi}^2}^{\Lambda^2} ds' \left\{ K_0(s, s') \text{Im} t_0^0(s') + K_1(s, s') \text{Im} t_1^1(s') + K_2(s, s') \text{Im} t_0^2(s') \right\} + a_0^0(s)$$

$$a = a_0^0, \quad b = (2a_0^0 - 5a_0^2)/(12M_{\pi}^2)$$

$$K_0(s, s') = \frac{1}{\pi(s' - s)} + \frac{2 \ln((s + s' - 4M_{\pi}^2)/s')}{3\pi(s - 4M_{\pi}^2)} - \frac{5s' + 2s - 16M_{\pi}^2}{3\pi s'(s' - 4M_{\pi}^2)}$$
Roy representation of $t_0^0$

Double-subtracted, crossing symmetric dispersion relation for $t_0^0$

$$t_0^0(s) = a + (s - 4M^2) b + \int_{4M^2}^{\Lambda^2} ds' \left\{ K_0(s, s') \text{Im} t_0^0(s') + K_1(s, s') \text{Im} t_1^1(s') + K_2(s, s') \text{Im} t_0^2(s') \right\} + a_0^0(s)$$

$$a = a_0^0, \quad b = \frac{2a_0^0 - 5a_0^2}{12M^2}$$

This representation allows one to evaluate $t_0^0$ in the complex plane – in its domain of validity on the first sheet.
Roy representation of $t_0^0$
Roy representation of $t_0^0$

Double-subtracted, crossing symmetric dispersion relation for $t_0^0$

$$t_0^0(s) = a + (s - 4M^2_\pi) b + \int_{4M^2_\pi}^{\Lambda^2} ds' \left\{ K_0(s, s') \text{Im} t_0^0(s') + K_1(s, s') \text{Im} t_1^1(s') + K_2(s, s') \text{Im} t_0^2(s') \right\} + a_0^0(s)$$

$$a = a_0^0, \quad b = \frac{(2 a_0^0 - 5 a_0^2)}{(12M^2_\pi)}$$

This representation allows one to evaluate $t_0^0$ in the complex plane – in its domain of validity on the first sheet.

Poles, however, are to be found on the second sheet.
Roy representation of $S_0^0$

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M^2_\pi}{s}} - 1t_0^0(s), \quad 0 \leq s \leq 4M^2_\pi$$
Roy representation of $S_0^0$

\[ S_0^0(s) = 1 - 2\sqrt{\frac{4M^2}{s}} - 1t_0^0(s), \quad 0 \leq s \leq 4M^2 \]

Unitarity implies that:

\[ S_0^0(s + i\epsilon) = [S_0^0(s - i\epsilon)]^{-1} \]
Roy representation of $S^0_0$

$$S^0_0(s) = 1 - 2\sqrt{\frac{4M^2_\pi}{s}} - 1t^0_0(s), \quad 0 \leq s \leq 4M^2_\pi$$

Unitarity implies that:

$$S^0_0^\dagger(s + i\epsilon) = \left[S^0_0^\dagger(s - i\epsilon)\right]^{-1}$$

The second sheet is reached by analytic continuation crossing the real axis from above:

(for $\epsilon$ infinitesimally small)

$$S^0_0^{\prime\prime}(s - i\epsilon) = S^0_0^\dagger(s + i\epsilon) = \left[S^0_0^\dagger(s - i\epsilon)\right]^{-1}$$
Roy representation of $S_0^0$

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M^2}{s}} - 1t_0^0(s), \quad 0 \leq s \leq 4M^2$$

Unitarity implies that:

$$S_0^0 I(s + i\epsilon) = [S_0^0 I(s - i\epsilon)]^{-1}$$

The second sheet is reached by analytic continuation crossing the real axis from above: (for $\epsilon$ infinitesimally small)

$$S_0^{II}(s - i\epsilon) = S_0^0 I(s + i\epsilon) = [S_0^0 I(s - i\epsilon)]^{-1}$$

By analytic continuation, it is then true everywhere that

$$S_0^{II}(s) = [S_0^0 I(s)]^{-1}$$

Poles on the second sheet correspond to zeros on the first sheet!

Caprini, GC and Leutwyler, PRL (06)
Summary: method to determine the pole position

- Roy equations provide an explicit representation of $t_0^0$ on the first sheet, in terms of the imaginary parts of the partial waves on the real axis and two subtraction constants:

$$t_0^0(s) = a + (s - 4M_{\pi}^2) b + \int_{4M_{\pi}^2}^{s\Lambda^2} ds' K_0(s, s') \text{Im} t_0^0(s') + \ldots$$
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- Unitarity implies that the S-matrix on the second sheet is equal to the inverse of the S-matrix on the first sheet

$$S_0^0 II(s) = \left[ S_0^0 I(s) \right]^{-1}$$
Summary: method to determine the pole position

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- Unitarity implies that the $S$-matrix on the second sheet is equal to the inverse of the $S$-matrix on the first sheet

$$S_0^0 II(s) = \left[ S_0^0 I(s) \right]^{-1}$$

- Using as input the imaginary parts of the partial waves and the two $S$-wave scattering lengths one can determine the position of the poles of the $S$-matrix on the second sheet
Importance of the scattering lengths

\[ \text{Ret}_0^0, \text{Imt}_0^0, \text{Subtraction term}, \text{Weinberg 1966} \]

Adler zero

\[ s \text{ in units of } M_{\pi}^2 \]
Zeros of $S_0^0$ (and $S_1^1$)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_{\sigma}^2 = (6.2 \pm 12.3) M_\pi^2 \quad m_{f_0}^2 = (51.4 \pm 1.4) M_\pi^2$$
Zeros of $S_0^0$ (and $S_1^1$)
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Error analysis: [at fixed $a_0^0$, $a_0^2$ and $\delta_A \equiv \delta_0^0(0.8\text{GeV})$]

$$m_\sigma = 441 \pm 4 - i(272 \pm 6) \text{ MeV}$$
Zeros of $S^0_0$ (and $S^1_1$)

Input: the imaginary parts from Roy solutions below 1.15 GeV [GC, Leutwyler, in preparation] and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m^2_\sigma = (6.2 \pm i 12.3) M^2_\pi \quad m^2_{f_0} = (51.4 \pm i 1.4) M^2_\pi$$

Error analysis:

$$m_\sigma = 441 \pm 4 - i(272 \pm 6) \text{ MeV} + (-2.4 + i 3.8) \Delta a^0_0$$

$$\Delta a^0_0 = \frac{a^0_0 - 0.220}{0.005}$$
Zeros of $S_0^0$ (and $S_1^1$)

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Error analysis:

$$m_\sigma = 441 \pm 4 - i(272 \pm 6) \text{ MeV} + (-2.4 + i3.8) \Delta a_0^0 + (0.8 - i4.0) \Delta a_0^2$$

$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005} \quad \Delta a_2^0 = \frac{a_0^0 + 0.0444}{0.001}$$
Zeros of $S_0^0$ (and $S_1^1$)

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$$\quad + (0.8 - i4.0) \Delta a_0^2 + (5.3 + i3.3) \Delta \delta_A$$

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Zeros of $S_0^0$ (and $S_1^1$)

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$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005} \quad \Delta a_2^0 = \frac{a_0^0 + 0.0444}{0.001} \quad \Delta \delta_A = \frac{\delta_A - 82.3}{3.4}$$

$$M_\sigma = 441^{+16}_{-8} \text{ MeV}, \quad \Gamma_\sigma = 544^{+18}_{-25} \text{ MeV}$$

Caprini, GC and Leutwyler, PRL (06)
Comparison to lattice results?

- dispersion relations + chiral symmetry
  \[\Rightarrow\] high precision determination of \(\sigma\) parameters

- this is the lowest resonance in QCD

- can the lattice provide a similar information?

- what method would allow one to calculate the \(\sigma\) parameters from first principles?

- see: lecture by C. Lang

Meißner, Rusetsky et al. 1007.0860, 1010.6018
Outline

\( \pi \pi \) scattering beyond LO
Experimental tests

The \( \sigma \) resonance

Finite volume effects
Asymptotic formulae
Resummation of higher exponentials
The pion mass to two loops

Summary
Preliminary remarks

CHPT: expansion in $m_{q_i}/\Lambda$ and $p/\Lambda$
In finite volume the momentum is quantized:
Condition of applicability of CHPT:

$$m_{q_i} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1, \quad L \gg 1\text{fm}$$

Two different physical regimes

$$LM_\pi \lesssim 1 \quad \Rightarrow \quad \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \quad \Rightarrow \quad p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$
Preliminary remarks

CHPT: expansion in $m_{q_i}/\Lambda$ and $p/\Lambda$

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$$LM_\pi \gg 1 \quad \Rightarrow \quad p-\text{regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$
**$p$-regime**

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

\[
G_L(\vec{x}, t) = \sum_{\vec{n}} G_\infty(\vec{x} + \vec{n}L, t)
\]
\( p \)-regime

**Examples:**

\[
M_\pi(L) = M_\pi \left[ 1 + \frac{1}{2N_f} \xi \tilde{g}_1(\lambda) + O(\xi^2) \right]
\]

\[
F_\pi(L) = F_\pi \left[ 1 - \frac{N_f}{2} \xi \tilde{g}_1(\lambda) + O(\xi^2) \right]
\]

with

\[
\lambda = M_\pi L, \quad \xi = \left( \frac{M_\pi}{4\pi F_\pi} \right)^2
\]

\[
\tilde{g}_1(\lambda) = \left( \frac{4\pi}{M_\pi} \right)^2 \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t)_{t=\vec{x}=0} = \sum_{\vec{n}^2=1} \frac{4m(|\vec{n}|)}{|\vec{n}|^2 \lambda} K_1(|\vec{n}| \lambda)
\]

Gasser and Leutwyler (88)
**p-regime**

**Examples:**

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\]

Asymptotic expansion of the Bessel function:

\[
K_1(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}
\]
Masses in finite volume

Loop-diagram
Masses in finite volume

Loop diagram with periodic boundary conditions
Masses in finite volume

Loop diagram with periodic boundary conditions
Masses in finite volume

Loop diagram with periodic boundary conditions

This diagram exists only for $L \neq \infty$

Its effect is of the order $\exp[-mL]$
Lüscher’s Formula
Lüscher has shown (86) that the leading exponential correction for $\lambda \gg 1$ has this general form:

$$M_{\pi,L} - M_\pi = C \int_{-\infty}^{\infty} dy \ e^{-\sqrt{M_\pi^2 + y^2}L} F(iy) + \ldots$$

$F(\nu)$ is the forward $\pi\pi$ scattering amplitude.
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- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)
- The formula extends almost trivially to other particles: what matters for the behaviour of the corrections is not the mass of the particle itself, but the mass of the pion
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- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)
- The formula extends almost trivially to other particles: what matters for the behaviour of the corrections is not the mass of the particle itself, but the mass of the pion
- e.g. both the corrections for the pion as well as those for the proton mass depend exponentially on $M_{\pi} L$
Cuts and poles in the scattering amplitude

Any scattering amplitude must have a cut at

\[ s, u = (M + m)^2 \Rightarrow \nu = \frac{s-u}{4m} = \pm M_\pi \]
Cuts and poles in the scattering amplitude

In addition it may have poles,
e.g. at $s, u = m^2 \Rightarrow \nu = \frac{s-u}{4m} = \mp \frac{M_\pi^2}{2m}$

Poles on the lhs of the imaginary axis **generate** an extra term in the Lüscher's formula

GC, Fuhrer, Lanz, 09
Cuts and poles in the scattering amplitude

In addition it may have poles,

or at \( s, u = m_X^2 \Rightarrow \nu = \frac{s-u}{4m} = \mp \frac{M^2}{2m} + \Delta m \left( 1 + \frac{\Delta m}{2m} \right) \)

\[ \Delta m = m_X - m \]

Poles on the rhs of the imaginary axis **do not generate** an extra term in the Lüscher’s formula, cf. Arndt and Lin (04)

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GC, Fuhrer, Lanz, 09
Corrections for $M_\pi$

$\pi\pi$ scattering amplitude with forward kinematics

$$F_{\pi\pi}(\nu) = T^{I=0}_I [0, 2M_\pi(M_\pi + \nu), 2M_\pi(M_\pi - \nu)] = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)$$
Lüscher’s formula or CHPT?

\[ \Delta M^L_{\pi \text{ Lüscher}} = \frac{-3}{16\pi^2\lambda} \int_{-\infty}^{\infty} dy \, F(iy) \, e^{-\sqrt{M^2_\pi + y^2}L} + O(e^{-ML}) \]

\[ \Delta M^L_{\pi \text{ CHPT}} = \frac{1}{4} \xi \tilde{g}_1(\lambda) + O(\xi^2) \]

\[ F(\nu) = -\frac{M^2_\pi}{F^2_\pi} + O(p^4), \quad \tilde{g}_1(\lambda) = \sum_{\tilde{n}^2=1}^{\infty} \frac{4m(|\tilde{n}|)}{|\tilde{n}|\lambda} K_1(|\tilde{n}|\lambda) \]

The leading exponential term at leading order in the chiral expansion is the same in both formulae.

Each formula gives the leading term in two different expansions.
Lüscher’s formula or CHPT?

\[ \Delta M^L_{\pi \text{ Lüscher}} = -\frac{3}{16\pi^2\lambda} \int_{-\infty}^{\infty} dy \ F(iy) \ e^{-\sqrt{M_{\pi}^2+y^2} L} + O(e^{-ML}) \]

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Lüscher’s formula or CHPT?

\[ \Delta M_{\pi}^{L} = \frac{-3}{16\pi^2\lambda} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{M_{\pi}^2+y^2}L} + O(e^{-ML}) \]

\[ \Delta M_{\pi}^{L} = \frac{1}{4} \xi \tilde{g}_1(\lambda) + O(\xi^2) \]
Lüscher’s formula vs CHPT: numerics

$$R_M = \frac{M_\pi L}{M_\pi} - 1$$

GC and S. Dürr 03
Extension of the Lüscher’s Formula
One-loop CHPT corrections

$$\tilde{g}_1(\lambda) = \left(\frac{4\pi}{M_\pi}\right)^2 \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t)|_{t = \vec{x} = 0} = \sum_{\vec{n}^2 = 1}^\infty \frac{4m(|\vec{n}|)}{|\vec{n}| \lambda} K_1(|\vec{n}| \lambda)$$
Extension of the Lüscher’s Formula

One-loop CHPT corrections

\[ \tilde{g}_1(\lambda) = \left( \frac{4\pi}{M_\pi} \right)^2 \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t) \big|_{t = \vec{x} = 0} = \sum_{\vec{n}^2 = 1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}| \lambda} K_1 (|\vec{n}| \lambda) \]

Analogously one can extend the Lüscher’s Formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

\[ M_{\pi, L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{\vec{n}^2 = 1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2 + y^2)L}} \]

The extension does not give all exponentially subleading terms!

but pushes the accuracy of the formula toward $O(e^{-2\lambda})$
Extension of the Lüscher’s Formula

\[ M_{\pi,L} - M_{\pi} = -\frac{1}{32\pi^2\lambda} \sum_{\vec{n}^2=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_{\pi}^2+y^2)L}} \]
Nonleading exp. terms in $M_{\pi, L}$
Nonleading exp. terms in $M_{\pi,L}$

![Graph showing the relationship between $M_{\pi,L}$ and $R_M$ for different values of $L$. The graph includes lines for L = 2 fm, L = 3 fm, and L = 4 fm, with markers for LO, NLO, and NNLO.](image-url)
Two loop effects beyond Lüsher
Diagrams with two propagators in finite volume are not included in Lüsher’s formula
Two loop effects beyond Lüscher
Diagrams with two propagators in finite volume are not included in Lüscher’s formula

However, also diagrams with one propagator in finite volume are not fully included
Two loop effects beyond Lüscher

\[
M_L - M_\infty = \sum_{\vec{n} \neq \vec{0}} \int d^4 \ell \, \Gamma(p, \ell, -\ell, -p) G_L(\ell) e^{i \vec{\ell} \cdot \vec{n} L} \\
G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}
\]
Two loop effects beyond Lüscher

\[ M_L - M_\infty = \sum_{\vec{n} \neq \vec{0}} \int d^4 \ell \ \Gamma(p, \ell, -\ell, -p) G_L(\ell) \]

\[ G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2} \]
The pion mass to two loops

\[ \Delta M_\pi = \text{Lüschers resummed} + \text{cut} + 2 \text{ FV-propagators} \]
The pion mass to two loops

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\[ \Delta M_\pi = \text{Lüscher resummed} + \text{cut} + 2 \text{ FV-propagators} \]
The pion mass to two loops

\[ \Delta M_\pi = \text{Lüscher resummed} + \text{cut} + 2 \text{ FV-propagators} \]

The contributions beyond the resummed Lüscher formula are negligibly small.
Outline

$\pi\pi$ scattering beyond LO
Experimental tests

The $\sigma$ resonance

Finite volume effects
Asymptotic formulae
Resummation of higher exponentials
The pion mass to two loops

Summary
Summary

As an illustration of the effective field theory method I have discussed a few applications:

- the analysis of the $\pi \pi$ scattering amplitude
- the determination of the $\sigma$ resonance parameters
- the calculation of finite volume effects