Introduction to Lattice Perturbation Theory

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Introduction

A spacetime lattice acts as a nonperturbative regularization: the lattice spacing a induces a momentum cutoff of order $1/a$

The other known regularizations (like dimensional regularization or Pauli-Villars) can only be defined order by order in perturbation theory

Moreover, the lattice regularization is not tied to any specific approximation

It allows calculations from first principles with numerical and analytical methods – no models need to be introduced, or additional parameters

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It allows calculations from first principles with numerical and analytical methods – no models need to be introduced, or additional parameters

Thus: the lattice machinery is introduced as a nonperturbative regularization scheme which enables nonperturbative computations

The lattice can probe the long-distance physics, which is otherwise unaccessible to investigations which use continuum QCD

Precisely for the study of low-energy nonperturbative phenomena the lattice was introduced by Wilson, who went on to prove quark confinement in the strong coupling regime

The words " lattice " and " perturbation theory " might then sound like a contradiction . . .

Introduction

. . . but this is not the case!

Perturbative calculations on the lattice are important, and for many reasons, both conceptual and practical

There are many instances where lattice perturbative calculations are useful – and in some cases even necessary

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Perturbation theory $\rightarrow$ weak coupling regime of the lattice

In principle all known perturbative results of continuum QED and QCD can also be reproduced using a lattice regularization instead of the more popular ones

However, calculating in such a way the corrections to the magnetic moment of the muon (to make an example) would be quite laborious

In most cases regularizations like Pauli-Villars or dimensional regularization are more suited and much easier to employ

Main virtue of the lattice regularization: to carry out nonperturbative investigations – which often need some perturbative calculations to be properly interpreted

Introduction

Main topics of these introductory lectures:

- Importance of lattice perturbative calculations
- Renormalization of operators *(matching to the continuum)*
- Feynman rules for the Wilson discretization of QCD
- Some technical issues
- Hypercubic group
- Mixing of operators
- Divergent integrals
- Algebraic reduction of integrals

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We will understand (among others):

- why perturbation theory on the lattice?
- in what lattice perturbation theory is different from continuum perturbation theory?
- what are the main complications?
- what special techniques are needed?

Motivation

Perturbation theory: essential aspect of computations on the lattice, especially for investigating the behavior of lattice theories near the continuum limit

Particularly important in connecting the outcome of Monte Carlo simulations to continuum physical numbers

This is called the matching to the continuum physical theory, and it involves calculations of renormalization factors of lattice matrix elements

Also needed: renormalization of the bare parameters of the Lagrangian, like coupling constants and masses

The precise knowledge of the renormalization of the strong coupling constant is essential for the determination of the Λ parameter of lattice QCD and its relation to its continuum counterpart, Λ_{QCD}

Every lattice action defines a different regularization scheme – and in principle there is a different Λ parameter for each lattice action

A Λ parameter in a given scheme specifies the value of the coupling constant in that scheme for any given scale μ, and all dimensionful quantities will be proportional to Λ

Motivation

Since the Λ parameter depends on the scheme, one has to compute the ratio $\Lambda_{lat}/\Lambda_{cont}$ for every lattice action

For example, for the pure gauge Wilson action one finds

$$\frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{lat}} = 28.80934(1)$$

Thus, one needs a complete set of these renormalization computations for each new lattice action that is used in Monte Carlo simulations

Lattice perturbation theory is important for many other aspects, e.g.to study the anomalies on the lattice – or the recovery in the limit $a \to 0$ of the continuum symmetries broken by the lattice regularization (like Lorentz or chiral symmetry)

In general perturbation theory is of paramount importance in order to establish the connection of lattice matrix elements to the physical continuum theory

Perturbative calculations are thus in many cases essential, and are the only possibility to have some analytical control over the continuum limit

Because of asymptotic freedom, for QCD one has indeed $g_0 \to 0$ as $a \to 0$

Motivation

The perturbative region is the one that must be necessarily “traversed” in order to reach the continuum limit

So, there is a strong connection between lattice perturbation theory and the continuum limit of discretized versions of QCD

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In discretizing a continuum field theory one has to give up Lorentz invariance (and in general Poincaré invariance)

The symmetry group of the lattice is the hypercubic group (discrete)

The internal symmetries can usually be preserved

In particular, gauge invariance can be maintained on the lattice for any finite value of the lattice spacing

$\Rightarrow$ this makes possible to define QCD

The fact that one is able to keep gauge invariance for any nonzero a is of great help in proving the renormalizability of lattice gauge theories

One cannot underestimate how important lattice perturbation theory was for proving that lattice gauge theories are renormalizable (T. Reisz, 1989)

Motivation

The phenomenological numbers that are quoted from lattice computations are very often the result of the combined effort of numerical simulations and analytic calculations, usually with some input from theory

In this context lattice PT has a wide and useful range of applications

In this respect, perturbative lattice renormalization is important by itself as well as a hint and a guide for the few cases in which one can also determine the renormalization constants nonperturbatively

This is even more important when some operator mixing is present

Since lattice symmetries are not as restrictive as those in the continuum, more mixings (of lattice operators) arise in general under renormalization

In fact, lattice mixing patterns, generally more complex than in the continuum, become in general more transparent when looked at using perturbative renormalization rather than nonperturbatively

Also, perturbative coefficients can be usually computed with good accuracy

Perturbative renormalization results can be useful in checking and understanding results coming from nonperturbative methods (where available)

Motivation

When short-distance quantities can be calculated using such diverse techniques, like lattice perturbation theory or Monte Carlo simulations, their comparison can give significant hints on the validity of perturbative and nonperturbative methods

In some cases a nonperturbative determination of the renormalization constants can turn out to be rather difficult to get

For nonperturbative renormalization to work, it is necessary that there is a plateau for the signal over a substantial range of momenta so that one can numerically extract the values of the renormalization factors

The nonperturbative renormalization methods can sometimes fail because a window which is large enough cannot be found

Moreover, where mixings are present these methods could come out to be useless because certain mixings are too small to be seen numerically, although still not so small to be altogether ignored

In these cases the only possibility to compute renormalization factors seems to be provided by the use of lattice perturbative methods

Motivation

There are cases in which lattice perturbation theory works rather well

For example, the scale evolutions of the strong coupling constant and quark masses computed in the Schrödinger Functional scheme
(quenched: $\Lambda = \Lambda_{\overline{\rm MS}} \cdot 0.48811(1) \quad \sim 116\ MeV$)

We can see that these scale evolutions are accurately described by perturbation theory for a wide range of energies

The perturbative and nonperturbative results are very close to each other, and almost identical even down to energy scales which are surprising low

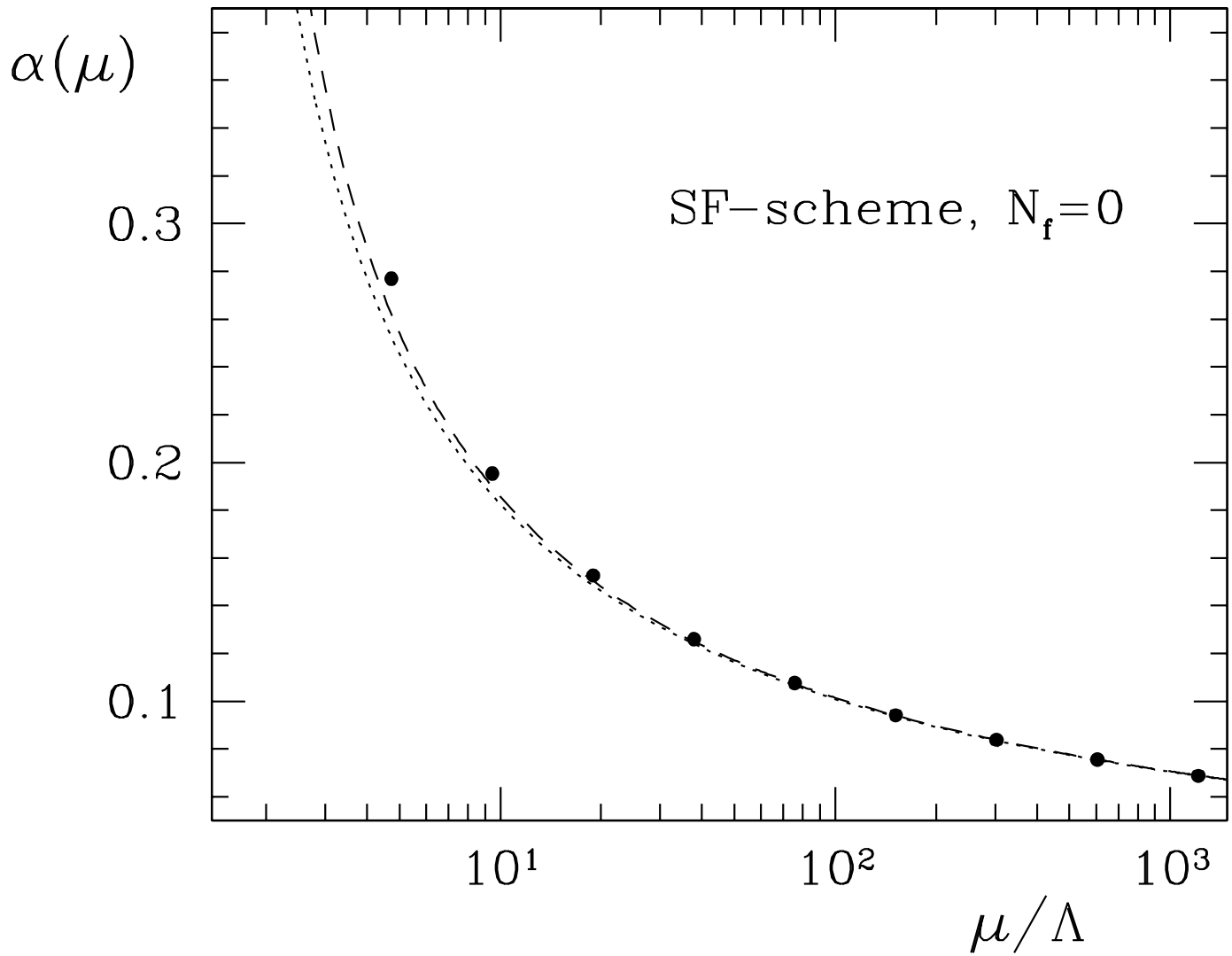
The best curves include the $b_2 g_0^7$ term of the β function and the $d_1 g_0^4$ term of the τ function – that is, the first nonuniversal coefficients

The other curves are lower-order approximations

These runnings are computed in the Schrödinger Functional, and depend on the details of the computational scheme employed – but it is interesting to see how close perturbation theory can come to nonperturbative results

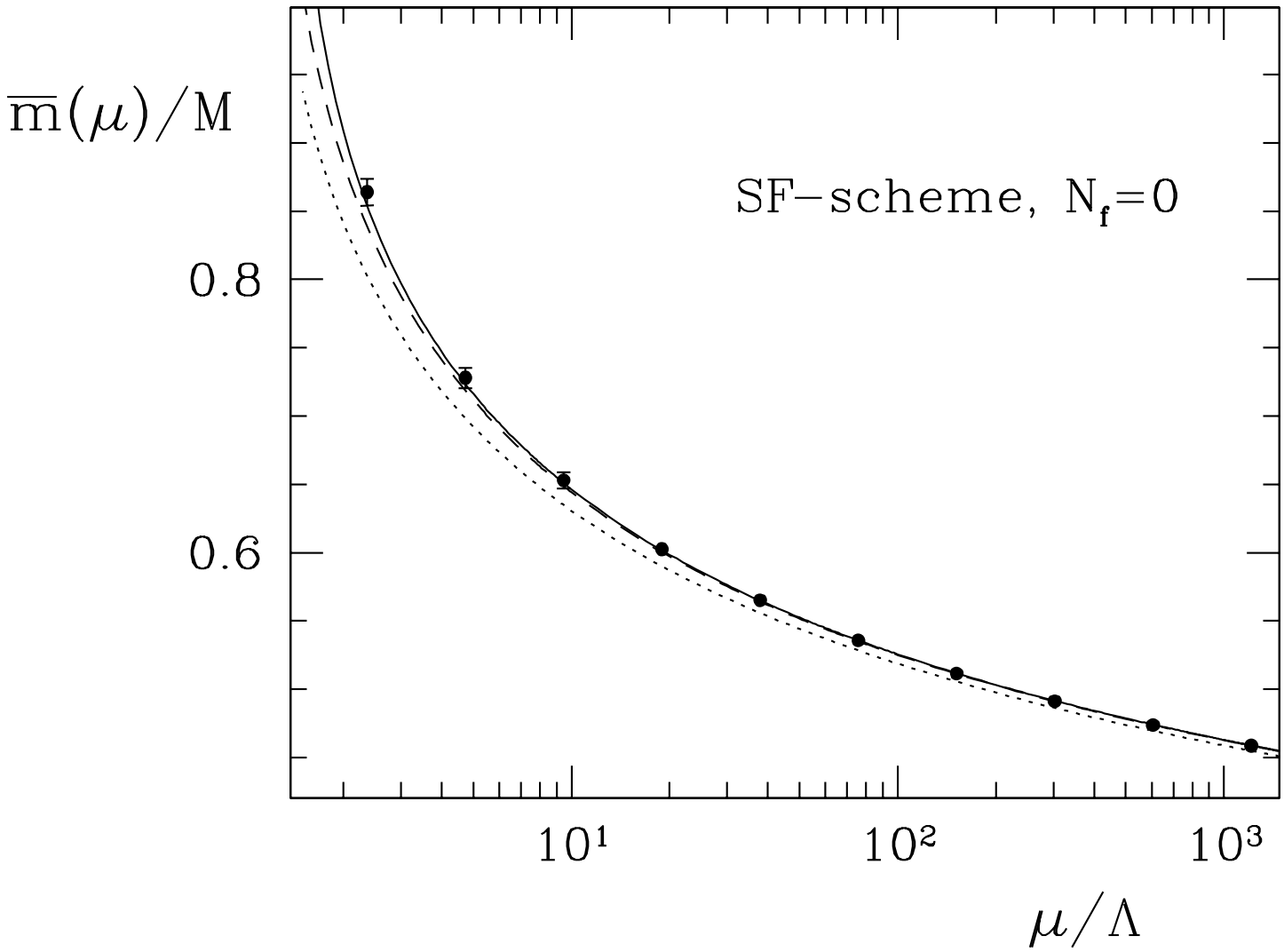
The choice of a scheme can have a lot of influence on the perturbative behavior of the coupling constant. . .

Motivation



From: S. C., M. Lüscher, R. Sommer, H. Wittig, Nucl. Phys. B544 (1999) 669

Motivation



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Motivation

Other example of the usefulness of LPT: understanding of the properties of minimally doubled fermions

"New" class of actions which maintain chiral symmetry for a doublet of quarks, and are almost as cheap as Wilson *(nearest-neighbors)*

Karsten (1981) and Wilczek (1987):

$$D(p) = i \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu + i\gamma_4 \sum_{k=1}^{\mathbf{3}} (1 - \cos p_k)$$

Two examples considered: Karsten-Wilczek and Boriçi-Creutz fermions

Great plus: a conserved axial current , which has a simple expression

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Great plus: a conserved axial current , which has a simple expression

However, simulating on a computer just the tree-level actions is not correct

By calculating self-energy of the quark and vacuum polarization at one loop, it was discovered that three counterterms to each action are needed
(S.C., M. Creutz, J. Weber, H. Wittig, 2010)

Reason: breaking of the hypercubic symmetry *(inequivalent directions)*

Motivation

Counterterms *(KW)* : $\overline{\psi}\,\gamma_4\,\psi \qquad \overline{\psi}\,\gamma_4\,D_4\,\psi \qquad \sum_{\mu\nu} F_{\mu 4} F_{4\nu}$

From perturbation theory:

$$\frac{g_0^2}{(16\pi^2}\,C_F \times \left\{ \begin{array}{cccc} -29.53230 & -0.12554 & -12.69766 & (KW) \\ 29.54170 & 1.52766 & -0.9094 & (BC) \end{array} \right.$$

Next: determine the coefficients of these counterterms nonperturbatively

Simulations with these new fermions are now being done in Mainz

So, most of what we know now of the properties of these new actions comes from perturbative calculations (something also from symmetry considerations)

Before our calculations: possibility of power divergences was hinted at, by P.F. Bedaque, M.I. Buchoff, B.C. Tiburzi and A. Walker-Loud (2008)

Perturbation theory has been essential for the discovery of some key features of this class of fermions

Even with that, it took some time to understand what was happening . . .

All this is also an example of the usefulness of perturbative techniques in helping to unfold theoretical aspects of *(new)* lattice formulations

Motivation

Lattice perturbation theory has now grown into a large and well-established subject

Often the perturbative calculations on the lattice are rather involved

$\Rightarrow$ use of computer codes for the analytic calculations

$\rightarrow$ and also for the computation of lattice integrals

The behavior of lattice perturbation theory is probably not worse than that of QCD in the continuum

asymptotic expansions, in some cases affected by large higher-order corrections

Actually, perturbation theory can be more accurately tested on the lattice than in a continuum regularization, because within a lattice scheme one can also have nonperturbative results to compare perturbation theory with

When lattice perturbation theory and nonperturbative numerical results do not agree, perhaps a look at the systematic errors coming from the numerical side can sometimes be worthwhile

Renormalization of operators

We are interested in matrix elements of operators – like the V and A currents

They are evaluated with numerical simulations on a lattice, but require some renormalization in order to be converted into meaningful physical quantities

Monte Carlo matrix elements can be viewed as (regulated) bare numbers

To get physical (= renormalized) results a lattice renormalization is required

Thus: matching of bare lattice results to some continuum scheme, often the $\overline{\rm MS}$ scheme of dimensional regularization

Why $\overline{\rm MS}$? In many physical problems one evaluates matrix elements of operators that appear in an operator product expansion

$$\langle\, A(x)B(0)\,\rangle \sim \sum_{N,i} c_{N,i}(x^2)\; x^{\mu_1}\cdots x^{\mu_N}\; \langle\, O^{(N,i)}_{\mu_1\cdots\mu_N}(0)\,\rangle \qquad \textit{(light-cone OPE)}$$

These matrix elements contain the long-distance physics of the system and are computed numerically on the lattice

The Wilson coefficients instead contain the short-distance physics and are obtained from perturbative calculations in the continuum

Renormalization of operators

In this situation, the operators evaluated on the lattice must at the end be matched to the same continuum scheme in which the Wilson coefficients are known

Therefore: very often one has to match lattice results to the $\overline{\rm MS}$ scheme

Example: moments of deep inelastic structure functions (parton distributions)

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Let us now consider the perturbative matching at one loop

It turns out that to extract physical continuum matrix elements from Monte Carlo simulations one needs lattice as well as continuum perturbative calculations

At tree level, for momenta much lower than the lattice cutoff, $p \ll \pi/a$, lattice operators have the same matrix elements as the original continuum operators

Renormalization of operators

Then at 1 loop one has *(generally, with a mixing)*

$$\langle q|O_i^{lat}|q\rangle = \sum_j \left(\delta_{ij} + \frac{g_0^2}{16\pi^2}\left(-\gamma_{ij}^{(0)}\log a^2p^2 + R_{ij}^{lat}\right)\right)\cdot\langle q|O_j^{tree}|q\rangle$$

$$\langle q|O_i^{\overline{\rm MS}}|q\rangle = \sum_j \left(\delta_{ij} + \frac{g_{\overline{\rm MS}}^2}{16\pi^2}\left(-\gamma_{ij}^{(0)}\log\frac{p^2}{\mu^2} + R_{ij}^{\overline{\rm MS}}\right)\right)\cdot\langle q|O_j^{tree}|q\rangle$$

Note: while R_{ij}^{lat} is the whole momentum-independent 1-loop correction, $R_{ij}^{\overline{\rm MS}}$ does not include the pole in ϵ and the factors proportional to γ_E and $\log 4\pi$

The lattice and continuum 1-loop finite constants, R_{ij}^{lat} and $R_{ij}^{\overline{\rm MS}}$, in general do not have the same value

This happens because lattice propagators and vertices are quite different from their continuum counterparts, especially for loop momenta of order $1/a$

Therefore the 1-loop renormalization factors on the lattice and in the continuum are in general not equal

Renormalization of operators

However: the 1-loop anomalous dimensions are the same (as expected)

From the previous equations, the connection between the original lattice numbers and the final continuum physical results is given by

$$\langle q|O_i^{\overline{\rm MS}}|q\rangle = \sum_j \left(\delta_{ij} - \frac{g_0^2}{16\pi^2} \left(-\gamma_{ij}^{(0)} \log a^2\mu^2 + R_{ij}^{lat} - R_{ij}^{\overline{\rm MS}} \right) \right) \cdot \langle q|O_j^{lat}|q\rangle$$

The differences $\Delta R_{ij} = R_{ij}^{lat} - R_{ij}^{\overline{\rm MS}}$ enter then in the matching factors

$$Z_{ij}(a\mu, g_0) = \delta_{ij} - \frac{g_0^2}{16\pi^2} \left(-\gamma_{ij}^{(0)} \log a^2\mu^2 + \Delta R_{ij} \right)$$

and represent the main objectives of the perturbative lattice calculations

While R^{lat} and $R^{\overline{\rm MS}}$ depend on the state $|q\rangle$, ΔR is independent of it, thus Z_{ij} depends only on $a\mu$

Renormalization of operators

This is as it should be: the renormalization factors are a property of the operators and are independent of the particular external states considered

This is the reason why we have left the state $|q\rangle$ unspecified

Furthermore: the matching factors between the lattice and the $\overline{\rm MS}$ scheme are gauge invariant

$\Rightarrow$ important checks of lattice perturbative calculations

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$\Rightarrow$ important checks of lattice perturbative calculations

Lattice operators have more possibilities of mixing than continuum ones, due to the lower symmetry of the lattice

There is no Lorentz invariance – and in many cases other symmetries, like chiral symmetry, are also broken

Thus, the matching factors are not in general square mixing matrices: $N_i \leq N_j$

To include all relevant operators one must be able to determine all the tree-level structures which appear when lattice radiative corrections are evaluated

Renormalization of operators

After having used both lattice and continuum perturbative techniques, at the end we obtain the renormalization factor $Z_O(a\mu)$ which converts the lattice operator $O(a)$ into the physical renormalized operator $\widehat{O}(\mu)$:

$$\widehat{O}(\mu) = Z_O(a\mu)\, O(a)$$

In this way one achieves the matching of the bare Monte Carlo results (obtained using a lattice regulator) directly to the physical renormalized results in the $\overline{\rm MS}$ scheme

As for any general quantum field theory, the process at the end of which physical numbers are obtained is carried out in two different steps

One first regularizes the ultraviolet divergences – in this case the regulator is given by the lattice itself

Then one renormalizes the regulated theory – on the lattice this results in a matching to a continuum scheme

Renormalization of operators

Finally, the lattice cutoff must be removed

This means that one has to go to the continuum limit $a \longrightarrow 0$ of the lattice theory, keeping some suitable quantity fixed

Only the scale μ brought in by the renormalization remains after all these steps

In our case, the scale μ at which the matrix elements are renormalized should be in the range

$$\Lambda_{QCD} < \mu < \frac{\pi}{a}$$

the lower bound ensures that perturbation theory is valid

the upper bound ensures that cutoff effects, proportional to positive powers of the lattice spacing, are small

Renormalization of operators

If one sets

$$\mu = \frac{1}{a}$$

a finite renormalization connects the lattice to the $\overline{\rm MS}$ scheme:

$$\langle q|O_i^{\overline{\rm MS}}|q\rangle = \sum_j \left(\delta_{ij} - \frac{g_0^2}{16\pi^2}\left(R_{ij}^{lat} - R_{ij}^{\overline{\rm MS}}\right)\right) \cdot \langle q|O_j^{lat}|q\rangle$$

Remember: the 1-loop anomalous dimensions are the same on the lattice and in the continuum

Every lattice action defines a different regularization scheme, and therefore these finite renormalization factors are in principle different when different actions are used

But also the bare numbers, that is the Monte Carlo results for a given matrix element, are different, and everything combines to give the same physical result

Discretization

In going from continuum to lattice actions one replaces integrals with sums

$$\int d^4x \to a^4 \sum_x$$

where on the right-hand side x means now sites: $x = an$

The momenta are restricted to an interval of range $2\pi/a$, the first Brillouin zone, and which can be chosen as

$$B_Z = \left\{ k : -\frac{\pi}{a} < k_\mu \leq \frac{\pi}{a} \right\}$$

B_Z: region of the allowed values of k, and domain of momentum integration

For a lattice of finite volume $V = L_0 L_1 L_2 L_3$, the allowed momenta in the first Brillouin zone become a discrete set, given by

$$(k_n)_\mu = \frac{2\pi}{a} \frac{n_\mu}{L_\mu} \qquad n_\mu = -L_\mu/2 + 1, \ldots, 0, 1, \ldots, L_\mu/2,$$

$\Rightarrow$ in principle also in momentum space one deals with sums

Discretization

However, in the infinite volume limit the sums over the modes of the first Brillouin zone become again integrals:

$$\frac{1}{V}\sum_k \longrightarrow \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_0}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_2}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk_3}{2\pi}$$

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The one-sided forward and backward lattice derivatives (also known as right and left derivatives) can be written as

$$\begin{aligned} \nabla_\mu \psi(x) &= \frac{\psi(x+a\hat{\mu}) - \psi(x)}{a} \\ \nabla^\star_\mu \psi(x) &= \frac{\psi(x) - \psi(x-a\hat{\mu})}{a} \end{aligned}$$

where $\hat{\mu}$ denotes the unit vector in the μ direction

It is easy to check that they are anti-conjugate to each other:

$$\begin{aligned} (\nabla_\mu)^\dagger &= -\nabla^\star_\mu \\ (\nabla^\star_\mu)^\dagger &= -\nabla_\mu \end{aligned}$$

Discretization

Therefore: in a lattice theory that is supposed to have a hermitian Hamiltonian only their sum, $\nabla_\mu + \nabla_\mu^\star$, which is anti-hermitian, can appear

$\rightarrow$ it acts as a lattice derivative operator extending over two lattice spacings:

$$\frac{1}{2}\left(\nabla + \nabla^\star\right)_\mu \psi(x) = \frac{\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})}{2a}$$

Note that the second-order differential operator $\nabla_\mu \nabla_\mu^\star = \nabla_\mu^\star \nabla_\mu$ is hermitian, and when μ is summed corresponds to the 4-dimensional lattice Laplacian,

$$\Delta\psi(x) = \sum_\mu \nabla_\mu^\star \nabla_\mu \psi(x) = \sum_\mu \frac{\psi(x + a\hat{\mu}) + \psi(x - a\hat{\mu}) - 2\psi(x)}{a^2}$$

For the discretization of continuum actions and operators (derivatives, . . .) and the practical setting of the corresponding lattice theory many choices are possible

Discretization

Since every lattice action defines a different regularization scheme, one needs for each action that is used a new complete set of renormalization computations, in order for the results which come out from Monte Carlo simulations to be used, interpreted and understood properly

Using different actions leads to different numerical results for the matrix elements computed in Monte Carlo simulations

Also the values of the renormalization factors, and of the Λ parameter, depend in general on the lattice action chosen

Even the number and type of counterterms required for the renormalization of operators can be different in each case

For example, for the renormalization of a weak operator more counterterms need to be computed when the Wilson action is used than when the overlap action is used, because chiral invariance is not broken in the second case

Of course all the differences that are seen at finite lattice spacing will disappear in the final numerical extrapolations to the continuum limit which must lead, within errors, to the same physical results

Feynman rules

Wilson action:

$$
\begin{aligned}
S_W &= S_W^f + S_W^g \\
S_W^g &= \frac{1}{g_0^2}\, a^4 \sum_{x,\mu\nu} \left[N_c - \mathrm{Re}\,\mathrm{Tr}\left[U_\mu(x) U_\nu(x+a\hat{\mu}) U_\mu^\dagger(x+a\hat{\nu}) U_\nu^\dagger(x) \right] \right] \\
S_W^f &= a^4 \sum_x \left[-\frac{1}{2a} \sum_\mu \left[\overline{\psi}(x)(r-\gamma_\mu) U_\mu(x) \psi(x+a\hat{\mu}) \right.\right. \\
&\qquad \left.\left. + \overline{\psi}(x+a\hat{\mu})(r+\gamma_\mu) U_\mu^\dagger(x) \psi(x) \right] + \overline{\psi}(x) \left(m_0 + \frac{4r}{a} \right) \psi(x) \right] \\
&= a^4 \sum_x \overline{\psi}(x) \left[\frac{1}{2} \left(\gamma_\mu (\widetilde{\nabla}_\mu^\star + \widetilde{\nabla}_\mu) - ar \widetilde{\nabla}_\mu^\star \widetilde{\nabla}_\mu \right) + m_0 \right] \psi(x)
\end{aligned}
$$

where we have introduced the lattice covariant derivative

$$
\widetilde{\nabla}_\mu \psi(x) = \frac{U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)}{a}
$$

Feynman rules

This action has only nearest-neighbor interactions

Other actions can have more complicated interactions, like overlap fermions

The first-order derivative in the Dirac operator is the symmetric one, given by $\frac{1}{2}\left(\nabla + \nabla^\star\right)_\mu \psi$ in the free case (after an integration by parts)

The fields $U_\mu(x)$ live on the links which connect two neighboring lattice sites – these variables are naturally defined in the middle point of a link

Link variables are unitary matrices – they do not depend linearly on the gauge potential $A_\mu(x)$

Reason: they belong to the **group** $SU(N_c)$ rather than to the corresponding **Lie algebra**, as is the case in the continuum

The relation of the $U_\mu(x)$ matrices to the gauge fields $A_\mu(x)$, the variables which have a direct correspondence with the continuum, is then given by

$$U_\mu(x) = \mathrm{e}^{\mathrm{i} g_0 a T^a A^a_\mu(x)} \qquad (a = 1, \ldots, N_c^2 - 1)$$

where the T^a are $SU(N_c)$ matrices in the fundamental representation

Feynman rules

In the weak coupling regime *(small g_0)* the functional integral is dominated by field configurations near the trivial gauge field $U_\mu(x) = 1$

Perturbation theory is then a saddle-point expansion around these classical vacuum configuration – and the degrees of freedom are given by the components of the potential, $A_\mu^a(x)$

Thus, while the fundamental gauge variables in Monte Carlo simulations are the U_μ's, and the action is relatively simple when expressed in terms of these variables, in perturbation theory the true dynamical variables are the A_μ's

This mismatch is responsible for many complications of lattice PT

Indeed: the Wilson action becomes very complicated when written in terms of the A_μ's:

$$U_\mu = 1 + \mathrm{i} g_0 a A_\mu - \frac{1}{2} g_0^2 a^2 A_\mu^2 + \cdots$$

Moreover, it consists of an infinite number of terms, which give rise to an infinite number of interaction vertices – *with an arbitrary number of fields*

$\rightarrow$ example: $\overline{\psi}\, A\, A\, \cdots\, A\, \psi$ *(not in the continuum. . .)*

Feynman rules

Is lattice gauge theory then non-renormalizable?

Feynman rules

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Fortunately, only a finite number of vertices is needed to any given order in g_0

All but a few vertices are "irrelevant" – they are proportional to some positive power of the lattice spacing a and so they vanish in the naive continuum limit

However, this does not mean that they can be thrown away in the computation of Feynman diagrams!

Quite on the contrary: they usually contribute to correlation functions in the continuum limit, through divergent loop corrections ($\sim 1/a^n$)

These irrelevant vertices are indeed important in many cases, they contribute to the renormalization of masses, coupling constants and wave-functions

All these vertices are in fact necessary to ensure the gauge invariance of physical amplitudes

Only when they are included can gauge-invariant Ward Identities be constructed, and the renormalizability of the lattice theory proven

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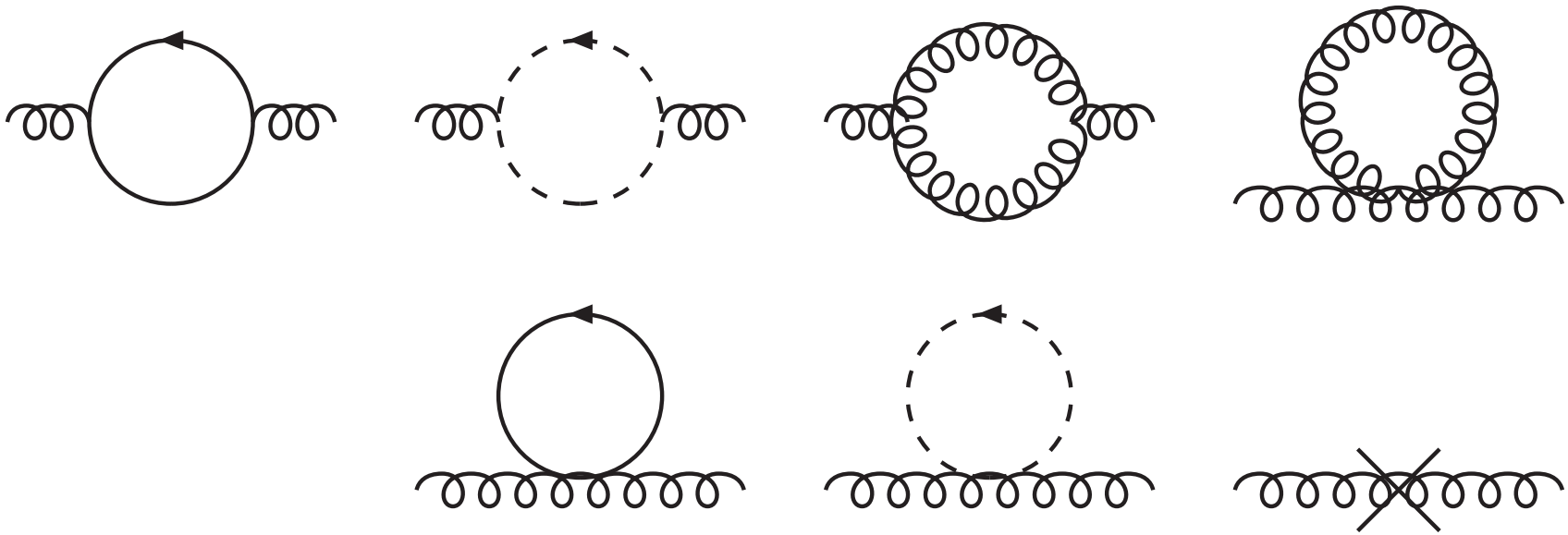
Example: diagrams contributing to the 1-loop gluon self-energy

Feynman rules



diagrams on the upper row: have a continuum analog

diagrams on the lower row: pure lattice artifacts – but necessary for the gauge invariance of the lattice theory, and for its renormalizability

If one takes only the upper row diagrams *(the ones that would also exist in the continuum)*, the lattice results diverge like $1/a^2$ *(unphysical)*

This divergence is removed only when the diagrams of the lower row are added

For this to happen, also a contribution coming from the measure is essential

Feynman rules

Only when the diagrams of the lower row are considered, gauge invariance is fully restored as well

In similar ways, terms of the type $p_\mu^2\,\delta_{\mu\nu}$, not Lorentz covariant, and often present in individual diagrams, disappear only after all diagrams have been considered and summed

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From what we have seen so far: a lattice regularization amount to much more than introducing in the theory a momentum cutoff

In fact, it is a far more complicated regularization than just introducing a cutoff

one has also to provide a gauge-invariant regularized action

Different discretizations of the same continuum action define different lattice regularizations

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Different discretizations of the same continuum action define different lattice regularizations

You cannot escape the complications of LPT: a gauge-invariant regularization requires the U's, but the degrees of freedom of LPT are the A's

Feynman rules

Lattice Feynman rules are much more complicated that in the continuum, and new interaction vertices appear which have no analog in the continuum

The structure of lattice integrals is also completely different – periodicity causes the appearance of trigonometric functions

The lattice integrands are then rational functions of trigonometric expressions

Thus, things are much more involved than in the continuum:

- there are more fundamental vertices
- there are more diagrams
- propagators and vertices are more complicated expressions
- often need Taylor expansions of trigonometric functions in the momenta

Consequence: final expressions containing a huge number of terms

Finally, one has also to evaluate more complicated integrals

For the calculation of all but the simplest matrix elements the help of computers is almost unavoidable

Feynman rules

Feynman diagrams in momentum space : we need the Fourier transforms on the lattice (in infinite volume, standard setting of perturbation theory) :

$$\psi(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}xp}\, \psi(p)$$

$$\overline{\psi}(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}xp}\, \overline{\psi}(p)$$

$$A_\mu(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}(x+a\hat{\mu}/2)k}\, A_\mu(k)$$

The Fourier transform of $A_\mu(x)$ is taken at the point $x + a\hat{\mu}/2$, halfway between x and the neighboring point $x + a\hat{\mu}$

This turns out to be quite important for the general economy of the calculations

The δ-functions in position and momentum space are

$$\delta_{xy} = a^4 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}(x-y)p}, \qquad \delta^{(4)}(p) = \frac{a^4}{(2\pi)^4} \sum_x \mathrm{e}^{-\mathrm{i}xp}$$

Feynman rules

Example: quark-quark-gluon vertex in momentum space, for Wilson fermions:
(euclidean space!)

$$S_{qqg} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p'}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+k-p')$$

$$\times \mathrm{i}g_0 \sum_\mu \overline{\psi}(p') \left(\gamma_\mu \cos \frac{a(p+p')_\mu}{2} - \mathrm{i}r \sin \frac{a(p+p')_\mu}{2} \right) A_\mu(k)\psi(p)$$

(color factor not included)

It is easy to see that in the formal $a \to 0$ limit this becomes the familiar continuum QCD vertex:

$$\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4p'}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+k-p') \cdot \mathrm{i}g_0 \sum_\mu \overline{\psi}(p') \gamma_\mu A_\mu(k) \psi(p)$$

In the following we will not explicitly write the δ-function of momenta present in each vertex and propagator

Convention for the vertices: all gluon lines are entering, and when there are quark or ghost lines there will always be an equal number of incoming and outgoing lines

Feynman rules

In the Wilson action the group elements $U_\mu(x)$ appear instead of the fundamental perturbative variables, the algebra elements $A_\mu(x)$

We want to derive the gluon vertices from the pure gauge action

$\Rightarrow$ expand the U_μ's of the plaquette in terms of the A_μ's

Thus, an infinite number of interaction vertices are generated

They express the self-interactions of n gluons, with arbitrary n

The power of the coupling constant which enters in these vertices grows with the number of gluons

$\Rightarrow$ only a finite number of vertices is needed to any given loop order

The expansion of the plaquette action in terms of the A_μ's can be derived by means of the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{A}\mathrm{e}^{B} = \exp\left\{ A+B+\frac{1}{2}\Big[A,B\Big]+\frac{1}{12}\Big[A-B,\Big[A,B\Big]\Big]+\frac{1}{24}\Big[\Big[A,\Big[A,B\Big],B\Big]\Big]+\cdots \right\}$$

Feynman rules

From the definition of the U's it follows that the entries of the matrices $ag_0 A_\mu(x)$ are angular variables – they take values between zero and 2π

In perturbation theory the range of integration of the fields $A^a_\mu(x)$ is extended to infinity

It is only after $A^a_\mu(x)$ has been decompactified that the tree-level propagators can be explicitly computed (Gaussian functional integral)

The 3-gluon vertex is *(with $p+q+r=0$, and gluons are all incoming and assigned clockwise)*:

$$W^{abc}_{\mu\nu\lambda}(p,q,r) = -\mathrm{i}g_0 f^{abc} \frac{2}{a} \left\{ \delta_{\mu\nu} \sin\frac{a(p-q)_\lambda}{2} \cos\frac{ar_\mu}{2} + \delta_{\nu\lambda} \sin\frac{a(q-r)_\mu}{2} \cos\frac{ap_\nu}{2} + \delta_{\lambda\mu} \sin\frac{a(r-p)_\nu}{2} \cos\frac{aq_\lambda}{2} \right\}$$

In the formal $a \to 0$ limit this reduces to the continuum expression

$$-\mathrm{i}g_0 f^{abc} \left\{ \delta_{\mu\nu} (p-q)_\lambda + \delta_{\nu\lambda} (q-r)_\mu + \delta_{\lambda\mu} (r-p)_\nu \right\}$$

Feynman rules

It is useful to introduce the shorthand notation $\widehat{k}_\mu = \frac{2}{a}\sin\frac{ak_\mu}{2}$

The 4-gluon vertex is quite complicated: $W^{abcd}_{\mu\nu\lambda\rho}(p,q,r,s) =$

$$
\begin{aligned}
&-g_0^2\Bigg\{\sum_e f_{abe}f_{cde}\Bigg[\delta_{\mu\lambda}\delta_{\nu\rho}\left(\cos\frac{a(q-s)_\mu}{2}\cos\frac{a(k-r)_\nu}{2}-\frac{a^4}{12}\widehat{k}_\nu\widehat{q}_\mu\widehat{r}_\nu\widehat{s}_\mu\right)\\
&-\delta_{\mu\rho}\delta_{\nu\lambda}\left(\cos\frac{a(q-r)_\mu}{2}\cos\frac{a(k-s)_\nu}{2}-\frac{a^4}{12}\widehat{k}_\nu\widehat{q}_\mu\widehat{r}_\mu\widehat{s}_\nu\right)\\
&+\frac{1}{6}\delta_{\nu\lambda}\delta_{\nu\rho}a^2\,\widehat{(s-r)}_\mu\,\widehat{k}_\nu\,\cos\frac{aq_\mu}{2}-\frac{1}{6}\delta_{\mu\lambda}\delta_{\mu\rho}a^2\,\widehat{(s-r)}_\nu\,\widehat{q}_\mu\,\cos\frac{ak_\nu}{2}\\
&+\frac{1}{6}\delta_{\mu\nu}\delta_{\mu\rho}a^2\,\widehat{(q-k)}_\lambda\,\widehat{r}_\rho\,\cos\frac{as_\lambda}{2}-\frac{1}{6}\delta_{\mu\nu}\delta_{\mu\lambda}a^2\,\widehat{(q-k)}_\rho\,\widehat{s}_\lambda\,\cos\frac{ar_\rho}{2}\\
&+\frac{1}{12}\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}a^2\sum_\sigma\widehat{(q-k)}_\sigma\widehat{(s-r)}_\sigma\Bigg]\\
&+(b\leftrightarrow c,\nu\leftrightarrow\lambda,q\leftrightarrow r)+(b\leftrightarrow d,\nu\leftrightarrow\rho,q\leftrightarrow s)\Bigg\}\quad+
\end{aligned}
$$

Feynman rules

$$+\frac{g_0^2}{12}\,a^4\left\{\frac{2}{3}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})+\sum_e(\delta_{abe}\delta_{cde}+\delta_{ace}\delta_{bde}+\delta_{ade}\delta_{bce})\right\}$$

$$\times\left\{\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\sum_\sigma\widehat{k}_\sigma\widehat{q}_\sigma\widehat{r}_\sigma\widehat{s}_\sigma-\delta_{\mu\nu}\delta_{\mu\lambda}\,\widehat{k}_\rho\widehat{q}_\rho\widehat{r}_\rho\widehat{s}_\mu\right.$$

$$-\delta_{\mu\nu}\delta_{\mu\rho}\,\widehat{k}_\lambda\widehat{q}_\lambda\widehat{s}_\lambda\widehat{r}_\mu-\delta_{\mu\lambda}\delta_{\mu\rho}\,\widehat{k}_\nu\widehat{r}_\nu\widehat{s}_\nu\widehat{q}_\mu-\delta_{\nu\lambda}\delta_{\nu\rho}\,\widehat{q}_\mu\widehat{r}_\mu\widehat{s}_\mu\widehat{k}_\nu$$

$$\left.+\delta_{\mu\nu}\delta_{\lambda\rho}\,\widehat{k}_\lambda\widehat{q}_\lambda\widehat{r}_\mu\widehat{s}_\mu+\delta_{\mu\lambda}\delta_{\nu\rho}\,\widehat{k}_\nu\widehat{r}_\nu\widehat{q}_\mu\widehat{s}_\mu+\delta_{\mu\rho}\delta_{\nu\lambda}\,\widehat{k}_\nu\widehat{s}_\nu\widehat{q}_\mu\widehat{r}_\mu\right\}$$

In the $a\to 0$ limit this expression becomes the four-gluon vertex of continuum QCD

To my knowledge, an explicit expression for the five-gluon vertex has not yet been given in the literature

General algorithms for the automated calculation of higher-order vertices (for a given configuration of external momenta) have been reported by Lüscher and Weisz (1986)

Feynman rules

The vertices containing five or more gluons are at least of order g_0^3, and thus they are not necessary for 1-loop calculations

For nonabelian gauge theories, the calculation of the pure gauge part of course does not end here

One has still to consider the gauge integration measure, which generates an infinite number of vertices with increasing powers of g_0, starting with a $1/a^2$ mass counterterm to order g_0^2

Furthermore, the Faddeev-Popov procedure has to be implemented on the lattice, from which the Feynman rules for the ghost propagator and the various ghost vertices can be derived

The effective ghost-gauge field interaction , at variance with the continuum, is not linear in the gauge potential A_μ

Thus, also in this sector we find an infinite number of new vertices that have no continuum analog, like for example the vertex involving two ghosts and two gluons

Feynman rules

Gauge fixing:

Although in some situations can be convenient, it is not in principle necessary on the lattice when one works with actions written in terms of the U_μ's (as done in Monte Carlo simulations)

Reason: unlike what happens in the continuum, the whole volume of the gauge group is finite – it is the product of a countable number of factors each equal to the volume of the $SU(3)$ group, $V = \prod_x v(SU(3))$

This factor cancels out in normalizing expectation values of operators

In perturbation theory: saddle-point approximation of the functional integral around $U_\mu = 1$, and the A_μ's become the actual degrees of freedom

Thus, gauge fixing is necessary in LPT

A gauge has to be fixed in order to eliminate the zero modes in the quadratic part of the action (expressed in terms of the A_μ's)

We can see why it is necessary to fix a gauge in perturbative lattice QCD also from the following argument

Feynman rules

Perturbation theory: expansion around the minimum of the plaquette action

Looking at the form of the Wilson action we see that $P_{\mu\nu}(x) = 1$ minimizes the pure gauge action – but this does not yet imply $U_\mu(x) = 1$

On the contrary, even if one fixes $U_\mu(x) = 1$ for each link from the beginning, a gauge transformation will lead to $1 \to \Omega(x)\,\Omega^{-1}(x + a\hat{\mu})$, a group element which can take any value

In order to avoid this, and for perturbation theory to be a weak coupling expansion around the configuration $U_\mu(x) = 1$, one must fix the gauge

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Gauge fixing is thus essential in lattice perturbative calculations, and can be implemented by a lattice Faddeev-Popov procedure, similarly to the continuum

The final result, however, is rather different

Indeed, as another consequence of the lattice gauge invariance, one obtains from the Faddeev-Popov procedure an infinite number of vertices

Although cumbersome, this procedure is perfectly consistent and gives a precise meaning to the lattice functional integral

Feynman rules

The result for the Faddeev-Popov determinant is indeed very reminiscent of the continuum:

$$\Delta_{FP}[A_\mu] = \det\left(-\nabla^\star_\mu \widehat{D}_\mu[A]\right)$$

The important difference with respect to the continuum case is that the lattice operator $\widehat{D}_\mu[A]$ is not linear in A

So, we get an infinite number of ghost-gluon vertices

Using the well-known formula for Grassmann spin-zero variables c and $\overline{c}$

$$\int \mathcal{D}(\overline{c}c)\, \exp^{-a^4 \sum_{ij} \overline{c}_i Q_{ij} c_j} = \det Q$$

we can write the Faddeev-Popov determinant in terms of an action involving ghosts:

$$\Delta_{FP}[A_\mu] = \int \left(\prod_{a,x} d\overline{c}^a(x) c^a(x)\right) \exp\left(a^4 \sum_x \overline{c}^a(x) \nabla^\star_\mu \widehat{D}^{ab}_\mu[A_\mu] c^b(x)\right)$$

Feynman rules

Lattice analog of the covariant Lorentz gauge:

$$\mathcal{F}_x^a[A_\mu, \chi] = \nabla_\mu^\star A_\mu^a(x) - \chi^a(x) = 0$$

(χ are some arbitrary fields)

The expectation value of a generic gauge-invariant operator is then

$$\langle O \rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\mathcal{D}\overline{c}\mathcal{D}c \cdot O \cdot \exp(-S_{QCD} + a^4 \sum_x \overline{c}^a(x)\nabla_\mu^\star \widehat{D}_\mu^{ab}[A]c^b(x) - S_{meas} - S_{gf})}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\mathcal{D}\overline{c}\mathcal{D}c \cdot \exp(-S_{QCD} + a^4 \sum_x \overline{c}^a(x)\nabla_\mu^\star \widehat{D}_\mu^{ab}[A]c^b(x) - S_{meas} - S_{gf})}$$

In the above equation the gauge-fixing term has been written, thanks to the δ-function $\delta(\mathcal{F}_x^a[A_\mu, \chi])$, as

$$S_{gf} = \frac{a^4}{2\alpha} \sum_x \left(\sum_\mu \nabla_\mu^\star A_\mu(x) \right)^2 = \frac{a^2}{2\alpha} \sum_x \left(\sum_\mu \left(A_\mu(x) - A_\mu(x - a\hat{\mu}) \right) \right)^2$$

Ghosts are spin-zero Grassmann variables transforming according to the adjoint representation of $SU(3)$

Feynman rules

The gluon propagator in the covariant gauge $\partial_\mu A_\mu = 0$ is

$$G^{ab}_{\mu\nu}(k) = \delta^{ab} \frac{1}{\frac{4}{a^2}\sum_\lambda \sin^2 \frac{ak_\lambda}{2}} \left\{ \delta_{\mu\nu} - (1-\alpha) \frac{\sin \frac{ak_\mu}{2} \sin \frac{ak_\nu}{2}}{\sum_\lambda \sin^2 \frac{ak_\lambda}{2}} \right\}$$

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In shorthand form it looks quite simple:

$$\delta^{ab} \cdot \frac{1}{\widehat{k}^2} \left[\delta_{\mu\nu} - (1-\alpha) \frac{\widehat{k}_\mu \widehat{k}_\nu}{\widehat{k}^2} \right]$$

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$$\delta^{ab} \cdot \frac{1}{\widehat{k}^2} \left[\delta_{\mu\nu} - (1-\alpha) \frac{\widehat{k}_\mu \widehat{k}_\nu}{\widehat{k}^2} \right]$$

So, it is easy to see that in the limit $a \to 0$ the lattice gluon propagator reduces to the well-known continuum expression

$$\delta^{ab} \cdot \frac{1}{k^2} \left[\delta_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2} \right]$$

Feynman rules

We stress once again that one needs to fix a gauge even on the lattice, in order to do perturbation theory

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The ghost propagator is

$$\delta^{ab} \cdot \frac{1}{\frac{4}{a^2} \sum_\lambda \sin^2 \frac{a k_\lambda}{2}}$$

The ghost-ghost-gluon vertex is

$$\mathrm{i} g_0 f_{abc} \, (\widehat{p}_2)_\mu \, \cos \frac{(a p_1)_\mu}{2}$$

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The ghost-ghost-gluon-gluon vertex is

$$\frac{1}{12} \, g_0^2 a^2 \, \{t^a, t^b\}_{cd} \, \delta_{\mu\nu} \, (\widehat{p}_1)_\mu \, (\widehat{p}_2)_\mu$$

This is a lattice artifact, which vanishes in the formal continuum limit $a \to 0$

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This is a lattice artifact, which vanishes in the formal continuum limit $a \to 0$

Vertices containing three or more ghosts are at least of order g_0^3 and do not enter in 1-loop calculations

Feynman rules

The definition of the gauge-invariant integration measure on the lattice turns out to be nontrivial for nonabelian gauge groups – and generates an infinite number of vertices

It is convenient to write this measure term in the form

$$\mathcal{D}U = \mathrm{e}^{-S_{meas}[A]}\,\mathcal{D}A$$

This can be done using $\det g = \exp(\mathrm{Tr}\,\log g)$, so that we obtain

$$\begin{aligned} S_{\mathrm{meas}}[A] &= -\frac{1}{2}\sum_{x,\mu}\mathrm{Tr}\ \log\frac{2(1-\cos(ag_0A^a_\mu t^a))}{(ag_0A^a_\mu t^a)^2} \\ &= -\frac{1}{2}\sum_{x,\mu}\mathrm{Tr}\ \log\left[1+2\sum_{l=1}^{\infty}\frac{(-1)^l}{(2l+2)!}\left(ag_0A^a_\mu t^a\right)^{2l}\right] \end{aligned}$$

The new factor 2 at numerator *(irrelevant constant)* is because we want $\log(1+x)$

From this expression one can read off all vertices

Feynman rules

To lowest order one gets

$$S_{meas}[A] = \frac{g_0^2}{8a^2} \sum_{x,a,\mu} (A_\mu^a)^2$$

This term, quadratic in A_μ, is part of the interaction and not a kinetic term, because of the presence of the factor g_0^2

It acts like a mass counterterm of order g_0^2, and is needed to restore gauge invariance in lattice Feynman amplitudes

For example, it cancels the quadratic divergence in the 1-loop gluon self-energy *(as we have seen before . . .)*

In momentum space this mass counterterm is

$$-\frac{g_0^2}{4a^2}\delta_{\mu\nu}\delta_{ab}$$

The higher orders (starting with g_0^3) give self-interaction vertices of the gluons

Feynman rules

The quark propagator can be computed by inverting the lattice Dirac operator in momentum space, and is given by

$$S^{ab}(k, m_0) = \delta^{ab} \cdot a \, \frac{-\mathrm{i} \sum_\mu \gamma_\mu \sin ak_\mu + am_0 + 2r \sum_\mu \sin^2 \frac{ak_\mu}{2}}{\sum_\mu \sin^2 ak_\mu + \left(2r \sum_\mu \sin^2 \frac{ak_\mu}{2} + am_0 \right)^2}$$

In the formal continuum limit it reduces to the well-known expression

$$\delta^{ab} \cdot \frac{-\mathrm{i} \sum_\mu \gamma_\mu k_\mu + m_0}{\sum_\mu k_\mu^2 + m_0^2}$$

The quark-antiquark-gluons vertices are obtained by expanding the fermionic part of the action in powers of $g_0 A$

We get again an infinite tower of vertices, which involve a $q\overline{q}$ pair and any number of gluons

Fortunately, only a finite number of these vertices is needed to any given order in g_0

Feynman rules

The expression of the quark-quark-gluon vertex is

$$(V_1^a)_\mu^{bc}(p_1,p_2) = -g_0(T^a)^{bc}\left(\mathrm{i}\gamma_\mu \cos\frac{a(p_1+p_2)_\mu}{2} + r\sin\frac{a(p_1+p_2)_\mu}{2}\right)$$

where p_1 and p_2 are the quark momenta flowing in and out of the vertex

For $a \to 0$ this becomes the familiar continuum QCD vertex $-g_0(T^a)^{bc}\,\mathrm{i}\gamma_\mu$

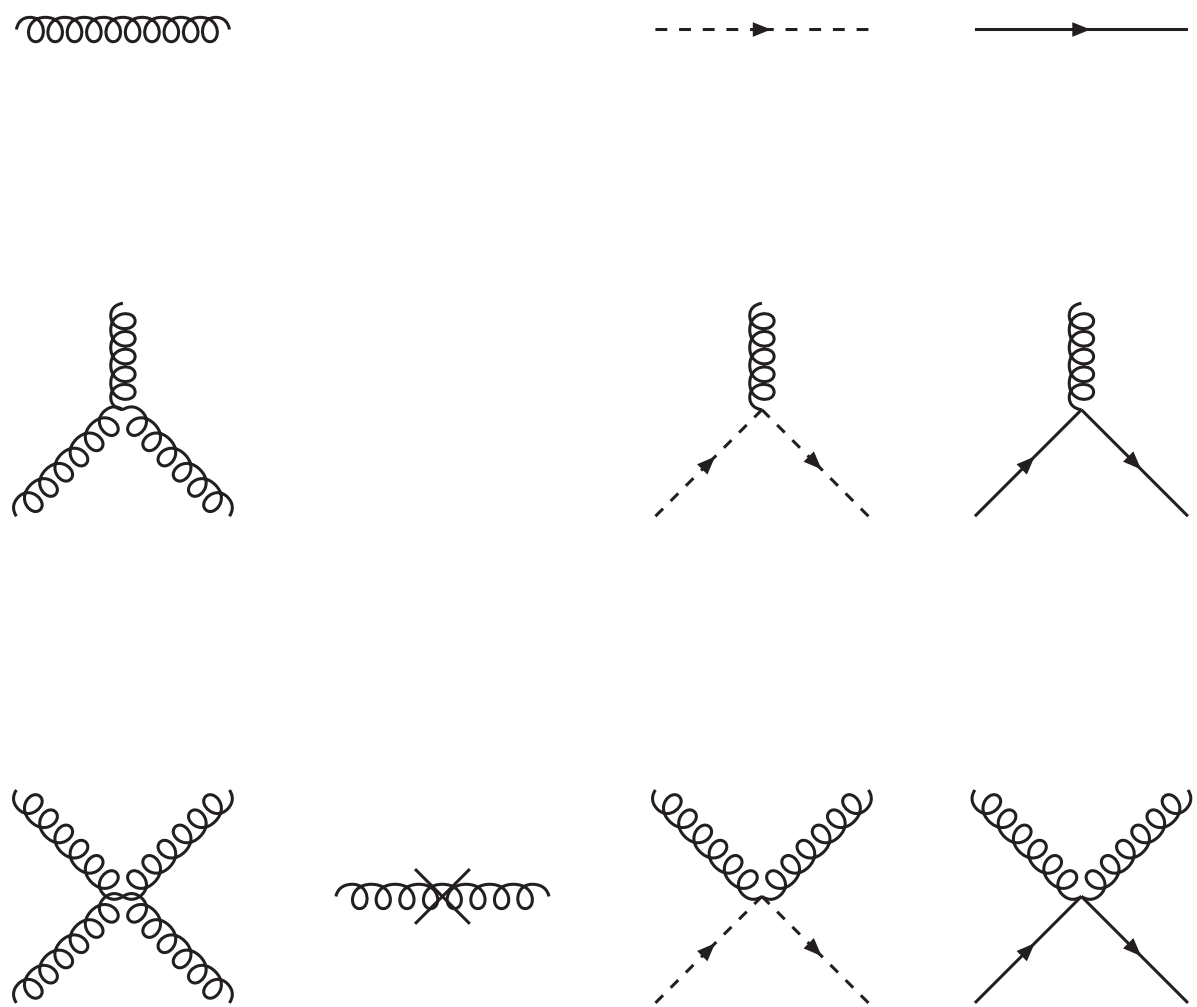
The quark-quark-gluon-gluon vertex is $(V_2^{ab})_{\mu\nu}^{cd}(p_1,p_2) =$

$$-\frac{1}{2}ag_0^2\,\delta_{\mu\nu}\left(\frac{1}{N_c}\delta^{ab} + d^{abe}T^e\right)^{cd}\left(-\mathrm{i}\gamma_\mu \sin\frac{a(p_1+p_2)_\mu}{2} + r\cos\frac{a(p_1+p_2)_\mu}{2}\right)$$

This vertex vanishes as $a \to 0$ and thus has no continuum analog

However it can still give nonvanishing contributions to a Feynman diagram, because of power divergences in loops ($\sim 1/a^n$) that can compensate the explicit factor a in front of V_2

Feynman rules



Propagators and vertices needed for 1-loop calculations in lattice QCD

Feynman rules

These are all quark-gluon vertices which are needed for 1-loop calculations

Vertices with two quarks and n gluons are associated with factors $a^{n-1} g_0^n$

In spite of the fact that the lattice gauge theory has an infinite number of fundamental vertices, it is still renormalizable

It turns out that the superficial degree of divergence of a Feynman diagram depends only on the number of its external lines (Kawai et al., 1981) :

$$D = 4 - E_G - E_g - \frac{3}{2} E_q$$

where E_G, E_g and E_q represent the number of external gluons, ghosts and quarks

Vertices of higher order in a and g_0 do not modify this continuum picture

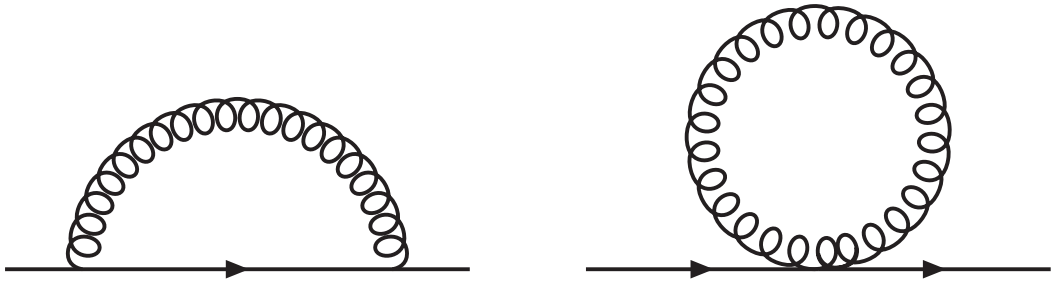
In the language of the renormalization group, they are irrelevant operators, and they do not affect the scaling behavior of the theory

The factors a^n in front of these irrelevant vertices contribute to keep the theory renormalizable

Tadpoles

Typical lattice diagrams are the tadpoles

Example: 1-loop quark self-energy



Tadpoles

Typical lattice diagrams are the tadpoles

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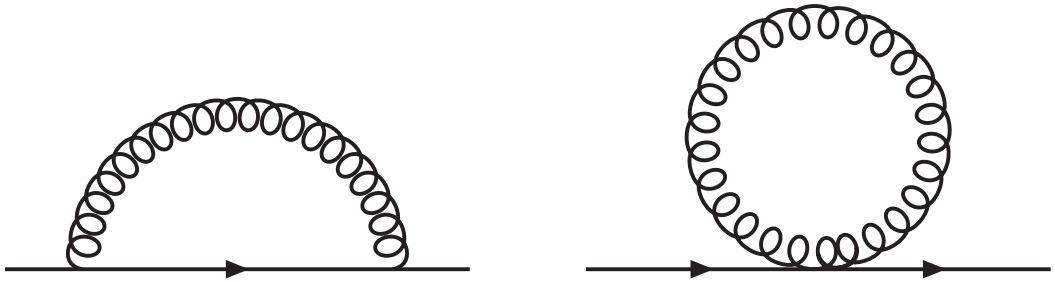


Wave-function renormalization:

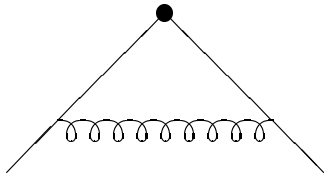
$$\frac{g_0^2}{16\pi^2} C_F \cdot \left(\log a^2 p^2 - 0.380646 \right) \qquad \frac{g_0^2}{16\pi^2} C_F \cdot \mathbf{12.23305}$$

Critical mass:

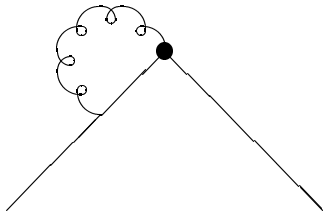
$$-\frac{1}{a}\,\frac{g_0^2}{16\pi^2} C_F \cdot 2.502511 \qquad -\frac{1}{a}\,\frac{g_0^2}{16\pi^2} C_F \cdot \mathbf{48.932201}$$

This shows an often occurring phenomenon in LPT: **tadpole dominance**

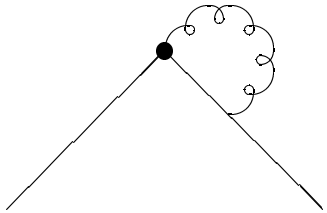
Diagrams



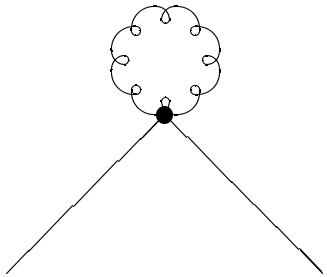
(a) vertex



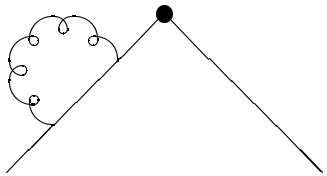
(b) sail



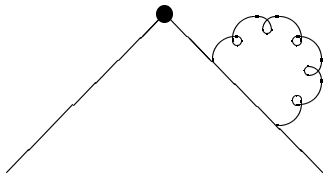
(c) sail



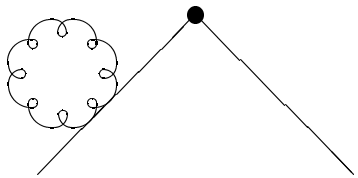
(d) operator tadpole



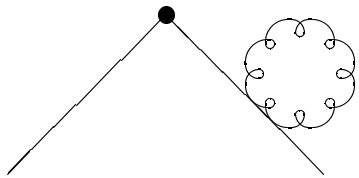
(e) leg self-energy
(excluding tadpoles)



(f) leg self-energy
(excluding tadpoles)

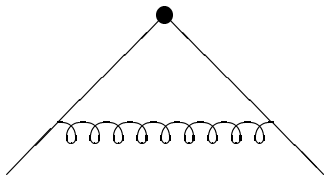


(g) leg tadpole

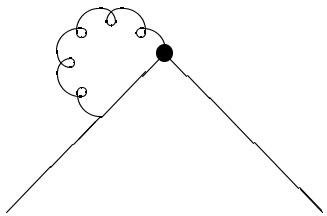


(h) leg tadpole

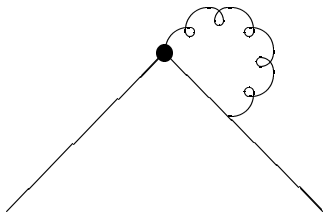
Diagrams



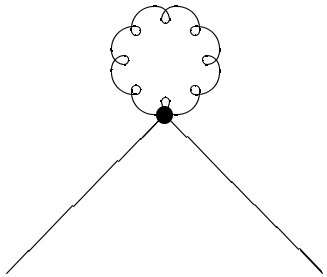
(a) vertex



(b) sail



(c) sail



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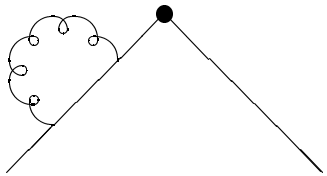

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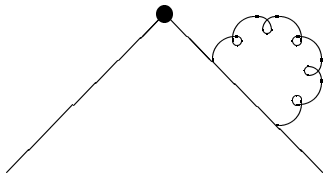
(excluding tadpoles)



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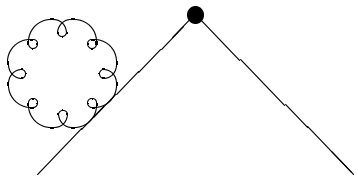

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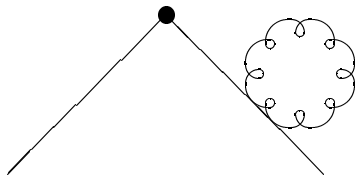

(h) leg tadpole

These are the 1-loop diagrams needed for the renormalization of currents and operators

In general, operators may contain gauge fields

Examples:

$$\overline{\psi}\,\gamma_\mu\,D_\nu\,\psi \quad \textit{(covariant derivative)}$$

$$\frac{1}{2}\Big(\overline{\psi}(x)\,(\gamma_\mu - r)\,U_\mu(x)\,\frac{\lambda^f}{2}\,\psi(x+a\hat{\mu}) + \overline{\psi}(x+a\hat{\mu})\,(\gamma_\mu + r)\,U_\mu^\dagger(x)\,\frac{\lambda^f}{2}\,\psi(x)\Big)$$

This generates the sails, and the operator tadpole

Bottom-half diagrams: corrections to the external legs...

Some technical issues

The Feynman rules on the lattice are quite different from the continuum ones

The number of Feynman diagrams is in general larger

The structure of the lattice integrals is also completely different

The integrands are periodic in the momenta, and the basic objects are trigonometric functions and not simple polynomials of the momenta

Many standard methods in continuum perturbation theory, like Feynman parameterization and partial integration, are then not of much use for perturbative lattice calculations.

Also the γ-algebra becomes soon too involved to be worked out still by hand

At the end of the day, perturbation theory on the lattice, even only for simple actions at the 1-loop level, is rather cumbersome

Compared to an equivalent continuum calculation, vertices and propagators are more complicated, usually there are more diagrams, and each diagram produces many more integrals (involving trigonometric functions)

$\Rightarrow$ fewer loops and fewer legs than in the continuum

Some technical issues

Computer codes are then necessary if one wants to compute but the simplest matrix elements
(for example: use the algebraic manipulation program FORM)

These codes take the Feynman rules for the particular combination of operators, propagators and vertices in each diagram, expand them in the lattice spacing a to the appropriate order, evaluate the γ-algebra on the lattice, and then work out everything until the final expressions are obtained

Due to the enormous number of terms in the initial stages of the manipulations one needs in many cases a large working memory

These codes turn out to be necessary also because they can provide the result of the analytic manipulations as an output file which is already formatted (for example in Fortran) as an input file for the numerical integration

Lattice perturbative calculations generally involve the manipulation of a huge number of terms, but often a large number of terms remain also in the final analytic expressions which have to be numerically integrated

The numerical integrations then require a lot of computer time, which in some cases can be of the order of thousands of hours

Some technical issues

The increasing complexities can be easily seen in the calculation of the renormalization of the moments of unpolarized structure functions

Reason: the covariant derivative is proportional to the inverse of the lattice spacing, $D \sim 1/a$, and so

$$\langle x^n \rangle \quad \sim \quad \langle \overline{\psi}\, \gamma_\mu\, D_{\mu_1} \cdots D_{\mu_n}\, \psi \rangle \quad \sim \quad \frac{1}{a^n}$$

This means that in order to compute the n-th moment, one needs to perform a Taylor expansion in a to order n of every single quantity (propagators, vertices, operator insertions, counterterms)

It is not difficult to see how many terms can come out of that

It is sufficient to look at the Wilson quark-quark-gluon vertex to order a^2

$$(V^a)^{bc}_\mu(k, ap) \quad = \quad -g_0\,(T^a)^{bc} \cdot \left\{ \mathrm{i}\gamma_\mu \left[\cos\frac{k_\mu}{2} - \frac{1}{2} a p_\mu \, \sin\frac{k_\mu}{2} - \frac{1}{8} a^2 p_\mu^2 \, \cos\frac{k_\mu}{2} \right] \right.$$

$$\left. + r \left[\sin\frac{k_\mu}{2} + \frac{1}{2} a p_\mu \, \cos\frac{k_\mu}{2} - \frac{1}{8} a^2 p_\mu^2 \, \sin\frac{k_\mu}{2} \right] + O\left(a^3 p_\mu^3\right) \right\}$$

or to the expansion of the Wilson quark propagator just only to order a:

Some technical issues

$$S^{ab}(k+aq,am_0) = \delta^{ab}\cdot\left\{\frac{-\mathrm{i}\sum_\mu \gamma_\mu \sin k_\mu + 2r\sum_\mu \sin^2\frac{k_\mu}{2}}{\sum_\mu \sin^2 k_\mu + \left[2r\sum_\mu \sin^2\frac{k_\mu}{2}\right]^2}\right.$$

$$+a\cdot\left[\frac{-\mathrm{i}\sum_\mu \gamma_\mu q_\mu \cos k_\mu + r\sum_\mu q_\mu \sin k_\mu + m_0}{\sum_\mu \sin^2 k_\mu + \left[2r\sum_\mu \sin^2\frac{k_\mu}{2}\right]^2}\right.$$

$$\left.\left.-\left(-\mathrm{i}\sum_\rho \gamma_\rho \sin k_\rho + 2r\sum_\rho \sin^2\frac{k_\rho}{2}\right)\frac{\sum_\mu q_\mu \sin 2k_\mu + 4r\sum_\mu \sin^2\frac{k_\mu}{2}\left(r\sum_\nu q_\nu \sin k_\nu + m_0\right)}{\left\{\sum_\mu \sin^2 k_\mu + \left[2r\sum_\mu \sin^2\frac{k_\mu}{2}\right]^2\right\}^2}\right]\right\}$$

$$+O\left(a^2 q_\mu^2\right)$$

The algebraic manipulations become thus quite complex

Overlap or domain-wall fermions, improved gauge actions, . . . , produce even more complicated expressions

Main consequence of all this: generation of a huge number of terms, at least in the initial stages of the manipulations, even in the case of matrix elements where all Lorentz indices are contracted

Some technical issues

The multiplication of two vertices and two quark propagators which have been expanded to order a generates about $4^2 \cdot 11^2 \sim 2000$ monomial terms

Initial expansions of Feynman diagrams for the operators of the second and third moment of parton distributions can easily reach the order of 10^6 terms

This slows down considerably the execution of the algebraic codes

Most of these terms either become zero after doing the Dirac algebra, or do not contribute to the sought Dirac structure, or are zero after integration

The terms which do not contribute to the final expression have to be discarded as early as possible to speed up the computations

Thus, the fact that an operator with n covariant derivatives requires Taylor expansions in a to order n also implies a limitation on the number of moments of structure functions that one can practically compute on the lattice

This is something different from the limitation coming from operator mixings, and the combination of these two computational challenges renders in practice the computation of the renormalization of the fourth moment or higher very difficult

Some technical issues

Typical structure of a lattice integral from a Feynman diagram:

$$\frac{1}{a^n}\int_\pi^\pi \left(c^{(0)} + c^{(1)}_\mu\, a p_\mu + \cdots + c^{(n)}_{\mu_1\cdots\mu_n}\, a^n p_{\mu_1}\cdots p_{\mu_n} + \cdots\right)$$

So, in general one can have power divergences

$$\sim \frac{1}{a^k}$$

Terms with a positive power of the lattice spacing are generally set to zero (*unless you are interested in Symanzik improvement. . .)*

Example:

$$\frac{1}{a}\int_\pi^\pi \left(c^{(0)} + c^{(1)}_\mu\, a p_\mu + c^{(2)}_{\mu\nu}\, a^2 p_\mu p_\nu + c^{(3)}_{\mu\nu\lambda}\, a^3 p_\mu p_\nu p_\lambda + \cdots\right)$$

$$= \frac{1}{a}\int_\pi^\pi c^{(0)} + p_\mu \int_\pi^\pi c^{(1)}_\mu \underbrace{+ a p_\mu p_\nu \int_\pi^\pi c^{(2)}_{\mu\nu} + a^2 p_\mu p_\nu p_\lambda \int_\pi^\pi c^{(3)}_{\mu\nu\lambda} + \cdots}_{\text{all these terms vanish}}$$

Be aware of this when doing the exercises!

Some technical issues

Lorentz symmetry is broken on the lattice

for example, one cannot make a rotation of an arbitrary degree

A whole new kind of problems stem from this fact, for example the non-validity of the Einstein summation convention ($k_\mu k^\mu = k^2$)

One of the biggest challenges of computer codes for lattice perturbation theory is to deal with the fact that the summation convention on repeated indices is suspended

FORM, and other similar symbolic manipulations programs, have been developed having in mind the usual continuum calculations

There are therefore many useful built-in features of FORM that are in principle somewhat of a hindrance when one does lattice perturbative calculations

These built-in functions cannot then be used straightforwardly in the lattice computations

Some technical issues

This is for example what FORM would do by default, because it assumes that two equal indices have to be contracted:

$$\sum_\lambda \gamma_\lambda p_\lambda \quad \longrightarrow \quad \not{p}$$

$$\sum_\lambda \gamma_\lambda p_\lambda \sin k_\lambda \quad \longrightarrow \quad \not{p} \sin k_\lambda$$

$$\sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\lambda \quad \longrightarrow \quad (\gamma \cdot \sin k) \cos^2 k_\lambda$$

$$\sum_{\lambda,\rho} \gamma_\rho \gamma_\lambda \gamma_\rho \sin k_\lambda \cos^2 k_\rho \quad \longrightarrow \quad -2 \sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\rho$$

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$$\sum_{\lambda,\rho} \gamma_\rho \gamma_\lambda \gamma_\rho \sin k_\lambda \cos^2 k_\rho \quad \longrightarrow \quad -2 \sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\rho$$

On the lattice however monomials typically contain more than twice the same index

Only the first case is then correctly handled by FORM

For example, in the last case the right answer is instead

$$-\sum_{\lambda,\rho} \gamma_\lambda \sin k_\lambda \cos^2 k_\rho + 2 \sum_\rho \gamma_\rho \sin k_\rho \cos^2 k_\rho$$

Some technical issues

For this reason one must develop special routines to deal with the γ-algebra on the lattice

One solution could be to introduce generalized Kronecker δ-symbols

$$\delta_{\mu_1 \mu_2 \ldots \mu_n}$$

which are equal to one only if all indices are equal, $\mu_1 = \mu_2 = \ldots = \mu_n$, and zero otherwise (Lüscher and Weisz, 1995)

In general one needs special routines, which introduce suitable modifications to the usual FORM commands, to properly treat Dirac matrices and handle terms like in the above examples

Since the computations cannot be carried out by hand and because of these special routines for the γ's, a number of additional checks is desirable

One can use different regularizations, like a mass regularization and dimensional regulation (in its various forms), and one can develop various routines which use different methods

Gauge invariance is also used as a check, by repeating the calculations in Feynman and Landau gauge

Hypercubic group

It is useful to discuss the symmetry group of the lattice and see what are the consequences of the breaking of Lorentz invariance

On the lattice one inevitably ends up with a discrete group

The symmetry group of the discrete rotations of a four-dimensional hypercubic lattice onto itself is a crystallographic group, denoted by W_4 and called the hypercubic group

It consists of $\pi/2$ rotations on the six lattice planes and reflections
(so that parity transformations are also included)

It has 384 elements and 20 irreducible representations

W_4 is a subgroup of the orthogonal group $O(4)$, which is the Lorentz group analytically continued to Euclidean space

A major difficulty in doing perturbative calculations on the lattice arises from the fact that the (Euclidean) Lorentz symmetry breaks down to the hypercubic W_4 symmetry

Since the lattice has a reduced symmetry with respect to the continuum, **more operator mixings** are allowed

Hypercubic group

Let us first consider the special hypercubic group, SW_4, consisting of proper rotations without reflections

It has 192 elements and 13 irreducible representations

5 of these representations (of dimensions 1, 1, 2, 3 and 3) are connected to the 4-element permutation group, S_4, because the latter is a subgroup of SW_4 and the 5 representations of S_4 can be taken as nonfaithful representations of SW_4

There are then four representations of SW_4 which can be identified by the fact that they correspond to representations of $O(4)$ which remain irreducible under SW_4:

$$(\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{1}),(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}),(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$$

The direct product of each the first three representations with the completely antisymmetric representation of the permutation group S_4 generates three other irreducible representations of SW_4 (which maintain the same dimensionality), while $(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$ is invariant under this operation

The representation $(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}})$ turns out to give the same hypercubic representation as $(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$

Hypercubic group

So far we have then been able to identify 12 representations

There is yet another representation, which has dimension 6

The complete list of the representations of the special hypercubic group SW_4 is thus given by

$$\mathbf{1_1}, \mathbf{1_2}, \mathbf{2}, \mathbf{3_1}, \mathbf{3_2}, \mathbf{3_3}, \mathbf{3_4}, \mathbf{3_5}, \mathbf{3_6}, \mathbf{4_1}, \mathbf{4_2}, \mathbf{6}, \mathbf{8}$$

where the subscripts label different representations with the same dimensionality

This group is a subgroup of $SO(4)$, the special orthogonal group

We now discuss the irreducible representations of W_4

Including the reflections doubles the number of group elements, but not the number of representations

Hypercubic group

This happens because, contrary to the cubic group in three dimensions, the hypercubic group is not the direct product of the rotation group and the reflection group

The reason is that the reflection of all four axes is still a rotation, which is not true for the reflection of three axes in three dimensions

Therefore, going from SW_4 to W_4 the number of representations only increases from 13 to 20

What happens is that 9 of these 13 representations just double (generating the representations with opposite parity), while the remaining 4, all of dimension 3, merge into two 6-dimensional representations which are reflection invariant

In particular, the $\mathbf{3_3}$ and $\mathbf{3_4}$ of SW_4 merge into the $\mathbf{6_1}$ of W_4, and the $\mathbf{3_5}$ and $\mathbf{3_6}$ of SW_4 merge into the $\mathbf{6_2}$ of W_4

We can then give the complete list of the representations of W_4:

$$\mathbf{1_1}, \mathbf{1_2}, \mathbf{1_3}, \mathbf{1_4}, \mathbf{2_1}, \mathbf{2_2}, \mathbf{3_1}, \mathbf{3_2}, \mathbf{3_3}, \mathbf{3_4}, \mathbf{4_1}, \mathbf{4_2}, \mathbf{4_3}, \mathbf{4_4}, \mathbf{6_1}, \mathbf{6_2}, \mathbf{6_3}, \mathbf{6_4}, \mathbf{8_1}, \mathbf{8_2}$$

Hypercubic group

The representation 4_1 is the canonical one, corresponding to an object with a Lorentz index , like $(\frac{1}{2}, \frac{1}{2})$ is in the continuum

When interested in the behavior of lattice operators which have more than one Lorentz index, we must identify the representations of the hypercubic group contained in the tensor products of the 4_1 with itself

Then, we compare the result with what happens in the continuum, where one has to consider the tensor products of the $(\frac{1}{2}, \frac{1}{2})$ with itself

The relation between these two expansions determines what kind of mixings arise when one computes radiative corrections of lattice matrix elements

All this of course apart from additional mixings due to the breaking of chiral symmetry or of other symmetries

More on this in:

- Baake, Gemünden and Oedingen, J. Math. Phys. **23** (1982) 944
- Baake, Gemünden and Oedingen, J. Math. Phys. **24** (1983) 1021
- Mandula, Zweig and Govaerts, Nucl. Phys. **B228** (1983) 91

Operator mixing

Since W_4 is a subgroup of $O(4)$, a continuum operator belonging to a given irreducible representation of the (Euclidean) Lorentz group becomes in general a sum of irreducible representations of the hypercubic group

The continuum operator can then belong to various distinct lattice representations, depending on the way in which its indices are chosen

This implies than on the lattice the possibilities for mixing under renormalization are larger than in the continuum

The number of independent renormalization factors in a lattice calculation is so in general larger than in the continuum

In particular, operators which are multiplicatively renormalizable in the continuum may lose this property on the lattice

This feature also occurs in other regularizations

For example, in continuum calculations using dimensional regularization in the version known as DRED, "evanescent" operators, coming from the additional -2ϵ dimensions, are generated in the intermediate stages of the calculations

Operator mixing

For Wilson fermions additional mixings *(beside those due to the breaking of Lorentz invariance)* can arise because of the breaking of chiral symmetry

For staggered fermions, the loss of flavor invariance also opens the door for more mixings

All these additional mixings are unphysical – are just lattice artifacts which have to be subtracted in order to extract physical results from the lattice

In practical terms the worst situation occurs in the case of mixings with operators of lower dimensions , with lattice renormalization factors containing a power divergent coefficient, proportional to $1/a^n$

These lattice artifacts ought to be subtracted nonperturbatively

In short, Lorentz breaking, as well as the possible breaking of chiral, flavor or other symmetries, may spoil in general the multiplicative renormalizability of continuum operators – in some cases even with power divergences

The necessary condition for not having any (hypercubic-related) mixing at all is that the operator belongs to an irreducible representation of W_4 – but this is sometimes not sufficient , as we will see shortly

Operator mixing

Let us see some examples involving operators which measure moments of unpolarized structure functions

These operators appear in the operator product expansion of two electromagnetic or weak hadronic currents, are symmetric in all their indices and traceless and have the form

$$O_{\{\mu\mu_1\cdots\mu_n\}}(x) = \overline{\psi}(x)\,\gamma_{\{\mu}\,D_{\mu_1}\cdots D_{\mu_n\}}\,\psi(x)$$

The operator $O_{\{\mu\mu_1\cdots\mu_n\}}$ measures the n-th moment, $\langle x^n \rangle$, of the unpolarized structure functions

All mixings which we discuss for these operators are artifacts of the lattice, and are only due to the breaking of Lorentz invariance

They have nothing to do with the breaking of chiral symmetry for Wilson fermions, and therefore they are still present, in exactly the same form, even when one uses Ginsparg-Wilson fermions

In the continuum each of the above operators belongs to an irreducible representation of the Lorentz group

Operator mixing

On the lattice instead they are in general reducible , and they become linear combinations of irreducible representations of the hypercubic group

This why mixings appear when radiative corrections are computed

Operator mixing

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This why mixings appear when radiative corrections are computed

First moment: the operator is $O_{\{\mu\nu\}} = \overline{\psi}\,\gamma_{\{\mu}\,D_{\nu\}}\,\psi$, symmetric and traceless

An object with a single Lorentz index belongs in the continuum to the $(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$ representation of the Euclidean Lorentz group $O(4)$, while on the lattice it belongs to the $\mathbf{4_1}$ representation of the hypercubic group W_4

The general decomposition of the 16 (nonsymmetrized) tensor components is in the continuum:

$$(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})\otimes(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}) = (\mathbf{0},\mathbf{0})\oplus(\mathbf{1},\mathbf{0})\oplus(\mathbf{0},\mathbf{1})\oplus(\mathbf{1},\mathbf{1})$$

while on the lattice is:

$$\mathbf{4_1}\otimes\mathbf{4_1} = \mathbf{1_1}\oplus\mathbf{3_1}\oplus\mathbf{6_1}\oplus\mathbf{6_3}$$

Operator mixing

Here we have essentially two choices for the symmetrized operators, that is the two indices can be different or can be equal

In the latter case, one has also to subtract the trace component

The first case can be exemplified by considering the operator $O_{\{01\}}$, which belongs to the $\mathbf{6_1}$ and is multiplicatively renormalizable

A representative of the second case is $O_{\{00\}} - \frac{1}{3}(O_{\{11\}} + O_{\{22\}} + O_{\{33\}})$, which belongs to the $\mathbf{3_1}$ and is also multiplicatively renormalizable

The subtracted trace part belongs to the $\mathbf{1_1}$

Finally, the antisymmetric components (which do not enter in the operator product expansion for the moments), for example the operator $O_{[01]}$, belong to the remaining representation in the expansion, the $\mathbf{6_3}$

Since they belong to different representations of W_4, the lattice renormalization factors of the operators $O_{\{01\}}$ and $O_{\{00\}} - \frac{1}{3}(O_{\{11\}} + O_{\{22\}} + O_{\{33\}})$ are different, as has been verified by explicit calculations

In the continuum however they are equal, as both operators belong to the $(\mathbf{1}, \mathbf{1})$

Operator mixing

From the point of view of Monte Carlo simulations choosing two different indices is worse, because in this case one component of the hadron momentum must be different from zero – and this leads to larger systematic effects due to the granularity of the lattice

Operator mixing

From the point of view of Monte Carlo simulations choosing two different indices is worse, because in this case one component of the hadron momentum must be different from zero – and this leads to larger systematic effects due to the granularity of the lattice

Second moment: the operator is $O_{\{\mu\nu\sigma\}} = \overline{\psi}\,\gamma_{\{\mu}\,D_\nu D_{\sigma\}}\,\psi$, symmetric and traceless

The general decomposition of the 64 (nonsymmetrized) tensor components of this rank-three operator is in the continuum:

$$(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})\otimes(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})\otimes(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}) = 4\cdot(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})\oplus 2\cdot(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})\oplus 2\cdot(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}})\oplus(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}})$$

while on the lattice is:

$$\mathbf{4_1}\otimes\mathbf{4_1}\otimes\mathbf{4_1} = 4\cdot\mathbf{4_1}\oplus\mathbf{4_2}\oplus\mathbf{4_4}\oplus 3\cdot\mathbf{8_1}\oplus 2\cdot\mathbf{8_2}$$

We have essentially three choices here for the symmetrized components

Operator mixing

One is represented by the operator $O_{\{123\}}$, which belongs to the $\mathbf{4_2}$ and is multiplicatively renormalizable

This choice however is quite unsatisfactory from the point of view of simulations, because two components of the hadron momentum have to be different from zero and from each other, leading to rather large systematic errors

One should minimize these systematic errors by including as few nonzero components of the hadron momentum as possible

From this point of view, the optimal choice is the operator $O_{\{111\}}$, which belongs to the $\mathbf{4_1}$

Unfortunately this operator mixes with $\overline{\psi}\,\gamma_1\,\psi$, which is a $\mathbf{4_1}$ as well

Moreover, the coefficient of this mixing can be seen from dimensional arguments to be power divergent , $1/a^2$, and thus this mixing cannot be resolved in perturbation theory

Operator mixing

There is an intermediate choice between having the indices all different or all equal, and is given by the operator

$$O_S = O_{\{011\}} - \frac{1}{2}(O_{\{022\}} + O_{\{033\}})$$

which does not have any power divergences due to the particular combination chosen

This operator belongs to an irreducible representation of W_4

Nonetheless, is not multiplicatively renormalizable, and undergoes a mixing with another operator

The way in which this happens is not trivial

The point is that the operator O_S belongs to the $\mathbf{8_1}$, but this representation is present three times in the lattice decomposition of $O_{\mu\nu\sigma}$

It turns out that two of these $\mathbf{8_1}$ representations mix with each other, at least at the 1-loop level

Operator mixing

This mixing can be best seen in the following way

The nonsymmetrized operators

$$
\begin{aligned}
O_A &= O_{011} - \frac{1}{2}(O_{022} + O_{033}) \\
O_B &= O_{101} + O_{110} - \frac{1}{2}(O_{202} + O_{220} + O_{303} + O_{330})
\end{aligned}
$$

turn out to have different 1-loop corrections on the lattice, and they renormalize with different numerical factors which form a nontrivial mixing matrix:

$$
\begin{aligned}
\widehat{O}_A &= Z_{AA}\, O_A + Z_{AB}\, O_B \\
\widehat{O}_B &= Z_{BA}\, O_A + Z_{BB}\, O_B
\end{aligned}
$$

Notice that the two covariant derivatives have the same index in O_A but two different indices in O_B, and the two operators have different tree levels, $\gamma_0 p_1^2 - \frac{1}{2}(\gamma_0 p_2^2 + \gamma_0 p_3^2)$ and $2\gamma_1 p_0 p_1 - (\gamma_2 p_0 p_2 + \gamma_3 p_0 p_3)$ respectively

Operator mixing

The operator that we want to measure,

$$O_S = O_{\{011\}} - \frac{1}{2}(O_{\{022\}} + O_{\{033\}}) = \frac{1}{3}(O_A + O_B),$$

does not transform into itself under 1-loop renormalization,

$$\widehat{O}_S = \frac{1}{3}(Z_{AA} + Z_{BA})\, O_A + \frac{1}{3}(Z_{AB} + Z_{BB})\, O_B,$$

because on the lattice $Z_{AA} + Z_{BA}$ is not equal to $Z_{AB} + Z_{BB}$
(as explicit calculations have shown)

In other words, the symmetric combination is lost and O_S mixes under renormalization with an operator of mixed symmetry (nonsymmetrized)

The choice of indices for this operator is thus very important , for the Monte Carlo simulations as well as for the calculation of renormalization factors

In the continuum all $O_{\{\mu\nu\sigma\}}$ cases discussed above, including $O_{\{111\}}$, belong to the $(\frac{3}{2}, \frac{3}{2})$

Thus, they have the same renormalization constant, and no mixing problem

Divergent integrals

It is convenient to use a method introduced by Kawai, Nakayama and Seo (1981) *(This article is a treasure mine)*

One makes a Taylor expansion in the external momenta, and computes on the lattice only integrals with vanishing momentum (technically much simpler)

Consider the case of a quadratically divergent integral, depending on two external momenta p and q:

$$I = \int \mathrm{d}k \; \mathcal{I}(k, p, q)$$

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Consider the case of a quadratically divergent integral, depending on two external momenta p and q:

$$I = \int \mathrm{d}k \; \mathcal{I}(k,p,q)$$

This integral can be split as

$$I = J + (I - J)$$

where

$$J = \int \mathrm{d}k \; \mathcal{I}(k,0,0) + \sum_{\rho,\sigma} \left[p_\rho q_\sigma \int \mathrm{d}k \; \frac{\partial^2 \mathcal{I}(k,p,q)}{\partial p_\rho \partial q_\sigma}\bigg|_{p=q=0} \right.$$

$$\left. + \frac{p_\rho p_\sigma}{2} \int \mathrm{d}k \; \frac{\partial^2 \mathcal{I}(k,p,0)}{\partial p_\rho \partial p_\sigma}\bigg|_{p=0} + \frac{q_\rho q_\sigma}{2} \int \mathrm{d}k \; \frac{\partial^2 \mathcal{I}(k,0,q)}{\partial q_\rho \partial q_\sigma}\bigg|_{q=0} \right]$$

is the Taylor expansion of the original integral to second order

Divergent integrals

The integrals appearing in J do not depend on the external momenta, and are so much easier to calculate on the lattice

The whole dependence on the external momenta stays in $I - J$ which, because of the subtraction, is ultraviolet-finite for $a \to 0$ and can be computed by taking the naive continuum limit *(power counting theorem of Reisz, 1988)*

Thanks to this, only zero-momentum integrals need to be actually computed on the lattice

For $p, q \neq 0$ and finite lattice spacing I is well defined, but J and $I - J$ are infrared divergent

To compute J and $I - J$ separately, one must then introduce an intermediate regularization

The associated divergences will at the end cancel out in the sum $J + (I - J)$

This intermediate regularization is completely independent from the main regularization used in the lattice theory – in particular can be different from it

It just comes out because the splitting is somewhat unnatural

Divergent integrals

Example: let us take the logarithmically divergent integral

$$I = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(\frac{4}{a^2}\sum_\mu \sin^2\frac{a(k-p)_\mu}{2}\right)\cdot\left(\frac{4}{a^2}\sum_\mu \sin^2\frac{ak_\mu}{2}\right)}$$

The splitting is made as follows:

$$J = I(p=0) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(\frac{4}{a^2}\sum_\mu \sin^2\frac{ak_\mu}{2}\right)^2}$$

$$I - J = \lim_{a\to 0} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{\left(\frac{4}{a^2}\sum_\mu \sin^2\frac{a(k-p)_\mu}{2}\right)\cdot\left(\frac{4}{a^2}\sum_\mu \sin^2\frac{ak_\mu}{2}\right)} - \frac{1}{\left(\frac{4}{a^2}\sum_\mu \sin^2\frac{ak_\mu}{2}\right)^2} \right\} = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{(k-p)^2\cdot k^2} - \frac{1}{(k^2)^2} \right\}$$

Taking common denominators, it is easy to see that the degree of divergence of $I - J$ is negative, and therefore it can be safely computed in the continuum

Divergent integrals

If we use dimensional regularization we have

$$\begin{aligned} J &= \frac{1}{16\pi^2}\left(\frac{2}{d-4} - \log a^2\mu^2 - \log 4\pi + F_0\right) \\ I - J &= \frac{1}{16\pi^2}\left(-\frac{2}{d-4} - \log\frac{p^2}{\mu^2} + \log 4\pi - \gamma_E - 2\right) \end{aligned}$$

where $\gamma_E = 0.5772156649\ldots$ and the lattice constant is $F_0 = 4.3692252338\ldots$

Notice: the integral of the second term in $I - J$ is zero in this regularization

Divergent integrals

If we use dimensional regularization we have

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$$I - J = \frac{1}{16\pi^2}\left(-\frac{2}{d-4} - \log\frac{p^2}{\mu^2} + \log 4\pi - \gamma_E - 2\right)$$

where $\gamma_E = 0.5772156649\ldots$ and the lattice constant is $F_0 = 4.3692252338\ldots$

Notice: the integral of the second term in $I - J$ is zero in this regularization

If we instead regularize adding a small mass term m^2 in the denominators, we obtain

$$J_m = \frac{1}{16\pi^2}\left(-\log a^2m^2 - \gamma_E + F_0\right)$$

$$(I - J)_m = \frac{1}{16\pi^2}\left(-\log\frac{p^2}{m^2} - 2\right)$$

With either regularization, adding up J and $I - J$ we obtain for our original integral the result

$$I = -\log a^2p^2 - \gamma_E + F_0 - 2$$

Divergent integrals

To summarize, in order to evaluate any divergent integral which depends on external momenta it is sufficient to compute some lattice integrals at zero momenta, and some continuum integrals

In computer programs, a convenient way to deal with a generic divergent integral (which has to be processed in an automated way) is to subtract from it a simple integral with the same divergent behavior, and for which the numerical value is exactly known

The difference is then finite and can be computed with reasonable precision using simple integration routines

This is extremely convenient in the case of actions which give rise to complicated denominators – for example, overlap fermions

In this case, Wilson integrals with the same divergence are subtracted from the original overlap integral, and then overlap denominators, which are much more complicated, appear only in the numerical calculation of finite integrals

The calculation of divergent integrals can then be accomplished only using Wilson fermions

Reduction of integrals

Using simple integration routines, 1-loop lattice integrals can be evaluated with a precision of 4–5 significant digits

Algebraic method (*for Wilson fermions*): any integral can be computed in a completely symbolic way and written in terms of a few basic constants

Once these few basic constants are determined with the desired precision, the original integral is just some appropriate linear combination of them

This means that a generic integral can then be calculated numerically with a very large precision with a very small effort

It is now possible to compute any bosonic integral with a very high precision, of $O(10^2)$ significant digits

Computing 1-loop integrals with such precisions is absolutely necessary if one wants to evaluate 2-loop integrals with at least ten significant decimal places ($\rightarrow$ *coordinate space methods (Lüscher and Weisz)*)

It is sufficient to apply this algebraic method only to integrals with zero external momenta, since the momentum-dependent part can be evaluated in the continuum (as we have just seen)

Reduction of integrals

The general zero-momentum integral for Wilson fermions can always be written as a sum of terms of the form

$$\mathcal{F}(p,q;n_x,n_y,n_z,n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\widehat{k}_x^{2n_x}\widehat{k}_y^{2n_y}\widehat{k}_z^{2n_z}\widehat{k}_t^{2n_t}}{D_F(k,m_f)^p D_B(k,m_b)^q}$$

The algebraic method allows to express a generic $\mathcal{F}(p,q;n_x,n_y,n_z,n_t)$ in terms of a certain number of basic integrals

The complete reduction is achieved using an iteration procedure which makes use of appropriate recursion relations (in noninteger dimensions)

At the end of this procedure:

- any purely bosonic integral can be expressed in terms of 3 constants
- any purely fermionic integral can be expressed in terms of 9 constants
- any general integral requires only $(\ 3+9\ +3\ =\ )\ 15$ constants

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The purely bosonic case (Wilson plaquette) is not too complicated
(Caracciolo, Menotti and Pelissetto, Phys. Lett. B260 (1991) 401 & Nucl. Phys. B375 (1992) 195)

Reduction of integrals

Any zero-momentum integral coming from the calculation of lattice Feynman diagrams in the pure gauge Wilson theory can be expressed as a sum of terms of the form

$$\mathcal{B}(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\widehat{k}_x^{2n_x} \widehat{k}_y^{2n_y} \widehat{k}_z^{2n_z} \widehat{k}_t^{2n_t}}{D_B(k,m)^p}$$

(p and n_i positive integers)

It is always possible to rewrite any numerator such that it contains only factors of $\sin^2 k_\mu/2$: use $\sin^2 k_\mu = 4\sin^2 k_\mu/2 - 4\sin^4 k_\mu/2$ *(similarly for cosines)*

The inverse bosonic propagator, taken in general to be massive in order to regularize the divergences coming from the separation in J and $I - J$, is

$$D_B(k,m) = \widehat{k}^2 + m^2$$

Due to the appearance of other kinds of singularities at some intermediate stages of the reductions, one must consider the more general integrals

$$\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\widehat{k}_x^{2n_x} \widehat{k}_y^{2n_y} \widehat{k}_z^{2n_z} \widehat{k}_t^{2n_t}}{D_B(k,m)^{p+\delta}}$$

where p is an arbitrary integer (not necessarily positive) and δ is a real number which will be set to zero at the end of the calculations

Reduction of integrals

To begin with, each integral $\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t)$ can be reduced through purely algebraic manipulations to a sum of integrals of the same type, but with $n_x = n_y = n_z = n_t = 0$ (i.e., pure denominators)

This is achieved by using the recursion relations

$$\begin{aligned}
\mathcal{B}_\delta(p;1) &= \frac{1}{4}\left[\mathcal{B}_\delta(p-1) - m^2\mathcal{B}_\delta(p)\right] \\
\mathcal{B}_\delta(p;x,1) &= \frac{1}{3}\left[\mathcal{B}_\delta(p-1;x) - \mathcal{B}_\delta(p;x+1) - m^2\mathcal{B}_\delta(p;x)\right] \\
\mathcal{B}_\delta(p;x,y,1) &= \frac{1}{2}\left[\mathcal{B}_\delta(p-1;x,y) - \mathcal{B}_\delta(p;x+1,y) - \mathcal{B}_\delta(p;x,y+1)\right. \\
&\qquad \left. -m^2\mathcal{B}_\delta(p;x,y)\right] \\
\mathcal{B}_\delta(p;x,y,z,1) &= \mathcal{B}_\delta(p-1;x,y,z) - \mathcal{B}_\delta(p;x+1,y,z) \\
&\qquad -\mathcal{B}_\delta(p;x,y+1,z) - \mathcal{B}_\delta(p;x,y,z+1) - m^2\mathcal{B}_\delta(p;x,y,z)
\end{aligned}$$

(when one of the arguments n_i is zero, it is omitted)

These recursion relations can be obtained from the trivial identity

$$D_B(k,m) = \sum_{i=1}^{4} \widehat{k}_i^2 + m^2$$

Reduction of integrals

With this first set of recursion relations, one can eliminate each numerator argument, n_i, of a $\mathcal{B}_\delta$ integral, provided that it has the value 1

When this is greater than 1, one has to lower its value until it reaches 1 – then it is possible to use the above set of recursion relations

One can lower n_i by employing other recursion relations:

$$\mathcal{B}_\delta(p;\ldots,r) = \frac{r-1}{p+\delta-1}\mathcal{B}_\delta(p-1;\ldots,r-1) - \frac{4r-6}{p+\delta-1}\mathcal{B}_\delta(p-1;\ldots,r-2) + 4\mathcal{B}_\delta(p;\ldots,r-1),$$

obtained integrating by parts the equation (for $r>1$)

$$\frac{(\widehat{k}_w^2)^r}{D_B(k,m)^{p+\delta}} = 4\frac{(\widehat{k}_w^2)^{r-1}}{D_B(k,m)^{p+\delta}} + 2\frac{(\widehat{k}_w^2)^{r-2}}{p+\delta-1}\sin k_w \frac{\partial}{\partial k_w}\frac{1}{D_B(k,m)^{p+\delta-1}}$$

Careful: for $p=1$ some coefficients in this recursion relation diverge as $1/\delta$ – therefore in order to correctly evaluate $\mathcal{B}_\delta(1;\ldots)$ for $\delta=0$ one needs to compute $\mathcal{B}_\delta(0;\ldots)$ including all terms of order δ

In general, one needs to keep all terms of order δ when computing the intermediate expressions for integrals $\mathcal{B}_\delta(p;n_x,n_y,n_z,n_t)$ with $p\leq 0$

Reduction of integrals

Using all recursion relations introduced so far, any integral $\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t)$ can be reduced to a sum of the form

$$\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) \;=\; \sum_{r=p-n_x-n_y-n_z-n_t}^{p} a_r(m, \delta)\, \mathcal{B}_\delta(r)$$

($a_r(m, \delta)$ are polynomials in m^2, which may diverge as $1/\delta$ for $p > 0$ and $r \leq 0$)

At this point, all that remains to do is to reexpress all $\mathcal{B}_\delta(p)$'s appearing in the above formula in terms of a small finite number of them

To accomplish this, one needs some other recursion relations, which can be obtained starting from the trivial identity

$$\mathcal{B}_\delta(p; 1, 1, 1, 1) - 4\mathcal{B}_\delta(p+1; 2, 1, 1, 1) - m^2\, \mathcal{B}_\delta(p+1; 1, 1, 1, 1) \;=\; 0$$

One applies to this identity the previous procedure until it is reduced to a relation between the $\mathcal{B}_\delta(r)$'s only

Reduction of integrals

So, one arrives to a nontrivial relation of the form

$$\sum_{r=p-4}^{p} b_r(p;\delta)\,\mathcal{B}_\delta(r) + \mathcal{S}(p;m,\delta) = 0$$

where $\mathcal{S}(p;m,\delta) = O(m^2)$ for $p \leq 2$, while for $p > 2$ it is a polynomial in $1/m^2$ (which is finite for $\delta \to 0$)

We can now use the last relation to express all $\mathcal{B}_\delta(p)$'s in terms of $\mathcal{B}_\delta(r)$'s which are only in the range $0 \leq r \leq 3$

To do this, when $p \geq 4$ we just write $\mathcal{B}_\delta(p)$ in terms of $\mathcal{B}_\delta(p-1), \ldots, \mathcal{B}_\delta(p-4)$ and iterate until needed

When $p \leq -1$, we solve the relation in terms of $\mathcal{B}_\delta(p-4)$, make the shift $p \to p+4$, and then use it to write $\mathcal{B}_\delta(p)$ in terms of $\mathcal{B}_\delta(p+1), \ldots, \mathcal{B}_\delta(p+4)$

Again we iterate until needed

Applying recursively these two relations we get, for $p \neq 0, 1, 2, 3$:

$$\mathcal{B}_\delta(p) = \sum_{r=0}^{3} c_r(p;\delta)\,\mathcal{B}_\delta(r) + \mathcal{T}(p;m,\delta)$$

where $\mathcal{T}(p;m,\delta)$ is a polynomial in $1/m^2$

Reduction of integrals

So, the entire procedure allows the general bosonic integral to be written, after a finite number of steps, as

$$\mathcal{B}_\delta(p;n_x,n_y,n_z,n_t) = A(\delta)\,\mathcal{B}_\delta(0)+B(\delta)\,\mathcal{B}_\delta(1)+C(\delta)\,\mathcal{B}_\delta(2)+D(\delta)\,\mathcal{B}_\delta(3)+E(m,\delta)$$

where $E(m,\delta)$ is a polynomial in $1/m^2$

It can be shown that the limit $\delta \to 0$ is safe at this stage, and one finally obtains

$$\mathcal{B}(p;n_x,n_y,n_z,n_t) = A(0) + B(0)\,\mathcal{B}(1) + C(0)\,\mathcal{B}(2) + D(0)\,\mathcal{B}(3) + E(m,0)$$

in terms of three basic constants : $\mathcal{B}(1)$, $\mathcal{B}(2)$ and $\mathcal{B}(3)$

This is a minimal set – no further reductions can be done

It is common practice to write the bosonic results in terms of the three constants Z_0, Z_1 and F_0, defined by

$$\begin{aligned} Z_0 &= \mathcal{B}(1)|_{m=0} \\ Z_1 &= \frac{1}{4}\,\mathcal{B}(1;1,1)|_{m=0} \\ F_0 &= \lim_{m\to 0}\left(16\pi^2\mathcal{B}(2) + \log m^2 + \gamma_E\right) \end{aligned}$$

Reduction of integrals

Explicitly:

$$Z_0 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{4\sum_\lambda \sin^2 \frac{k_\lambda}{2}}$$

$$Z_1 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin^2 \frac{k_1}{2} \sin^2 \frac{k_2}{2}}{\sum_\lambda \sin^2 \frac{k_\lambda}{2}}$$

and

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(4\sum_\lambda \sin^2 \frac{k_\lambda}{2}\right)^2 + m^2} = \frac{1}{16\pi^2}\left(-\log m^2 - \gamma_E + F_0\right)$$

($\gamma_E = 0.57721566490153286\ldots$ *is Euler's constant*)

Values of these basic integrals:

Z_0	0.154933390231060214084837208
Z_1	0.107781313539874001343391550
F_0	4.369225233874758

Reduction of integrals

The two basic constants Z_0 and Z_1 are now known with an incredible high precision – about 400 significant decimal places

Rewriting $\mathcal{B}(1)$ and $\mathcal{B}(2)$ in terms of F_0 and Z_0 is rather trivial

For $\mathcal{B}(3)$ one has

$$\mathcal{B}(3) = \frac{1}{32\pi^2 m^2} - \frac{1}{128\pi^2}\left(\log m^2 + \gamma_E - F_0\right) - \frac{1}{1024} - \frac{13}{1536\pi^2} + \frac{Z_1}{256}$$

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In general, $d-1$ basic constants are enough for all bosonic integrals in d dimensions, and $d-2$ if one only considers finite integrals

This becomes expecially interesting in two spacetime dimensions

On a 2-dimensional lattice, any finite bosonic integral can be written in terms of rational numbers and factors $1/\pi^2$ only

One constant (basic integral) has to be added in case of divergent integrals

Reduction of integrals

Example of reduction:

$$
\begin{aligned}
\mathcal{B}(2;2,1) &= \frac{1}{3}\Big(\mathcal{B}(1;2)-\mathcal{B}(2;3)\Big) \\
&= \frac{1}{3}\Bigg(\Big(\frac{1}{\delta}\mathcal{B}(0;1)-\frac{2}{\delta}\mathcal{B}(0)+4\mathcal{B}(1;1)\Big) \\
&\qquad -\Big(2\mathcal{B}(1;2)-6\mathcal{B}(1;1)+4\mathcal{B}(2;1)\Big)\Bigg) \\
&= \frac{1}{3}\Big(\frac{1}{\delta}\mathcal{B}(0;1)-\frac{2}{\delta}\mathcal{B}(0)+4\mathcal{B}(1;1) \\
&\qquad -\frac{2}{\delta}\mathcal{B}(0;1)+\frac{4}{\delta}\mathcal{B}(0)-8\mathcal{B}(1;1)+6\mathcal{B}(1;1)-4\mathcal{B}(2;1)\Big) \\
&= \frac{1}{3}\Big(-\frac{1}{\delta}\mathcal{B}(0;1)+\frac{2}{\delta}\mathcal{B}(0)+2\mathcal{B}(1;1)-4\mathcal{B}(2;1)\Big)
\end{aligned}
$$

where the limit $\delta = 0$ has been taken when safe

Now,

$$\mathcal{B}(0;1) = \frac{1}{4}\mathcal{B}(-1)$$

Reduction of integrals

. . . however we need an expression for $\mathcal{B}(-1)$ which includes terms of order δ

By applying the recursion relations to the identity

$$\mathcal{B}_\delta(3;1,1,1,1) - 4\mathcal{B}_\delta(4;2,1,1,1) - m^2\,\mathcal{B}_\delta(4;1,1,1,1) \;=\; 0$$

we obtain

$$\mathcal{B}(-1) = 8 + \delta\cdot\Big(-20Z_0 - 48Z_1 + 8\Big) + O(\delta^2)$$

Then:

$$-\frac{1}{\delta}\mathcal{B}(0;1)+\frac{2}{\delta}\mathcal{B}(0) = -\frac{1}{4\delta}\left(8+\delta\Big(-20Z_0-48Z_1+8\Big)\right) + \frac{2}{\delta} = 5Z_0+12Z_1-2$$

which is finite (as it should be) in the limit $\delta \to 0$

The remaining integrals are very simple *(use the first recursion relation)*:

$$\mathcal{B}(1;1) = \frac{1}{4}\mathcal{B}(0) = \frac{1}{4}; \qquad \mathcal{B}(2;1) = \frac{1}{4}\mathcal{B}(1;0) = \frac{1}{4}Z_0$$

So, we can finally obtain the result of the decomposition:

$$\boxed{\mathcal{B}(2;2,1) = \frac{4}{3}Z_0 + 4Z_1 - \frac{1}{2}}$$

Reduction of integrals

The general integral which includes quarks in 1-loop calculations of Wilson fermions requires a much more complicated procedure
(Burgio, Caracciolo and Pelissetto, Nucl. Phys. B478 (1996) 687)

Any lattice zero-momentum integral coming from the calculation of lattice Feynman diagrams in the general Wilson case can be written as a sum of terms of the form

$$\mathcal{F}(p,q;n_x,n_y,n_z,n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\widehat{k}_x^{2n_x}\widehat{k}_y^{2n_y}\widehat{k}_z^{2n_z}\widehat{k}_t^{2n_t}}{D_F(k,m_f)^p D_B(k,m_b)^q}$$

(p, q and n_i positive integers)

Unlike the bosonic case, a description of the reduction steps is too long

For fermions, the procedure requires several sets of recursion relations (and rather complicated, too)

It turns out that every $\mathcal{F}_\delta(p,q;n_x,n_y,n_z,n_t)$ with $q \leq 0$ (i.e., a purely fermionic integral) can be expressed iteratively in terms of nine purely fermionic integrals:

$\mathcal{F}(1,0)$, $\mathcal{F}(1,-1)$, $\mathcal{F}(1,-2)$, $\mathcal{F}(2,0)$, $\mathcal{F}(2,-1)$, $\mathcal{F}(2,-2)$, $\mathcal{F}(3,-2)$, $\mathcal{F}(3,-3)$ and $\mathcal{F}(3,-4)$

Reduction of integrals

Purely fermionic integrals can always be expressed in terms of integrals of the same type – this is a general property of all recursion relations

The integral $\mathcal{F}(2,0)$ appears only in the case of divergent integrals

This is associated to a constant Y_0, which appears in the logarithmic divergent integral

$$\mathcal{F}(2,0) = -\frac{1}{16\pi^2}\left(\log m^2 + \gamma_E - F_0\right) + Y_0$$

Only eight constants are then needed if the original purely fermionic integral is finite (i.e., $q \leq 0$ and $p \leq 1$)

In the general case in which q can be positive (mixed fermionic-bosonic integrals) one needs three additional constants, Y_1, Y_2 and Y_3:

$$\begin{aligned} Y_1 &= \frac{1}{8}\,\mathcal{F}(1,1;1,1,1) \\ Y_2 &= \frac{1}{16}\,\mathcal{F}(1,1;1,1,1,1) \\ Y_3 &= \frac{1}{16}\,\mathcal{F}(1,2;1,1,1) \end{aligned}$$

Then, including the "bosonic" constants Z_0, Z_1 and F_0, we complete the set needed for Wilson fermions

Reduction of integrals

Values of the (new) basic integrals:

$\mathcal{F}(1,0)$	0.08539036359532067914
$\mathcal{F}(1,-1)$	0.46936331002699614475
$\mathcal{F}(1,-2)$	3.39456907367713000586
$\mathcal{F}(2,-1)$	0.05188019503901136636
$\mathcal{F}(2,-2)$	0.23874773756341478520
$\mathcal{F}(3,-2)$	0.03447644143803223145
$\mathcal{F}(3,-3)$	0.13202727122781293085
$\mathcal{F}(3,-4)$	0.75167199030295682254
Y_0	$-$ 0.01849765846791657356
Y_1	0.00376636333661866811
Y_2	0.00265395729487879354
Y_3	0.00022751540615147107

This method depends on the form of the quark propagator, but not on the vertices

It can thus be applied to $O(a)$ improved Wilson fermions as well

Reduction of integrals

Using these reduction methods, it is possible to give a purely algebraic result for the 1-loop quark self-energy

The 1-loop quark self-energy

$$\Sigma(p^2,m^2) = g_0^2 C_F \left(\widetilde{m}_c + \mathrm{i}\not{p}\, \widetilde{\Sigma}_1(p^2,m^2) + m\, \widetilde{\Sigma}_2(p^2,m^2) \right)$$

in terms of the basic constants is (in Feynman gauge) *(Burgio et al., 1996)* :

$$\widetilde{m}_c = -Z_0 - 2\mathcal{F}(1,0) \approx -0.32571411742170157236$$

$$\begin{aligned}
\widetilde{\Sigma}_1(p^2,m^2) \quad = \quad & \frac{1}{16\pi^2}(2G(p^2a^2,m^2a^2) + \gamma_E - F_0) + \frac{1}{8}Z_0 + \frac{1}{192} \\
& - \frac{1}{32\pi^2} - Y_0 + \frac{1}{4}Y_1 - \frac{1}{16}Y_2 + 12\,Y_3 - \frac{1}{768}\mathcal{F}(1,-2) \\
& - \frac{1}{192}\mathcal{F}(1,-1) + \frac{109}{192}\mathcal{F}(1,0) - \frac{1}{768}\mathcal{F}(2,-2) + \frac{25}{48}\mathcal{F}(2,-1) \\
& \approx \frac{1}{8\pi^2}G(p^2a^2,m^2a^2) + 0.0877213749
\end{aligned}$$

Reduction of integrals

$$\begin{aligned}
\widetilde{\Sigma}_2(p^2,m^2) &= \frac{1}{4\pi^2}(F(p^2a^2,m^2a^2)+\gamma_E-F_0)+\frac{1}{48}-\frac{1}{4\pi^2} \\
&\quad -4\,Y_0+\,Y_1-\frac{1}{4}Y_2-\frac{1}{192}\mathcal{F}(1,-2)-\frac{1}{48}\mathcal{F}(1,-1) \\
&\quad -\frac{83}{48}\mathcal{F}(1,0)-\frac{1}{192}\mathcal{F}(2,-2)+\frac{49}{12}\mathcal{F}(2,-1) \\
&\quad \approx \frac{1}{4\pi^2}F(p^2a^2,m^2a^2)+0.0120318529
\end{aligned}$$

where

$$\begin{aligned}
F(p^2a^2,m^2a^2) &= \int_0^1 dx\,\log[(1-x)(p^2x+m^2)a^2] \\
G(p^2a^2,m^2a^2) &= \int_0^1 dx\,x\,\log[(1-x)(p^2x+m^2)a^2]
\end{aligned}$$

The importance of having explicit expressions like these cannot be underestimated

Thanks to them, the 1-loop self-energy $\Sigma(p^2,m^2)$ can be computed with many significant decimal places, provided the basic constants are determined with sufficient accuracy

Some References on Lattice Perturbation Theory

Books:

- I. Montvay and G. Münster, *Quantum Fields On A Lattice*
- H. Rothe, *Lattice Gauge Theories: An Introduction*
- J. Smit, *Introduction To Quantum Fields On A Lattice: A Robust Mate*
- T. DeGrand and C. DeTar, *Lattice Methods for Quantum Chromodynamics*

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- H. Kawai, R. Nakayama and K. Seo, *Comparison of the lattice Λ parameter with the continuum Λ parameter in massless QCD*, Nuclear Physics B 189 (1981) 40
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