Finite Temperature Lattice QCD

Introduction

The QCD transition

The transition temperature

The Equation of State

Screening masses

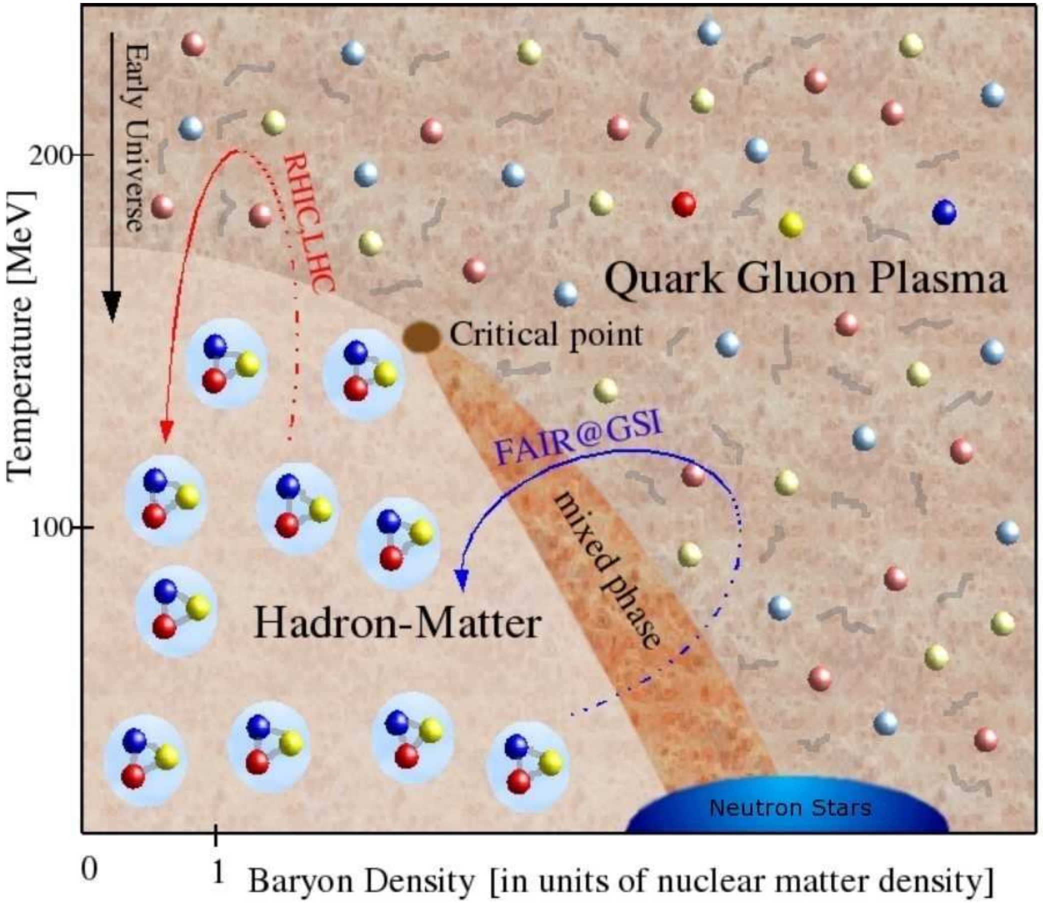
finite density $\longrightarrow$ Ch. Schmidt

topological aspects

$\longrightarrow$ M. Müller-Preußker

Introduction

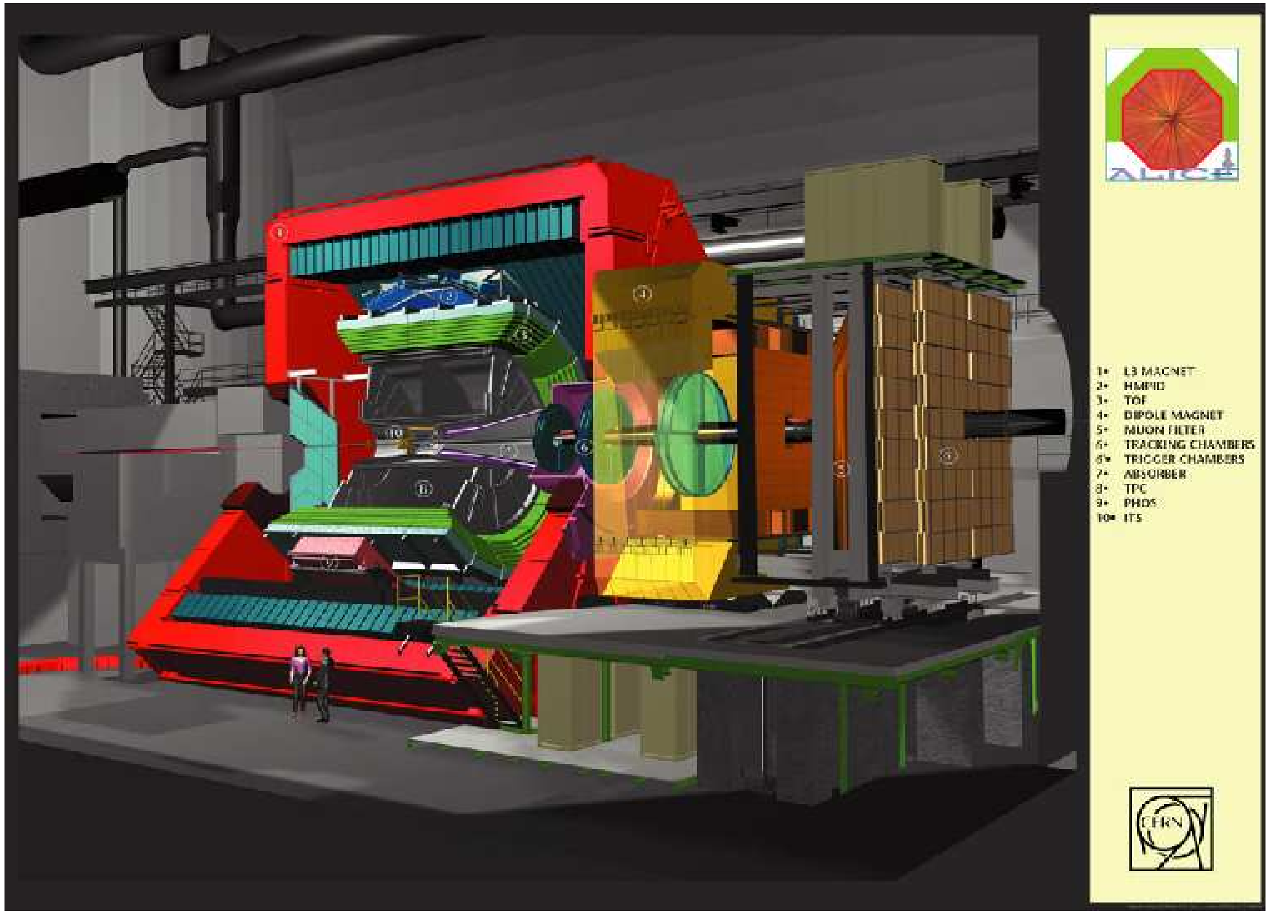
asymptotic freedom $\alpha_s(Q \to \infty) = 0$, at finite T: $Q \sim T$

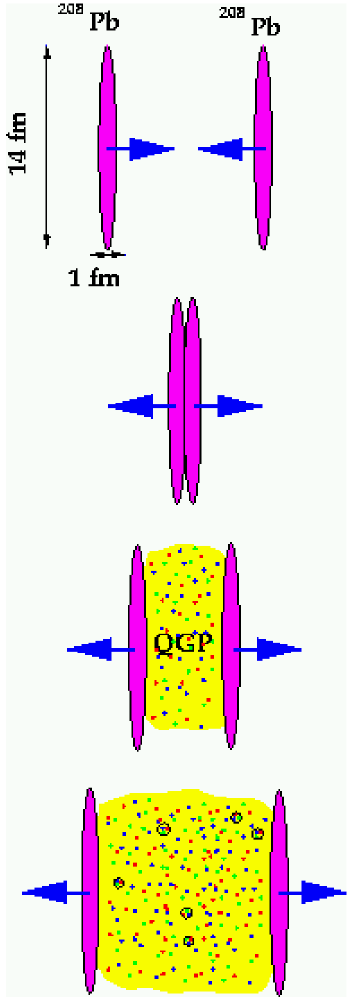


- universe $n_B/n_\gamma \simeq 10^{-9}$ $\rightsquigarrow$ cosmology
- at high density
  - probably rich phase structure (color superconductivity, color-flavor locking ...)
  - neutron stars $\rightsquigarrow$ astrophysics
  - difficult for the lattice (sign problem) $\rightarrow$ lectures by G. Aarts, Ch.Schmidt
- heavy ion collisions (LHC, RHIC, FAIR)

  FAIR: $N_B/S \simeq 0.03$, $\mu_B/T \simeq 2.5$

  RHIC: $N_B/S \simeq 0.003$, $\mu_B/T \simeq 0.25$





– at mid rapidity, baryon density is small

– short equilibration time $\tau_{\text{equi}} \lesssim 1$ fm (strong interaction !)

– expansion is described by hydrodynamics $\longrightarrow$ Equation of State estimates from a simple model: bag model

hadron view

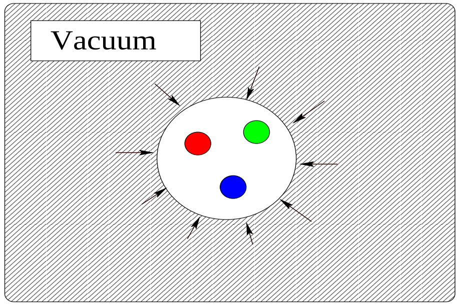
$$p_H = d_H \frac{\pi^2}{90} T^4$$

$$d_H = \#\,\text{d.o.f.} = 3 \quad (\pi^\pm, \pi^0)$$

quark-gluon view

$$p_P = d_P \frac{\pi^2}{90} T^4 - B$$

$$d_P = 2 \cdot 8 + 7/8 \cdot 2 \cdot 2 \cdot 3 \cdot N_F \quad (G, q)$$



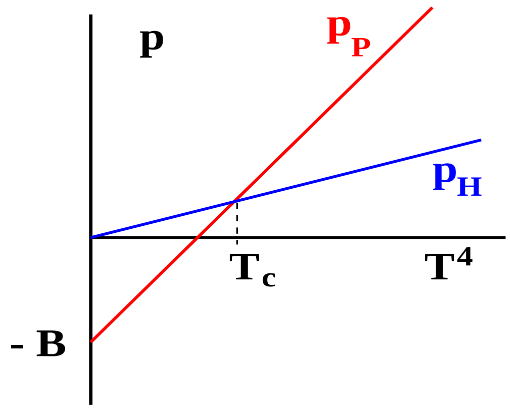
at $T \neq 0$, instead of energy E, free energy

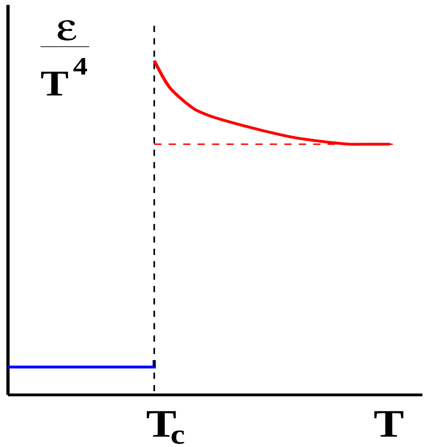
$F = E - TS = -pV$ minimal

$\Rightarrow$ here **1st order phase transition**

since $\epsilon = E/V \sim T^2 \partial(p/T)/\partial T$

has a discontinuity





$$p_H(T_c) = p_P(T_c) \;\Leftrightarrow\; T_c = \left(\frac{90}{\pi^2(d_P - d_H)}\right)^{1/4} B^{1/4} \simeq 0.7 B^{1/4} \simeq 140 \text{ MeV}$$

$$M_H = \frac{16\pi}{3} R^3 B$$

NOTE: we have a scale here, the temperature T

Quantum statistics in equilibrium:

the central quantity is the **partition function** $Z = \mathrm{tr}\left\{\exp(-\hat{H}/T)\right\} = \Sigma_n \langle n| \exp(-\hat{H}/T)|n\rangle$

(at $\mu = 0$)

compare with time evolution operator $\exp(-it\hat{H})$

– (inverse) temperature direction ≡ imaginary time $\tau = it$

– temporal extent limited by $0 \leq \tau \leq 1/T$

– trace requests periodic b.c. in time for bosons $\phi(\tau = 1/T) = \phi(0)$ (commutators)
anti-periodic b.c. in time for fermions $\psi(\tau = 1/T) = -\psi(0)$ (anti-commutators)

→ Feynman path integral

$$Z(T, V) = \int_{\text{periodic}} \mathcal{D}\phi(\vec{x}, \tau) \exp\left\{- \int_0^{1/T} d\tau \int_V d^3\vec{x}\, \mathcal{L}_E[\phi(\vec{x}, \tau)]\right\}$$

where $\mathcal{L}_E$ is the euclidean Lagrange density

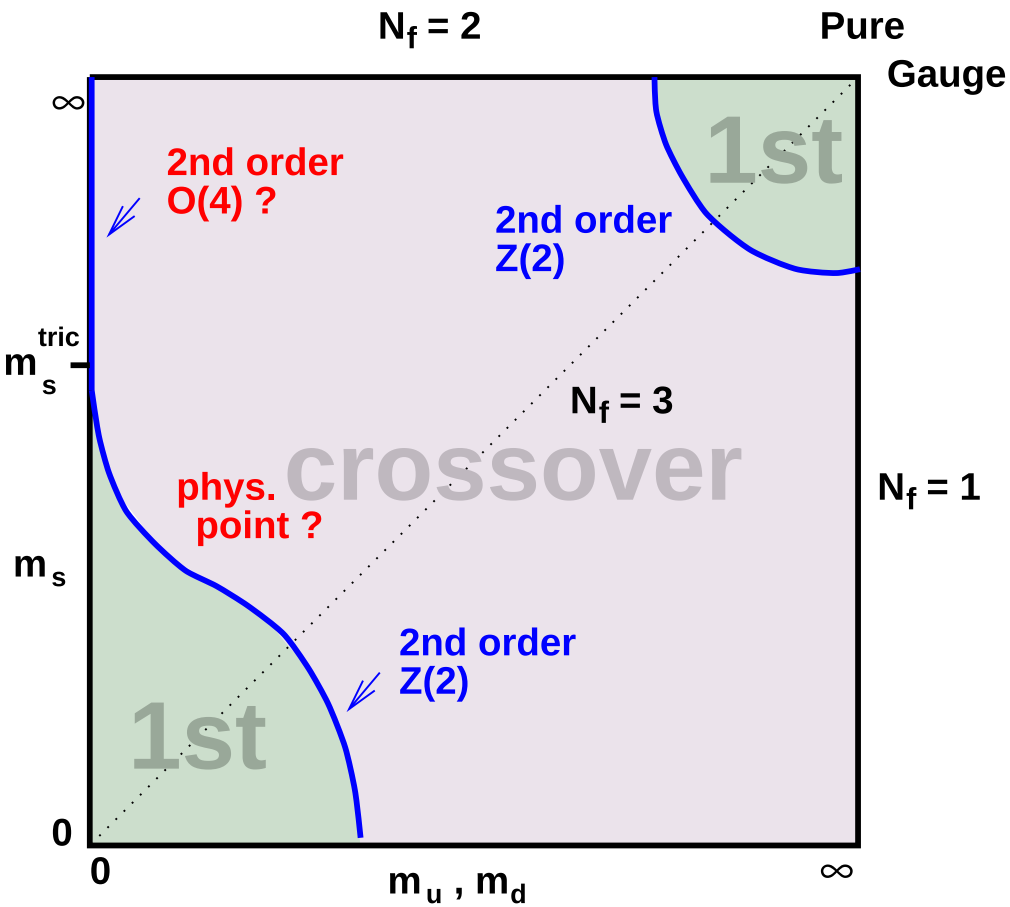
apply standard thermodynamic relations, e.g. energy density $\epsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T}\Big|_V$

specific heat $c_V = \frac{1}{VT^2}\frac{\partial^2 \ln Z}{\partial(1/T)^2}\Big|_V$

on the **lattice** as at zero temperature, but $a(\beta)T = \frac{1}{N_\tau}$ and $LT = \frac{N_\sigma}{N_\tau}$

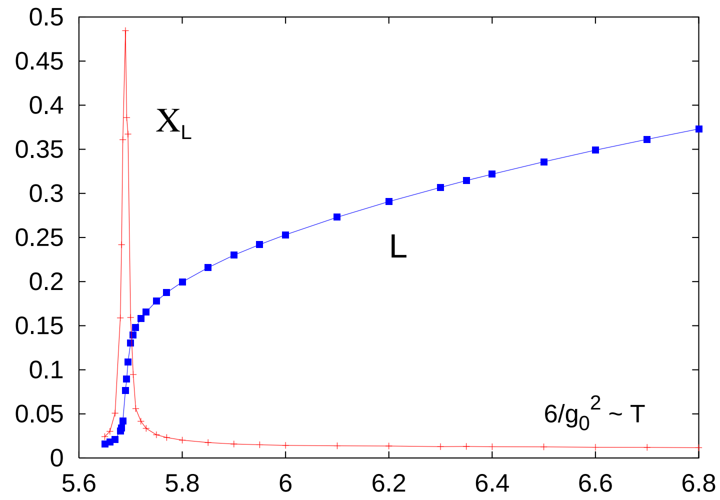
- charm at $m_c/T_c = \mathcal{O}(10)$ presumably too heavy for the dynamics at T_c
- leaves light $m_u \simeq m_d \equiv m_l$ and strange m_s quark

- the nature of the transition is expected to depend strongly on m_l and m_s :



- the action is invariant under the global transformation $U_\tau(\vec{x}, \tau = N_\tau) \rightarrow zU_\tau(\vec{x}, \tau = N_\tau)$
  with $z \,\epsilon\, Z(3) = \{\exp(in2\pi/3), n = 0, 1, 2\}$, the center group of $SU(3)$
- the Polyakov loop $L(\vec{x}) = tr\, \Pi_\tau\, U_\tau(\vec{x}, \tau)$ is not

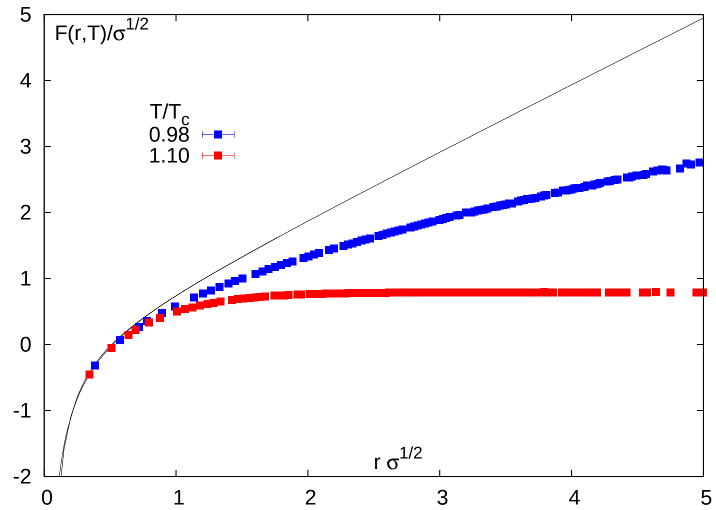
$\Rightarrow$ $\langle L(\vec{x})\rangle$ is an order parameter for the spontaneous breakdown of the $Z(3)$ symmetry



– the rise of L signals a phase transition

– the susceptibility $\chi_L \sim \langle L^2\rangle - \langle L\rangle^2$
measures thermal fluctuations
which grow large at T_c

corrolary $\langle L(r)L(0)\rangle \sim \exp(-F_{Q\bar{Q}}(r)/T)$



measures the free energy of static quarks at separation r

$$F_{Q\bar{Q}}(r \rightarrow \infty) \rightarrow \infty \quad \text{for } T < T_c$$

$$F_{Q\bar{Q}}(r \rightarrow \infty) \rightarrow \text{finite} \quad \text{for } T > T_c$$

i.e. in pure Yang-Mills, $\langle L\rangle$ detects the confinement - deconfinement transition

with fermions, $\langle L\rangle$ is not an order parameter, $Z(3)$ broken by fermion action

- at and in the vicinity of a phase transition thermal fluctuations grow large or diverge
- i.e. correlation lengths ξ become large or diverge
- in this limit microscopic interaction details don't matter $\rightsquigarrow$ series of "spin blockings" and RG flow
- what counts are global symmetries and the dimension

a universality class consists of all those models which flow into the same critical fixed point under repeated renormalization group transformations

$\longrightarrow$ hence the same critical behavior in terms of scaling with universal critical exponents and certain other universal quantities/functions

- equilibrium QCD (Yang-Mills) at high T is expected to be 3 dimensional
  generically: $\phi(\tau) = \sum_n \phi(\omega_n) \exp(i\omega_n \tau)$ with $\omega_n = 2\pi T n \rightsquigarrow$ only static modes ($n = 0$) survive

$\Rightarrow$ compare with (classical) 3d systems with the same global symmetries,

- pure $SU(3)$ Yang-Mills: the same universality class as the 3d $Z(3)$ (3-state) Potts spin model
  (weakly) 1st order

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* the theory behind this is Wilson's renormalization group
there is a huge condensend matter literature
for introductory lectures see e.g. J. Engels at
`http://www.physik.uni-bielefeld.de/igs/schools/Fall2006/phase.html`

the free energy density $f = -\frac{T}{V} \ln Z$ consists of a regular and a singular part, $f = f_{\rm reg} + f_{sing}$, where f_{sing} develops the singularities which drive the transition

scaling: in the vicinity of a critical point $(T_c, H = 0)$, under an arbitrary scale change b

$$f_{sing}(t,h) = b^{-d} f_{sing}(b^{y_t} t, b^{y_h} h)$$

where y_t, y_h are universal critical exponents

$t = \frac{1}{t_0}\frac{T-T_c}{T_c}$ (normalized) reduced temperature

$h = \frac{1}{h_0} H$ (normalized) external symmetry breaking, e.g. magnetic field, m_q in QCD

t_0, h_0 non-universal metric factors i.e. need to be adjusted for each model

$$f_{sing}(t,h) = b^{-d} f_{sing}(b^{y_t} t, b^{y_h} h)$$

- choose $b = |t|^{-1/y_t}$: $\quad f_{sing}(t,h) = |t|^{d/y_t} f_{sing}(\pm 1, |t|^{-y_h/y_t} h)$

  $\Rightarrow$ for magnetization $M = -\partial f/\partial H$ find at $t < 0, H = 0$: $\quad M \sim (-t)^{\beta}$ with $\beta = (d - y_h)/y_t$

  $\Rightarrow$ for susceptibility $\chi = \partial M/\partial H \sim \langle (\delta M)^2 \rangle$ find at $H = 0$: $\quad \chi \sim |t|^{-\gamma}$ with $\gamma = (2y_h - d)/y_t$

- choose $b = |h|^{-1/y_h}$: $\quad f_{sing}(t,h) = |h|^{d/y_h} f_{sing}(t|h|^{-y_t/y_h}, \pm 1)$

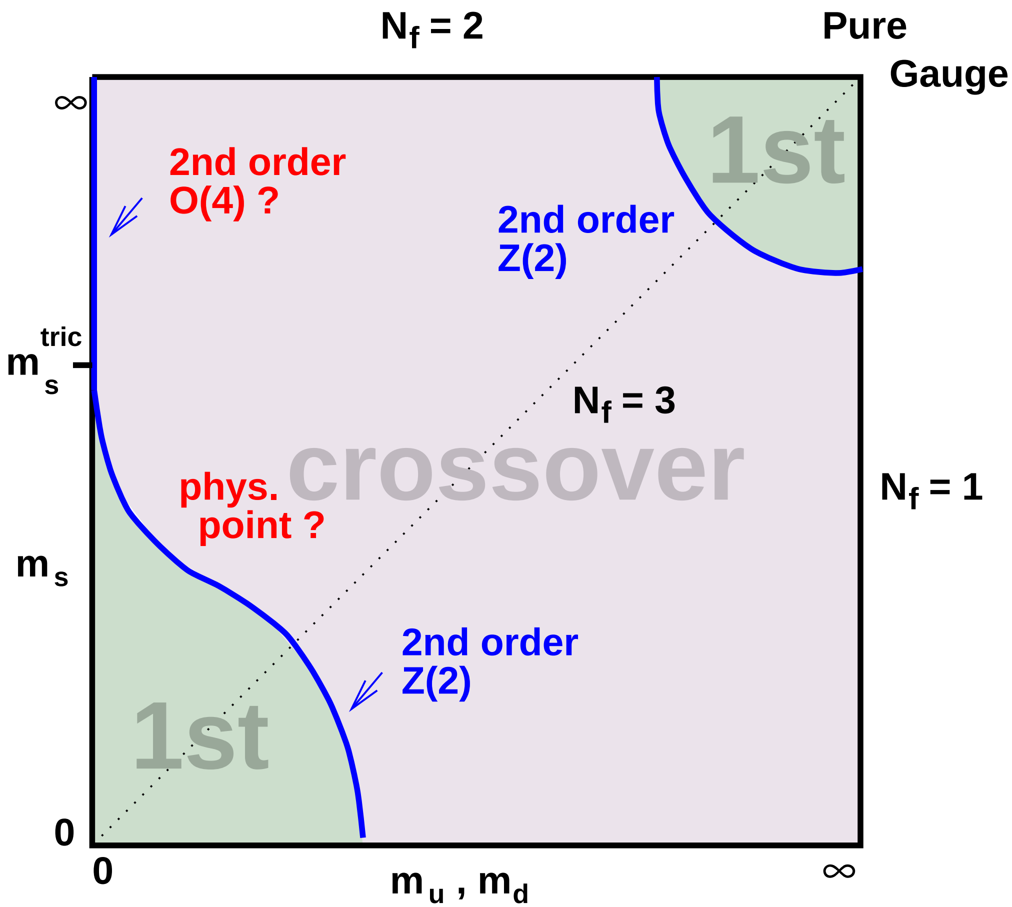
  $\Rightarrow$ for magnetization find at $t = 0$: $\quad M \sim |h|^{1/\delta}$ with $1/\delta = d/y_h - 1$

  $\Rightarrow$ note that $\quad \beta + \gamma = y_h/y_t = \beta\delta \quad$ "hyperscaling relation"

  $\Rightarrow$ at $t \simeq 0$: $\quad M(t,h) = |h|^{1/\delta} f_G(z)$ with $f_G(z) = f_{sing}(z) + z f'_{sing}(z)$ universal and $z = t|h|^{-1/\beta\delta}$

- similarly, e.g. the so-called Binder cumulant $B_4 = \dfrac{\langle (\delta M)^4 \rangle}{\langle (\delta M)^2 \rangle^2}$ with $\delta M = M - \langle M \rangle$ is a universal number

- keep in mind though that corrections to scaling might be important, and there are also the regular terms conjectured landscape of transitions



arising from the chiral symmetry of QCD at $m_q = 0$

$$SU_L(N_f) \times SU_R(N_f) \times U_A(1)$$

– its spontaneous breakdown at low T

– its restoration at high T

– and its modelling by σ models (universality)

– note: $U_A(1)$ broken by triangle anomaly but could be effectively restored at high T (no topologically non-trivial configurations

$N_F = 2$ Wilczek, Pisarski

– if $U_A(1)$ effectively restored then 1st order

– if 2nd order then in $SU(2) \times SU(2) \simeq O(4)$ class

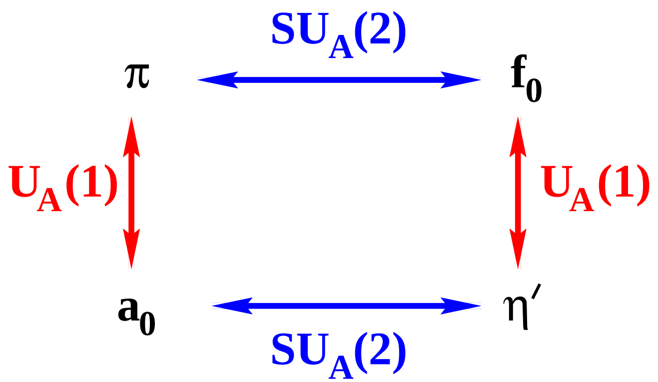
$N_F = 3$

– 1st order Wilczek, Pisarski

– at critical end point: $Z(2)$ class Gavin et al.

$N_F = 2$

- conflicting results for critical behavior of $M = \langle \bar{q}q \rangle$, all from coarse lattices with un-improved actions
  - Wilson M scales $\sim O(4)$ in $m \sim H$ at $T > T_c$ [Iwasaki et al.]
  - staggered χ_m, χ_t do not scale $\sim O(4)$ in m [Karsch,EL; JLQCD; MILC]
  - staggered M scales $\sim O(4)$ in L [Engels et al.]
  - staggered c_V scales as 1st order [Di Giacomo et al.]
  - staggered M is as in $O(2)$ at finite L [Kogut, Sinclair]
- $U_A(1)$:
  - if effectively restored, then degeneracy in 2-point function (mass spectrum) [Shuryak; Cohen et al;, ...]



there are indications that $U_A(1)$ is not restored

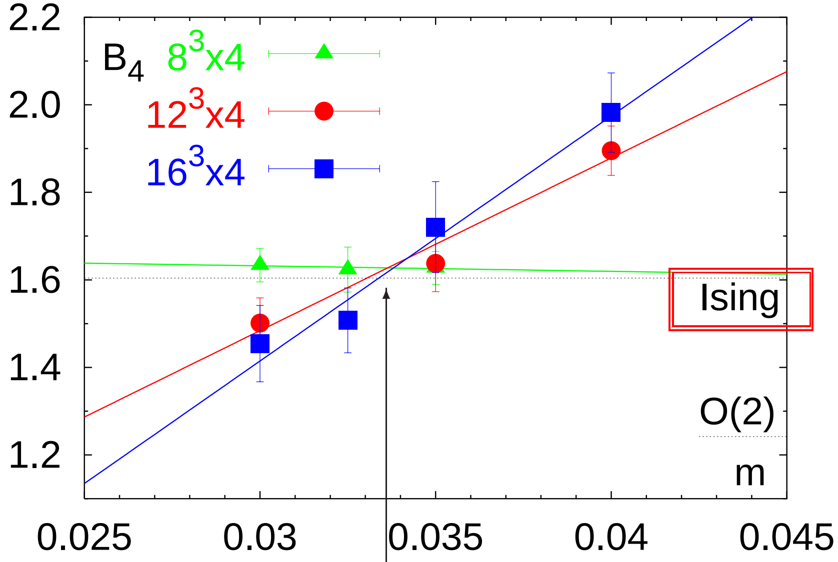
- current projects with Wilson-type quarks [QCDSF-DIK,tmfT,WHOT] have not yet addressed this question

$N_F = 3$

Binder cumulant B_4

- intersection for various V yields critical value of m
- value of B_4 is universal
- corrections from V finite and 'order parameter not matched correctly'

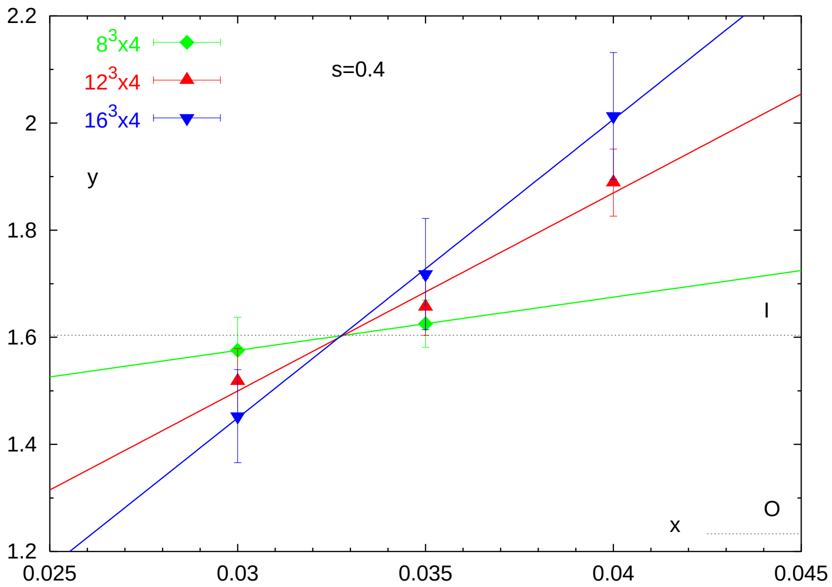
standard action



$m_c \simeq 4\, m_u^{phys} \Leftrightarrow m_{PS} \simeq 290$ MeV

[Bielefeld; deForcrand,Philipsen]

but: – m_c not universal
– improved action: $m_{PS} \simeq 70$ MeV



magnetization-like order parameter $\mathcal{M}$

not identical with chiral condensate $\langle \bar{q} q \rangle$

(chiral symmetry broken by $m_q \neq 0$ anyway)

$$\mathcal{M} = \langle \bar{q} q \rangle \; + \; s\, S$$

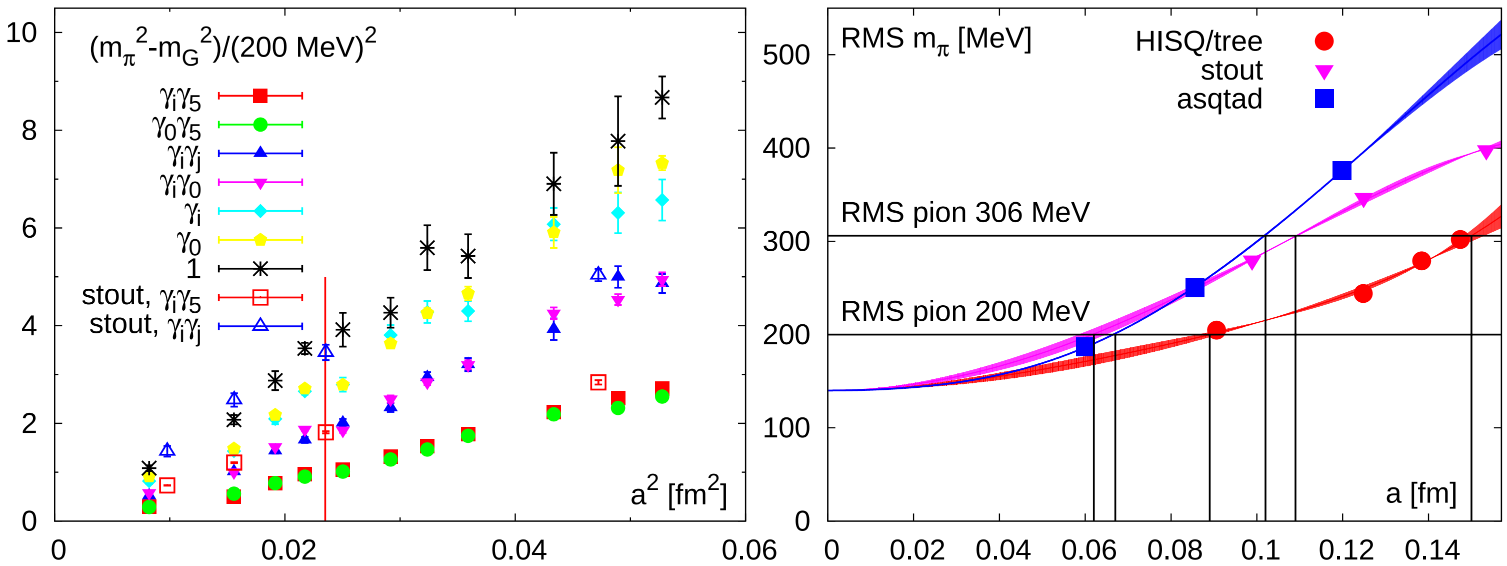
$N_F = 2 + 1$

recent projects all use improved staggered quarks (p4fat3, asqtad, stout, HISQ) *

at finite lattice spacing: taste violations $\mathcal{O}(a^2\alpha_s)$

$\Rightarrow$ the pion 16-multiplet $\longrightarrow$ 8 different representations, $stagg\chi PT$: 5 different reps.

only 1 pion is true Goldstone boson also at $a \neq 0$: $m^2_{\gamma_5} = m^2_G \sim m_q$ in chiral limit



$m_\pi^{RMS} = \sqrt{\sum_{i=1}^{16} m^2_{\pi,i}}$

$\Rightarrow$ the expected $O(4)$ for 2 light flavors in the limit $m_l \longrightarrow 0$ goes over into $O(2)$ at $a \neq 0$

* first results from DWF fermions, not yet on the nature of the transition [Christ, Karsch et al.]
first results from improved Wilson quarks, mainly equation of state [WHOT]

in $O(N)$ spin models

"magnetic equation of state" $M(t,h) = h^{1/\delta} f_G(z)$ with $z = t/h^{1/\beta\delta}$ and f_G universal

$\Rightarrow$ $M \sim (-t)^\beta$ at $h = 0, t < 0$

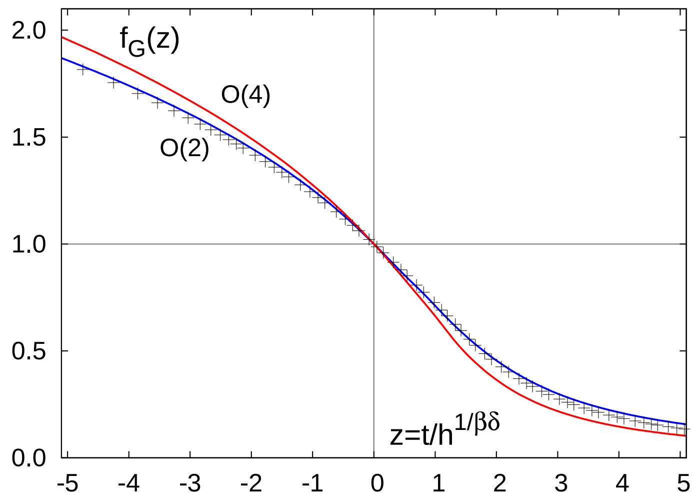
$M \sim h^{1/\delta}$ at $t = 0$

$M \simeq h^{1/\delta}(-z)^\beta[1 + \tilde{c}_2\beta\,(-z)^{-\beta\delta/2} + \ldots] = M(t,0) + c(t)\sqrt{h} + \ldots$ at $z \to -\infty$ Zia, Wallace

$\sqrt{h}$ behavior is known as Goldstone effect in d=3 ($\sim h \ln h$ in d=4) Gasser, Leutwyler, Hasenfratz

critical exponents β, δ and $f_G(z)$ known from spin model simulations [Engels et al.]

N	β	δ	$\tilde{c}_2$	z_p
2	0.349	4.7798	0.592(10)	1.56(10)
4	0.380	4.824(9)	0.666(6)	1.33(5)



the chiral condensate $\langle \bar{q}q \rangle \sim \langle \mathrm{Tr} D_q^{-1} \rangle$

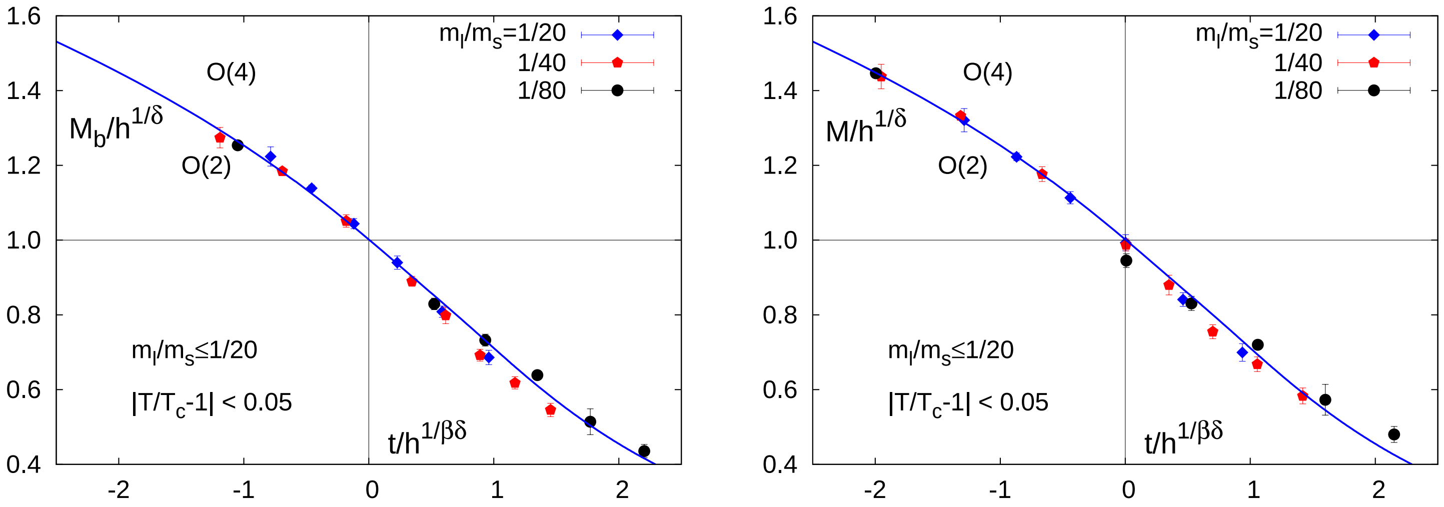
is the order parameter in the limit $m_l \to 0$, i.e. takes the role of the magnetization M

complications: – needs multiplicative renormalization $\langle \bar{q}q \rangle^R = Z_m^{-1} \langle \bar{q}q \rangle$
where Z_m is mass renormalization $m_q^R = Z_m m_q$

– away from chiral limit receives additive power divergence $\langle \bar{q}q \rangle(m_q) = \langle \bar{q}q \rangle(0) + m_q/a^2$

$\rightsquigarrow$ subtracted condensate $M = m_s \left[\langle \bar{l}l \rangle - \frac{m_l}{m_s} \langle \bar{s}s \rangle \right]$, difference to $M_b = m_s \langle \bar{l}l \rangle$ small and vanishing with $m_l \to 0$

fit data to the scaling function f_G with 3 unknowns: t_0, h_0, the metric constants
T_c, the critical temperature in the chiral limit



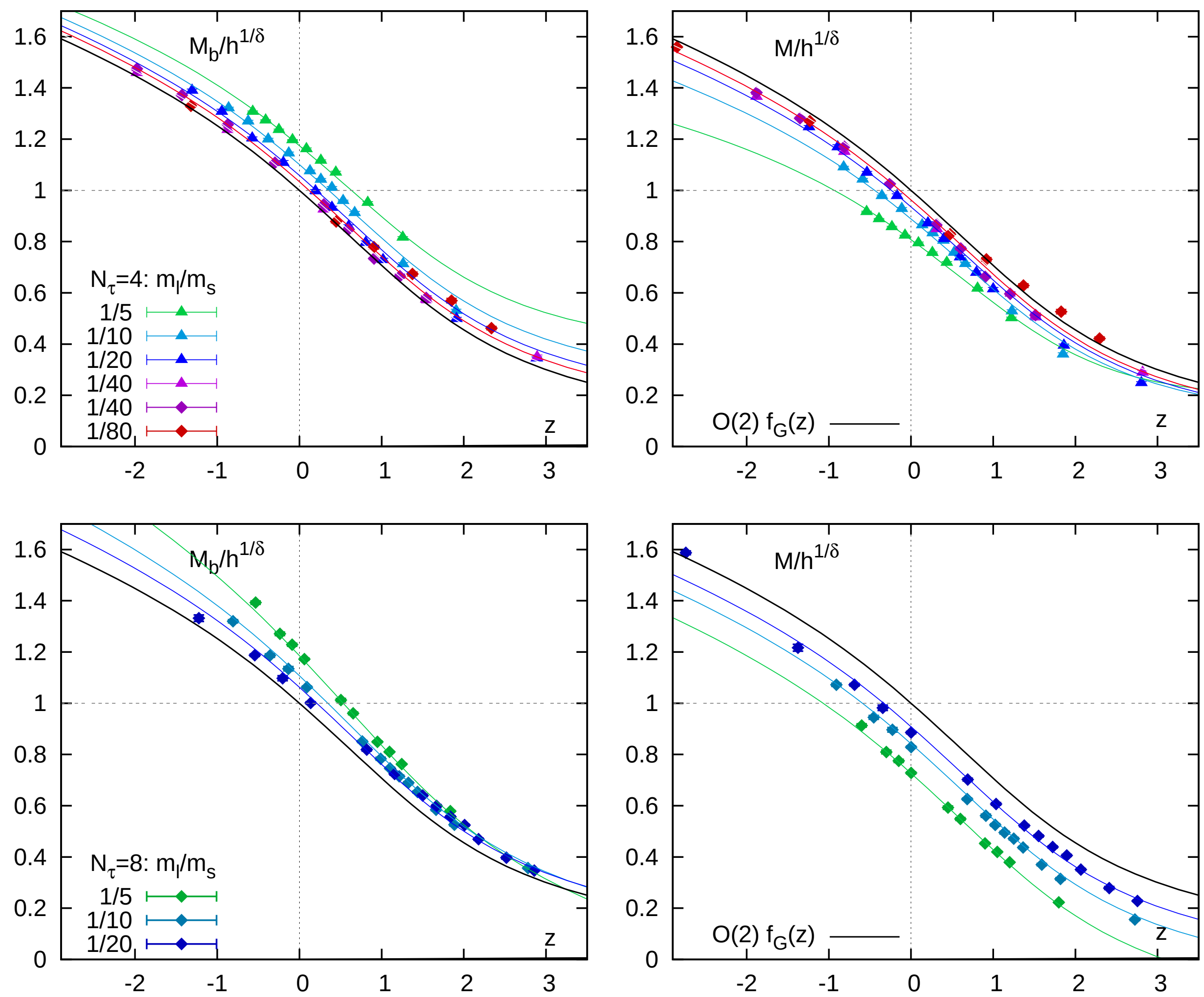
data is well described by $O(2)$ scaling note: $(m_l/m_s)^{phys} \simeq 1/27$

at $m_l/m_s \geq 1/10$ corrections to scaling need to be taken into account

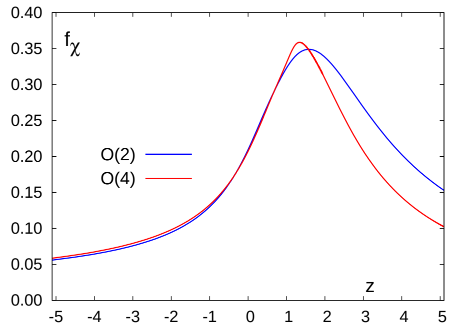
$$M(t,h) = h^{1/\delta} f_G(t/h^{1/\beta\delta}) + a_t t h + b_1 h + b_3 h^3 + b_5 h^5$$

$N_\tau = 4$

$N_\tau = 8$



$b_1 \simeq 0$ for (subtracted) M



the chiral susceptibility χ_M is now fixed completely

$$\chi_M(t,h) = \frac{\partial M}{\partial H} = \frac{1}{h_0} h^{1/\delta - 1} f_\chi(z) \quad \text{with} \quad f_\chi(z) = \frac{1}{\delta}\left(f_G(z) - \frac{z}{\beta} f_G'(z) \right)$$

v.v., reconstruct scaling fct f_χ from the data on χ_M: $f_\chi = \chi_M \, h_0 \, h^{1-1/\delta}$

recall, in QCD $\langle \bar{q}q\rangle(m_q) = \langle \bar{q}q\rangle(0) + \sqrt{m_q} + m_q/a^2 + \cdots$

moreover, in QCD $\chi_M = 2\chi_{dis} + \chi_{con}$ with

$$\chi_{\rm dis} = \frac{1}{16 N_\sigma^3 N_\tau} \left\{ \langle (\mathrm{Tr} D_l^{-1})^2 \rangle - \langle \mathrm{Tr} D_l^{-1} \rangle^2 \right\}$$ – measures fluctuations of the chiral condensate

$$\chi_{\rm con} = \frac{1}{4} \sum_x \langle \, D_l^{-1}(x,0) D_l^{-1}(0,x) \, \rangle$$ – arises from the explicit m_q dependence of $D_q = D + m_q$

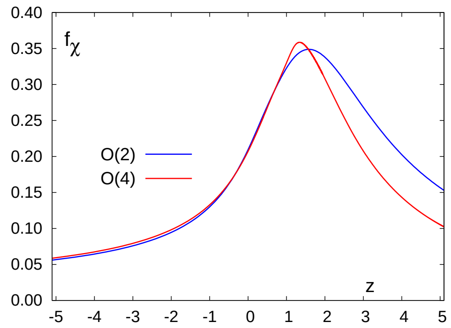
– integral over quark-line connected correlator of the scalar isovector a_0

$\Rightarrow$ quadratically UV divergent $\sim a^{-2}$

– in $O(N)\chi PT$ it can be shown that χ_{con} vanishes for $N = 4$ but not for $N = 2$ [Smilga, Stern, Verbaarsc

– in $stagg\chi PT$ taste violations lead to IR Goldstone contribution to $\chi_{con} \sim 1/\sqrt{m_q}$ [DeTar et al.]

– χ_{con} does not contribute to f_χ in the chiral limit because of $h^{1-1/\delta}$ and $\delta > 2$



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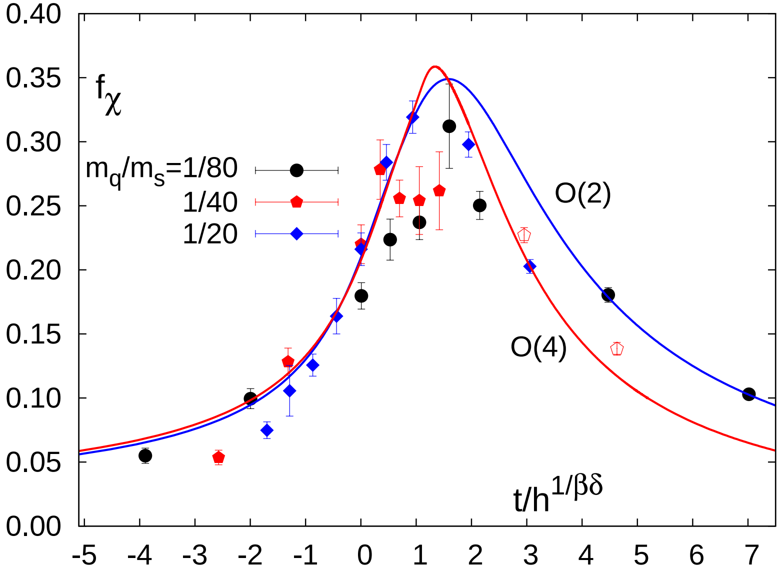
v.v., reconstruct scaling fct f_χ from the data on χ_M: $f_\chi = \chi_M \, h_0 \, h^{1-1/\delta}$

and compare with f_χ from the $O(N)$ models:



p4fat3

peaks in χ_M at pseudocritical temperatures T_p are caused by the peak in f_χ at $z = z_p$

$$\frac{t_p}{h^{1/\beta\delta}} = \frac{\frac{1}{t_0}\frac{T_p - T_c}{T_c}}{\left(\frac{H}{h_0}\right)^{1/\beta\delta}} = z_p \quad \Rightarrow \quad \frac{T_p(H) - T_c}{T_c} = z_p \, H^{1/\beta\delta} \, \frac{t_0}{h_0^{1/\beta\delta}}$$

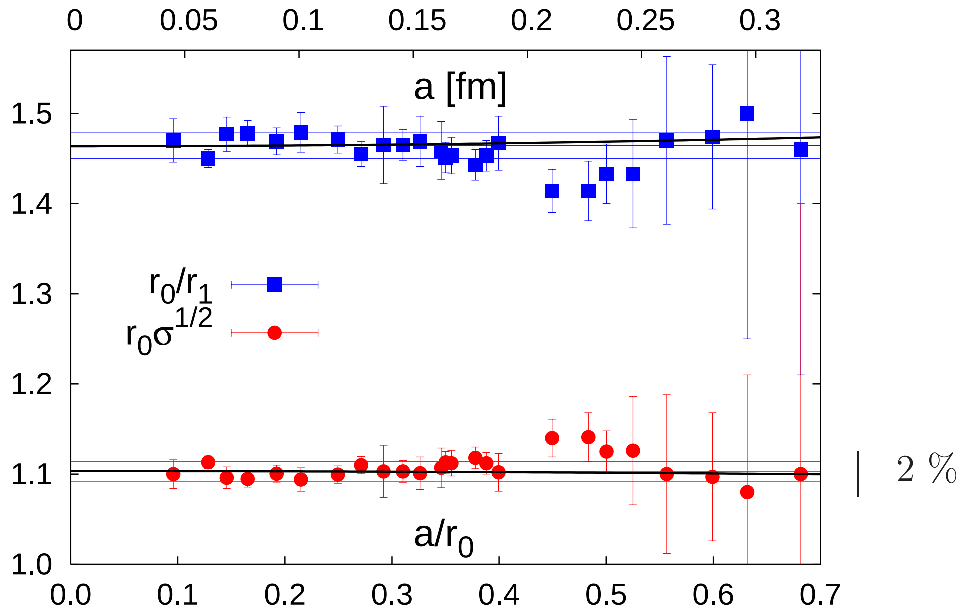
$\longrightarrow$ predict T_p from $\langle \bar{q} q \rangle$

The transition temperature

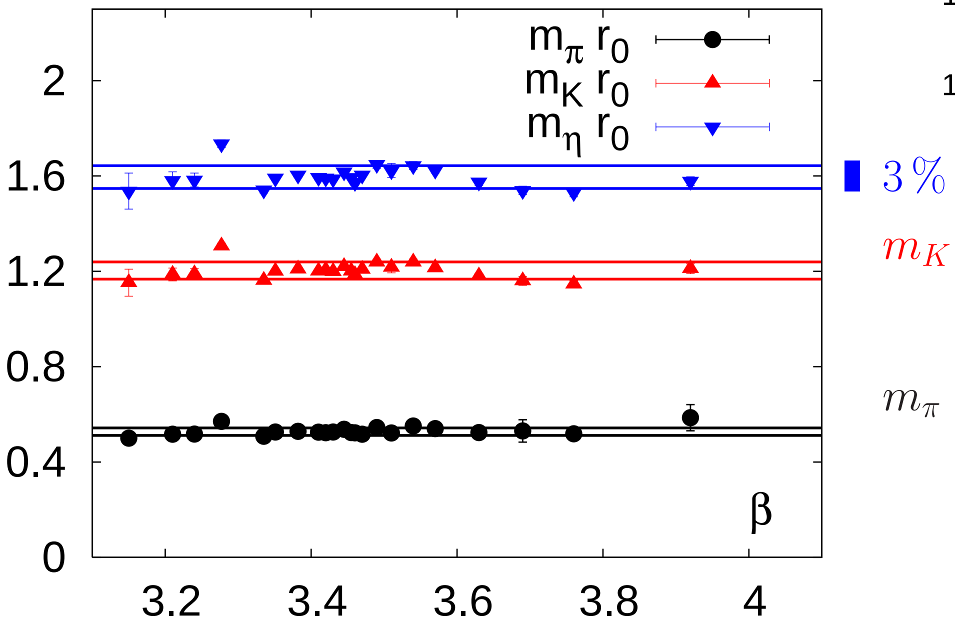
- ⋆ scan through $\beta \longrightarrow a(\beta) \longrightarrow T = 1/N_\tau a(\beta)$
- ⋆ T = 0 scale taken from $\Upsilon\, 2S - 1S$ splitting [A. Gray et al.] via the heavy quark potential $V(r)$

$$r_{0,1}/a \quad \text{from} \quad r^2 \frac{dV(r)}{dr}\bigg|_{r=r_{0,1}} = 1.65(1.0)$$

for absolute values (in MeV) we use $r_0 = 0.469(7)$ fm [A. Gray et al.]



- ⋆ fine tune $\hat{m}_{l,s}(\beta)$ such that $m_{\pi,K} = const$



$m_K \simeq m_K^{\text{phys}}$

$m_\pi \simeq 220$ MeV

plots for p4fat3 action at $m_l/m_s = 0.1$
similar for other actions, different m_l/m_s
may take other scale setting quantity e.g. f_K

⋆ investigate quantities sensitive to the transition

renormalized chiral condensate

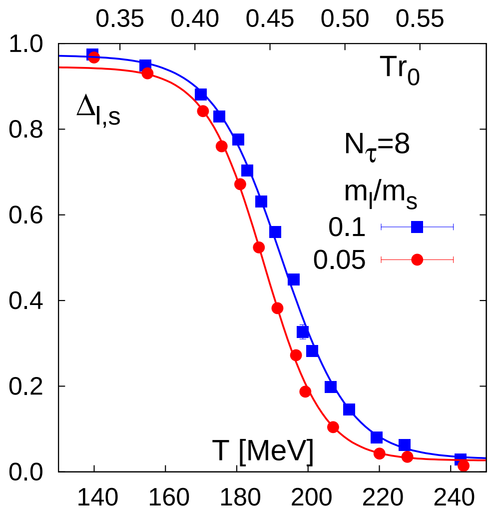
$$\Delta_{l,s}(T) = \frac{\langle\bar\psi\psi\rangle_{l,T} - \frac{m_l}{m_s}\langle\bar\psi\psi\rangle_{s,T}}{\langle\bar\psi\psi\rangle_{l,0} - \frac{m_l}{m_s}\langle\bar\psi\psi\rangle_{s,0}}$$

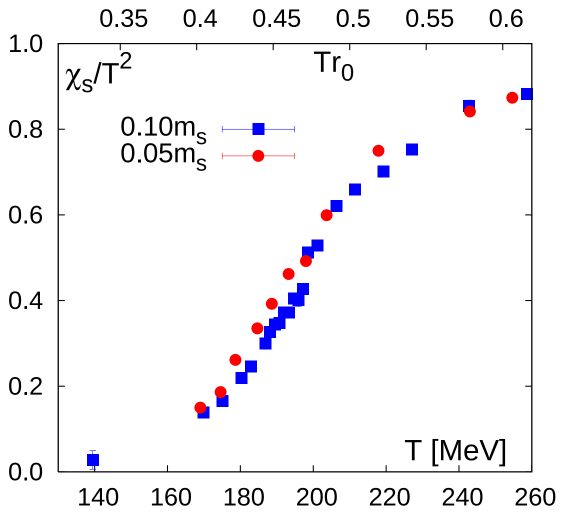
subtracting power-law additive and cancelling multiplicative UV divergencies

strange number susceptibility

$$\chi_s(T)/T^2 \sim \langle n_s^2\rangle - \langle n_s\rangle^2$$

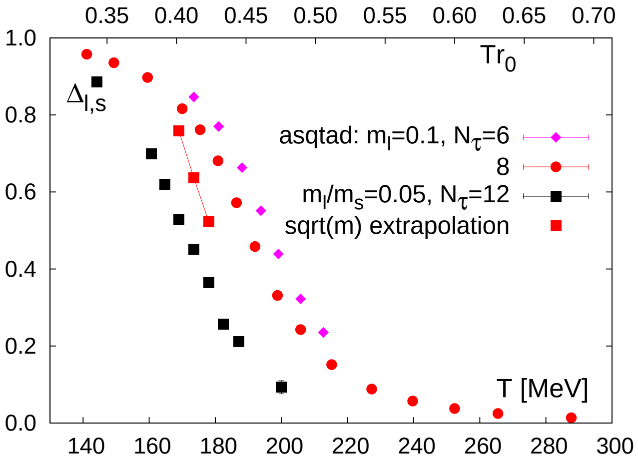
measures strangeness fluctuations indicative of deconfinement and depends on f_{sing}

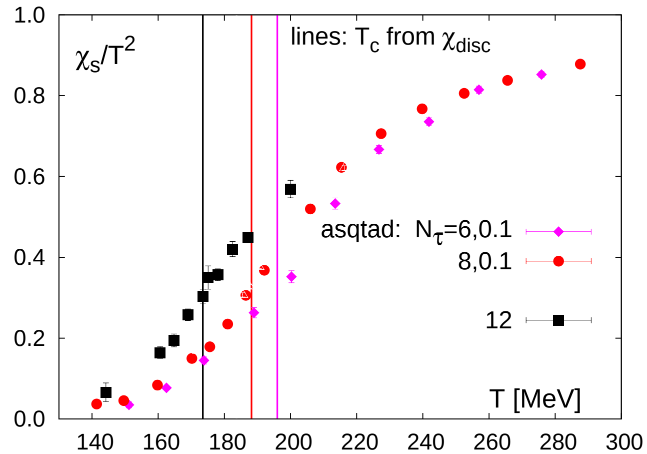


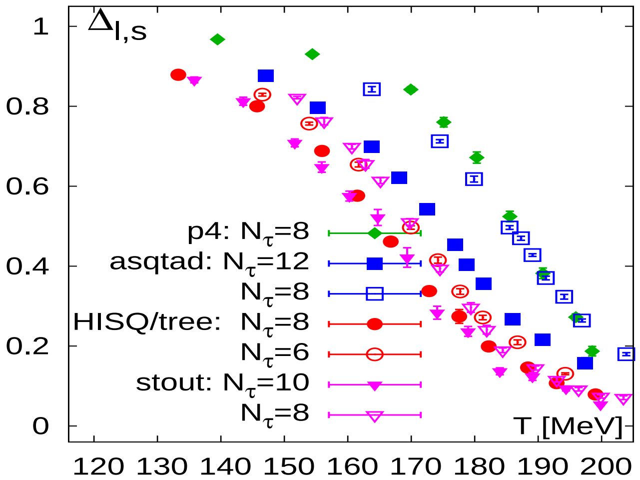


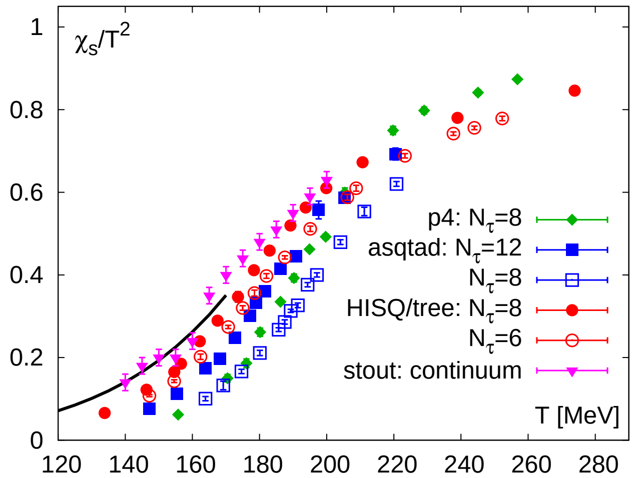
p4fat3 action

location of (rapid) change in both variables moves to lower T when m_l is decreased similarly, location of (rapid) change in both variables moves to lower T when a is decreased





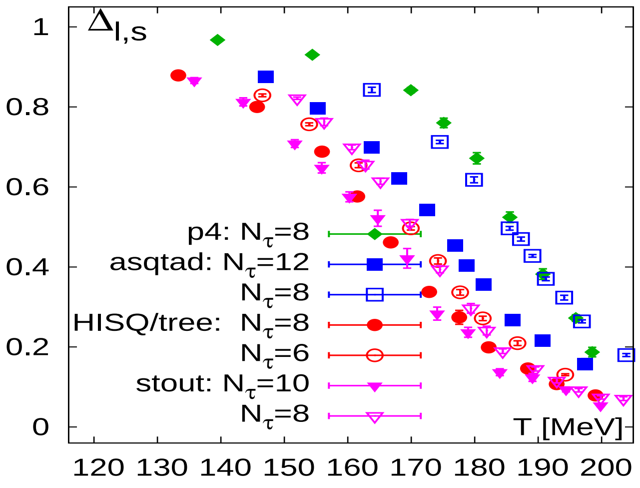




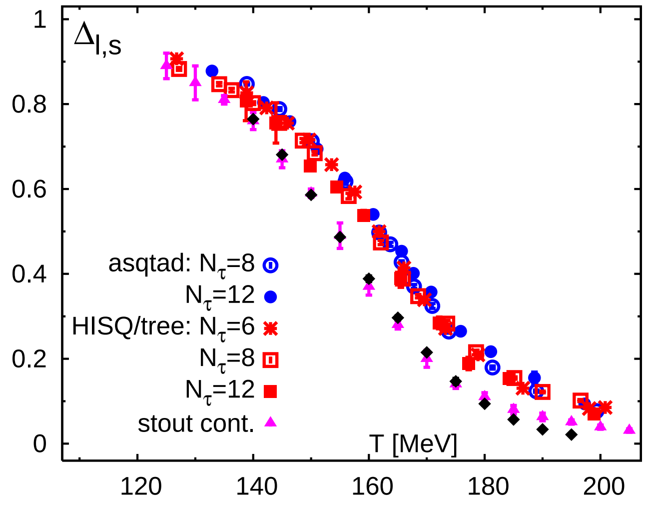
difficult to identify a transition temperature e.g. by locating an inflection point:

curvature = 2nd derivative changes sign $\Rightarrow$ 1st one has maximum situation may look better when a different quantity is used to set the T scale

$r_{0,1}$ scale



f_K scale

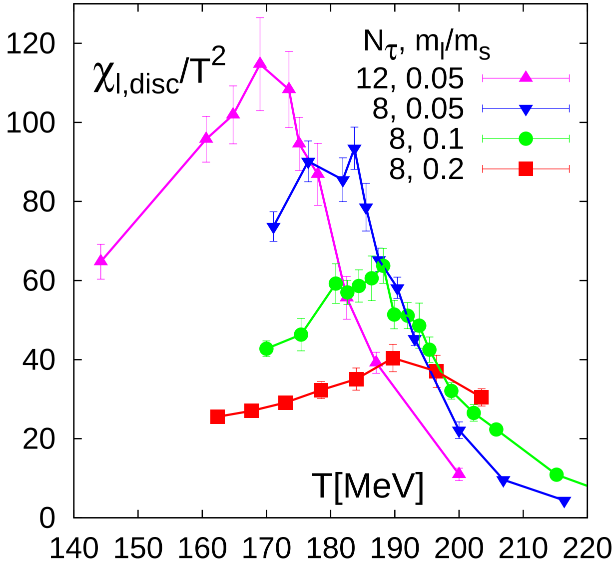


explanation: f_K is affected by taste violations roughly in the same way as $\Delta_{l,s}$

r_i i.e. static quark potential much less sensitive $\rightarrow$ in principle more reliable

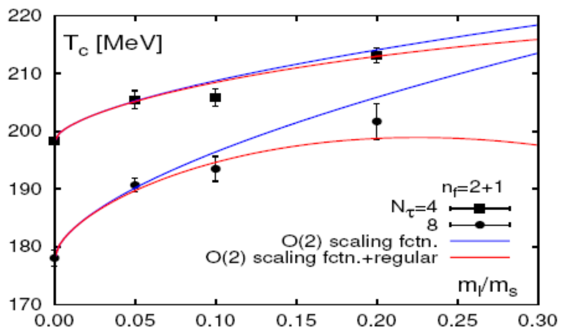
in any case, a continuum extrapolation is needed, and a better way to locate the transition

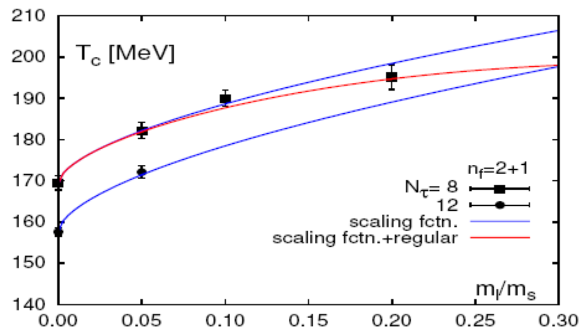
$\rightsquigarrow$ determine $T_p(m_l, N_\tau)$ from the location of peaks of the susceptibility χ_{dis}



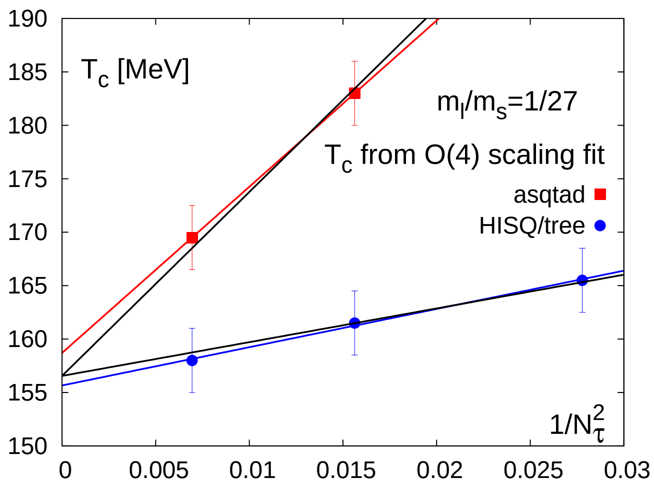
asqtad action

and compare with predictions from $O(N)$ scaling fits to $\langle \bar{q}q \rangle$





extrapolation to continuum limit at physical mass $m_l/m_s = 1/27$



$T_c = 157$ (4 stat) (3 extrapolation) (1 scale) MeV — asqtad, HISQ

$T_c = 147$ (2 stat) (3 syst) MeV — stout χ_M

$T_c = 155$ (3 stat) (3 syst) MeV — stout $\Delta_{l,s}$