Heavy flavour physics from lattice QCD

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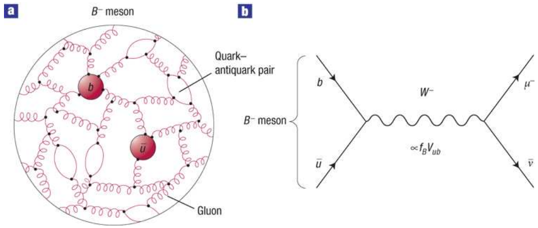
(Lattice) QCD and the weak interaction

New Physics effects expected in the quark flavour sector, because most extensions of the Standard Model contain

- new CP-violating phases
- new quark flavour-changing interactions



Changes of quark flavour inside a hadron are weak interaction processes

$\rightarrow$ Due to confinement, QCD corrections to the decay rate are significant

$\rightarrow$ Non-perturbative QCD effects typically absorbed into hadronic matrix elements such as decay constants, form factors and bag parameters

$\Rightarrow$ A task for lattice QCD

Scope of heavy quark physics from LQCD

Current computations / studies include

- Spectroscopy (charmonium, bottomium, beauty-hadrons)
- Heavy quark masses (m_c, m_b)
- Leptonic B-meson decays & B-meson mixing

  (e.g., to understand CP-violation in the Standard Model and beyond)
- Semi-leptonic decay form factors of D's & B's
- Analyses of the CKM matrix via the theoretical formula

$$\begin{pmatrix} \text{measured} \\ \text{quantity} \end{pmatrix} = \begin{pmatrix} \text{kinematic} \\ \text{factor} \end{pmatrix} \begin{pmatrix} \text{short-distance} \\ \text{factor} \end{pmatrix} \begin{pmatrix} \text{QCD} \\ \text{factor} \end{pmatrix}$$

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The need for an *effective* theory

$m_c \approx 1.27\,\text{GeV}$ $m_b \approx 4.2\,\text{GeV}$ $m_{J/\psi} \approx 3.1\,\text{GeV}$ $m_{\Upsilon} \approx 9.46\,\text{GeV}$

$\rightarrow$ To reliably describe these states and treat them in numerical simulations, the lattice cutoff $1/a$ should be larger than their m's

$\rightarrow$ Comfortable spatial volumes to accomodate heavy hadrons would then amount to lattice sizes $\gtrsim O(100)^4$

$\Rightarrow$ Central idea:
remove the heavy (valence) quark mass as the dominant scale

The CKM matrix . . .

. . . encodes the mixing between quark flavours under weak interactions

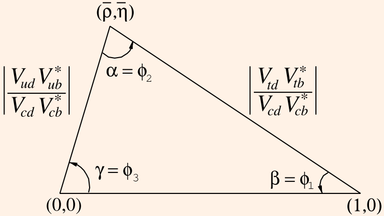
$$\underbrace{\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}}_{\text{weak int.}} = V_{CKM} \underbrace{\begin{pmatrix} d \\ s \\ b \end{pmatrix}}_{\text{strong int.}} \qquad V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Wolfenstein parametrization of the CKM matrix

- Empirically, matrix elements are largest among the diagonal
  $\rightarrow$ hierarchy gets explicit by expansion in powers of $|V_{us}| = \lambda \simeq 0.22$
- $\exists$ unitarity relations such as $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$
  $\rightarrow$ V_{CKM} represented as unitarity triangle in the complex (ρ, η)–plane

up to $O(\lambda^4)$:

$$\begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$



Impact of LQCD on precision heavy flavour physics

Heavy quark sector constrains UT: angles & sides are related to hadronic matrix elements of $\mathcal{H}_{\rm weak}^{\rm (eff)}$, corresponding to mesonic decays/transitions

$$\Delta m_d \propto F_{B_d}^2 \widehat{B}_{B_d} |V_{td} V_{tb}^*|^2 \qquad \frac{\Delta m_s}{\Delta m_d} = \xi^2 \frac{m_{B_s}}{m_{B_d}} \frac{|V_{ts}|^2}{|V_{td}|^2} \qquad \xi = F_{B_s} \sqrt{\widehat{B}_{B_s}} \Big/ F_{B_d} \sqrt{\widehat{B}_{B_d}}$$

- $\exists$ large number of experimental data from heavy flavour-factories (CLEO, BaBar, Belle, LHCb, . . .)
- Inputs of theory and predominantly LQCD computations needed to
  - interpret results of experimental measurements
  - determine / pin down heavy quark masses & CKM matrix elements
  - overconstrain unitarity relations $\leftrightarrow$ unveiling New Physics effects

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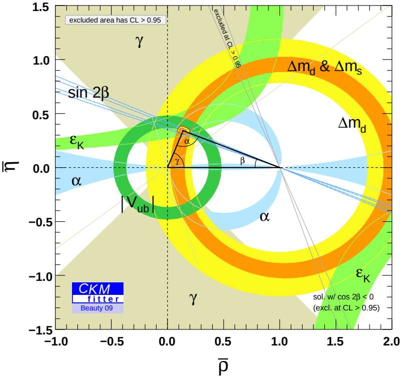
$$\begin{pmatrix} \mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\ \pi \to \ell\nu & K \to \ell\nu & B \to \pi\ell\nu \\ & K \to \pi\ell\nu & \\ \mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\ D \to \ell\nu & D_s \to \ell\nu & B \to D\ell\nu \\ D \to \pi\ell\nu & D \to K\ell\nu & B \to D^*\ell\nu \\ \mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \\ B_d \leftrightarrow \overline{B}_d & B_s \leftrightarrow \overline{B}_s & \end{pmatrix}$$

"Gold-plated" lattice processes

- 1 hadron in the initial state, 0 or 1 hadron in the final state
- stable hadrons (or narrow, far from theshold)
- controlled χ-extrapolation

Impact of LQCD on precision heavy flavour physics



- Constrain apex $(\bar\rho, \bar\eta)$ as precisely as possible by independent processes
- Theory & Exp. sufficiently precise
  $\Rightarrow$ New Physics $=$ inconsistent $(\bar\rho, \bar\eta)$
- LQCD inputs from the heavy sector:
  - B-meson decays & mixing: F_B, B_B
  - $B \to D^{(*)}$ decays:
    $F(1), G(1) \hookrightarrow |V_{cb}|$
  - semi-leptonic B-meson decays:
    $f_+(q^2) \hookrightarrow |V_{ub}|$

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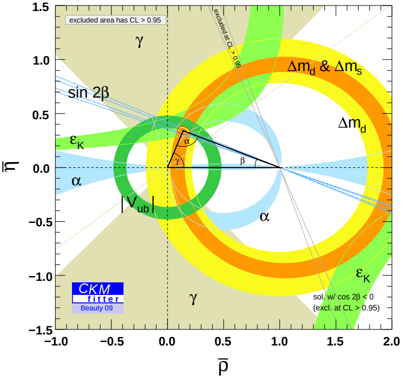

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What is the required precision for key contributions to phenomenology ?

- Experiments reach few-% level, even $\lesssim 5\%$ $\Rightarrow$ theory error dominates
  $\Delta m_{d,s}$: $< 1\%$ [PDG,CDF], $\mathcal{B}(D_{(s)} \to \mu\nu)$: $\lesssim 4\%$ [CLEO-c], $\mathcal{B}(B \to D^* \ell\nu)$: 1.5% [HFAG]
- Lattice calculations with an accuracy of $O(5\%)$ or better required
  $\to$ incl. *all* systematics (unquenching, extrapolations, renormalization, . . .)
- Verification/Agreement of results using different formulations crucial !

Outline

1. Lecture 1: Introduction to heavy quarks on the lattice
   - Lattice QCD: Basics & Challenges
   - Effective theories for heavy quarks
     - Heavy Quark Effective Theory (HQET) $\rightarrow$ 2nd lecture
     - Non-Relativistic QCD (NRQCD)
     - ”Fermilab” approach
   - Overview of lattice heavy quark formalisms
2. Lecture 2: Non-perturbative Heavy Quark Effective Theory
   - Introduction to HQET
   - Non-perturbative formulation of HQET
   - Mass dependence at leading order in $1/m$
   - Strategy to determine HQET parameters at $O(1/m)$
   - First physical results in two-flavour QCD
3. Conclusions & Outlook

Lecture 1

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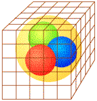
Lattice QCD — The principle

'Ab initio' approach to determine standard model parameters

$$\mathcal{L}_{\mathrm{QCD}}[g_0, m_f] = -\frac{1}{2g_0^2}\,\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\} + \sum_{f=u,d,s,\ldots} \overline{\psi}_f\{\gamma_\mu(\partial_\mu + g_0 A_\mu) + m_f\}\psi_f$$

$$\underbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_K \\ m_D \\ m_B \end{bmatrix}}_{\text{Experiment}} \overset{\mathcal{L}_{\mathrm{QCD}}[g_0, m_f]}{\Longrightarrow}$$



$$\underbrace{\begin{bmatrix} \Lambda_{\mathrm{QCD}} \\ M_u, M_d \\ M_s \\ M_c \\ M_b \end{bmatrix}}_{\text{QCD parameters (RGIs)}} + \underbrace{\begin{bmatrix} F_D \\ F_B \\ B_K, B_B \\ \xi \\ \ldots \end{bmatrix}}_{\text{Predictions}}$$

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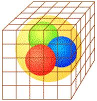
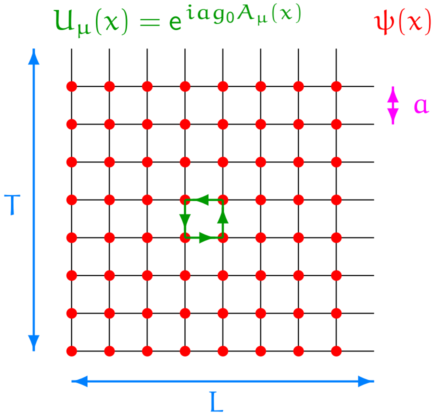


Sources of systematic uncertainties in LQCD computations:

- Part of the vacuum polarization effects is missed, as long as u, d, s (and ideally also c) sea quarks are not incorporated
- Extrapolations to $m_{u,d}$ guided by χPT to connect to the physical world
- Discretization errors, notably from heavy quarks: $O\left[(am_Q)^n\right]$ effects
- Perturbative vs. non-perturbative renormalization



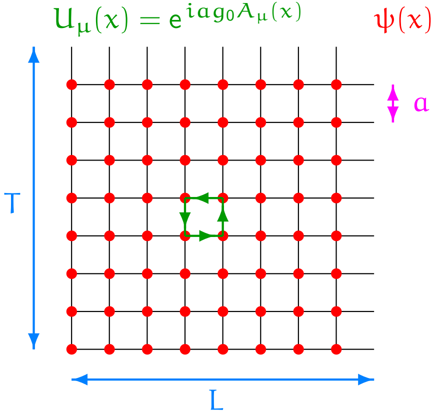
- Lattice cutoff $a^{-1} \sim \Lambda_{UV}$
- Finite volume $L^3 \times T$
- Lattice action

$$S[U, \overline{\psi}, \psi] = S_G[U] + S_F[U, \overline{\psi}, \psi]$$

$$S_G = \frac{1}{g_0^2} \sum_p \mathrm{Tr}\{1 - U(p)\}$$

$$S_F = a^4 \sum_x \overline{\psi}(x)\, D[U]\, \psi(x)$$

- Physical quantities:
  Expectation values,
  represented as path integrals



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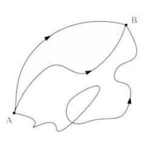
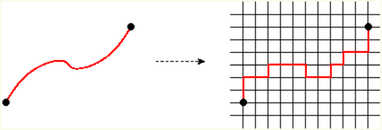
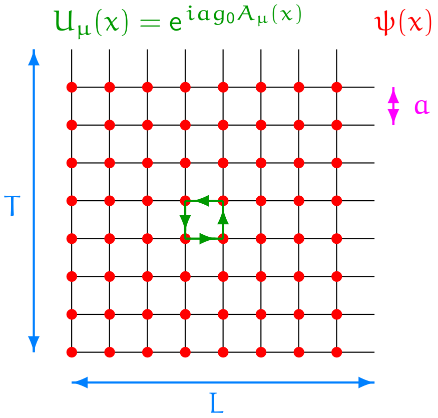
Feynman path (resp. functional) integral



$$\langle B | e^{-H\tau} | A \rangle = \int \mathcal{D}[x(t)]\, e^{-S_E[x]}$$

$$x(0) = A\,,\; x(t) = B$$





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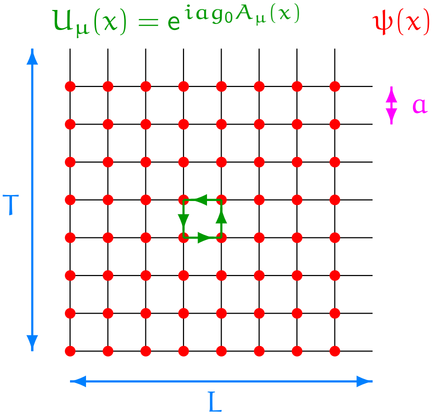
$$\mathcal{Z} = \int \mathcal{D}[U]\mathcal{D}[\overline{\psi},\psi]\, e^{-S[U,\overline{\psi},\psi]} = \int \mathcal{D}[U] \prod_f \det\left(\not{D} + m_f\right) e^{-S_G[U]}$$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \prod_{x,\mu} dU_\mu(x)\, O \prod_f \det\left(\not{D} + m_f\right) e^{-S_G[U]} \;\hat{=}\; \text{thermal average}$$



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Stochastic evaluation with *Monte Carlo (MC) methods*

→ Observables $\langle O \rangle = \frac{1}{N} \sum_{n=1}^{N} O_n \pm \Delta_O$ from numerical simulations

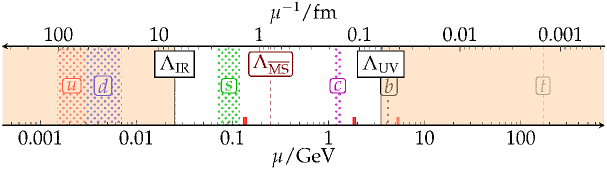
One of the challenges: The multi-scale problem

Predictivity in a quantum field theory relies upon a large scale ratio

interaction range $\ll$ physical length scales

momentum cutoff $\gg$ physical mass scales : $\Lambda_{\text{cut}} \sim a^{-1} \gg E_i, m_j$

This is a challenge in QCD, which has many physical scales:



hierarchy of disparate physical scales to be covered:

$$\Lambda_{IR} = L^{-1} \ll m_\pi, \ldots, m_D, m_B \ll a^{-1} = \Lambda_{UV}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\left\{ O(e^{-Lm_\pi}) \Rightarrow L \gtrsim \frac{4}{m_\pi} \sim 6\,\text{fm} \right\} \curvearrowright \quad L/a \gtrsim 120 \quad \curvearrowleft \left\{ am_D \lesssim \frac{1}{2} \Rightarrow a \approx 0.05\,\text{fm} \right\}$$

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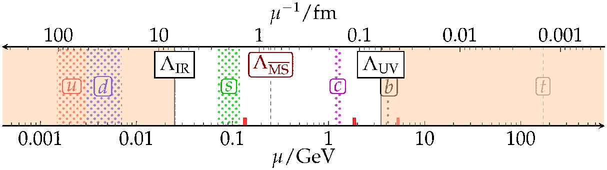


$\Rightarrow$ Difficult to satisfy simultaneously, clever technologies are required

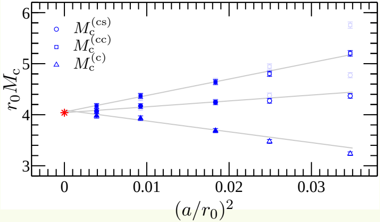
- charm just doable, but lattice artefacts may be substantial
- given the today's computing resources, it seems impossible to work directly with relativistic b-quarks (resolving their propagation) on the currently simulated lattices
- the b-quark scale ($m_b/m_c \sim 4$) has to be separated from the others in a theoretically sound way before simulating the theory

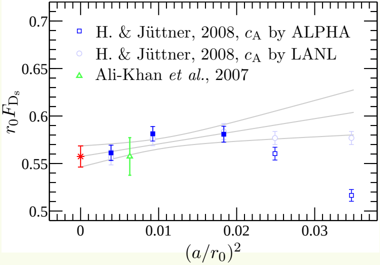
Illustration: Cutoff effects in the charm sector

High-precision computation of the charm quark's mass and F_{D_s} $(N_f = 0)$

- Large volume and small lattice spacings: $a \approx (0.09 - 0.03)$ fm
- $O(a, am_{q,c})$ cutoff effects relevant & removed NP'ly [ALPHA Collaboration]
- Controlling the CL demands scaling study down to very fine lattices

Lattice artefacts may be large for charm physics [H. & Jüttner, 2009]





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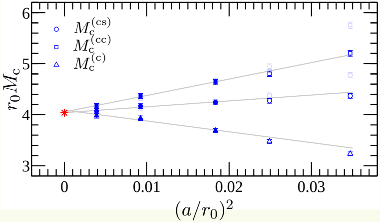
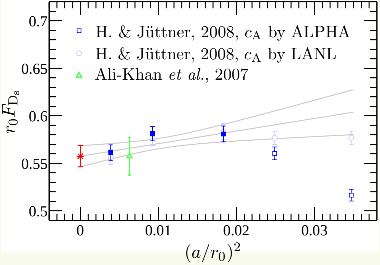
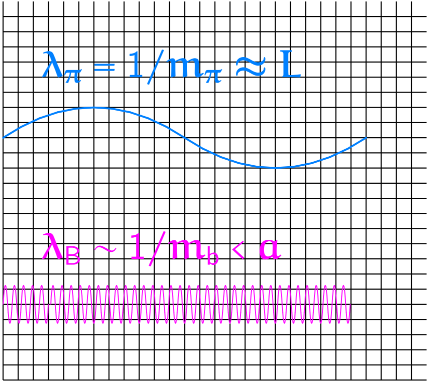




$\Rightarrow$ *Warning* from F_{D_s}:
Symanzik programme works for charm, but $a < 0.08$ fm mandatory

- Note: small lattice spacings are challenging for $N_f > 0$
- $M_b \simeq 4M_c$ s.th. beauty is not yet accomodated
  $\rightarrow$ for b-quarks: can't control $a \rightarrow 0$ his way, *effective theory* needed

Effective theories for heavy quarks — Why ?



- Light quarks: too light
  - Widely spread objects
  - Finite-volume errors due to light pions
- b-quark: too heavy
  - Extremely localized object
  - B-mesons with a propagating b-quark on the lattice require finest resolutions ($am_b \ll 1$), beyond today's computing resources; otherwise:
    - ◇ large discretization errors
    - ◇ "they fall through the lattice"

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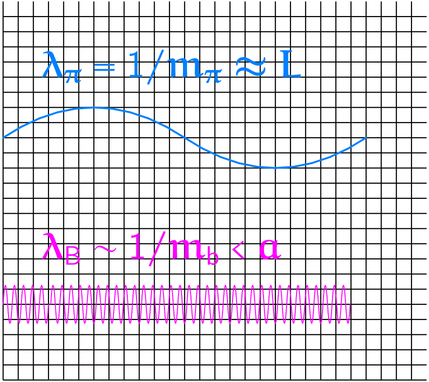

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⇒ Discretize an *effective* theory for the b-quark in heavy-light systems:

Heavy Quark Effective Theory

[Eichten, 1988; Eichten & Hill, 1990]

⇒ Discretize a non-relat. *effective* Lagrangian for heavy-heavy systems:

Non-Relativistic QCD

[Caswell & Lepage, 1986; Lepage & Thacker, 1988 & 1991]

Philosophy behind effective field theories (EFTs)

- EFTs have become increasingly popular in particle physics, because
  - they provide a realization of Wilson renormalization group ideas
  - they fully exploit the properties of local quantum field theories
- An EFT is a quantum field theory with the following properties:
  - it contains the *relevant* aspects (DOFs) of the ”full theory” to describe phenomena occuring in a certain limited range of energies & momenta, while ignoring the *irrelevant* ones
  - it contains an intrinsic energy scale Λ (e.g., $\Lambda_{\rm QCD}$) that sets the limit of its applicability

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- Lagrangian $\mathcal{L}_{\rm EFT}$ is organized in operators of increasing dimension

  $\Rightarrow$ it is in general non-renormalizable in the usual sense, but
  - it can be made finite to any finite order in $1/\Lambda$ by renormalizing (*matching*) the constants (*matching coefficients*) in front of the operators in $\mathcal{L}_{\rm EFT}$
  - more renormalization conditions needed as order in $1/\Lambda$ increases
- Fixing these constants, e.g., by some experimental input(s), reduces but does not spoil the predictive power of the EFT

Concepts of EFTs for heavy quarks

- As effects of a heavy particle get irrelevant at low energy, it's useful to construct some "easier" *low-energy EFT*, where it no longer appears
  - particle physics example: Fermi's theory of weak interactions
  - limitation: with increasing E, structure of intermediate particles and interactions is more and more resolved s.th. EFT is no longer adequate
- Technically, integrate out the heavy field's DOFs in the generating functional of the Green functions of the theory
  - *non-local* effective action S_{EFT}, rewritten as series of *local* terms (OPE)
  - disentangle physics at long distances (i.e. low E), where S_{EFT} correctly reproduces the full theory, from that at short distances (i.e. high E)

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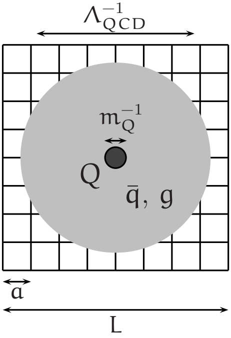
  - particle physics example: Fermi's theory of weak interactions
  - limitation: with increasing E, structure of intermediate particles and interactions is more and more resolved s.th. EFT is no longer adequate
- Technically, integrate out the heavy field's DOFs in the generating functional of the Green functions of the theory
  - *non-local* effective action S_{EFT}, rewritten as series of *local* terms (OPE)
  - disentangle physics at long distances (i.e. low E), where S_{EFT} correctly reproduces the full theory, from that at short distances (i.e. high E)
- HQET & NRQCD distinguished by the way they classify interactions, as dictated by the physics (underlying dynamics) of heavy-light and heavy-heavy hadrons
  - HQET applies to heavy-light systems only
  - NRQCD can be used for both heavy-light and heavy-heavy systems (e.g., heavy quarkonia)
  - both receive power-law and logarithmic m_Q–dependences
  - only in HQET: $1/m_Q$–terms can be dealt with as operator insertions

HQET — The physical picture

Energy scale governing dynamics of quarks & gluons inside *light* hadrons:

- QCD scale Λ_{QCD}, characterizing the momentum scale where the QCD coupling α_s becomes large

A *heavy* quark (Q) introduces a new scale $\rightarrow$ different QCD dynamics :



In the heavy-light meson the motion of the heavy quark of mass m_Q is hardly affected by the light DOFs of typical momentum Λ_{QCD}, if $m_Q \gg \Lambda_{QCD}$

$$p_Q^\mu = m_Q v^\mu + k^\mu \qquad v^2 = 1$$

with residual momentum: $k \sim O(\Lambda_{QCD}) \ll m_Q v$

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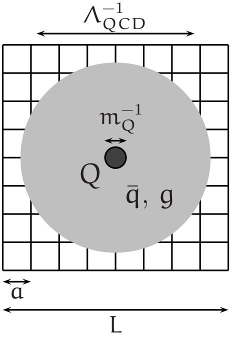

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Typical momentum scales in heavy-light ($Q\bar{q}$) mesons:

- Q almost at rest at bound state's center, surrounded by the light DOFs
- Motion of the heavy quark is suppressed by Λ_{QCD}/m_Q

HQET — The physical picture

More formally, e.g., in case of the B-meson system:

$\mathcal{L}_{\mathrm{HQET}} =$ asymptotic $1/m_b$–expansion of continuum QCD

$$\overline{\psi}_b\{\gamma_\mu D_\mu + m_b\}\psi_b \quad \longrightarrow$$

$$\mathcal{L}_{\mathrm{HQET}}(x) = \overline{\psi}_h(x)\Big[\underbrace{D_0 + m_b}_{\text{static limit}} - \underbrace{\omega_{\mathrm{kin}}}_{\sim \frac{1}{m_b}} \mathbf{D}^2 - \underbrace{\omega_{\mathrm{spin}}}_{\sim \frac{1}{m_b}} \boldsymbol{\sigma}\cdot\mathbf{B}\Big]\psi_h(x) + \ldots$$

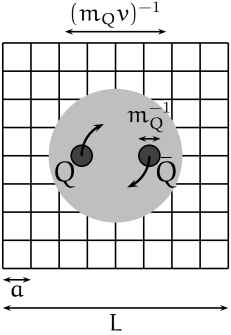
$$= \mathcal{L}_{\mathrm{stat}}(x) + \mathrm{O}\left(\frac{1}{m_b}\right)$$

- $P_+\psi_h = \psi_h$ with $P_+ = \frac{1}{2}(1+\gamma_0)$ $\Rightarrow$ only 2 independent DOFs
- $\mathcal{L}_{\mathrm{stat}}$ represents a non-moving heavy quark, acting only as a static color source
- Systematic & accurate expansion for $m_b/\Lambda_{\mathrm{QCD}} \gg 1$
- Respects heavy quark spin-symmetry in the static limit
- . . .

$\rightarrow$ 2nd lecture

NRQCD — The physical picture

The dynamics of *quarkonium* is governed by different energy scales:



Classically, in the heavy-heavy meson the non-relativistic kinetic energy $\langle p^2 \rangle / (2m_Q)$ and the potential energy $-\frac{4}{3}\alpha_s \left\langle \frac{1}{r} \right\rangle$ have to be balanced, and the heavy quarks move around each other

$\langle p \rangle$: typ. size of the relative spatial momentum

$\left\langle \frac{1}{r} \right\rangle$: typ. distance between Q and $\overline{Q}$

$\Rightarrow$ $\langle p \rangle \sim \alpha_s m_Q$ (uncertainty relation $\langle p \rangle \sim \langle 1/r \rangle$)

$\Rightarrow$ typical velocity $v \sim \langle p \rangle / m_Q \sim O(\alpha_s)$

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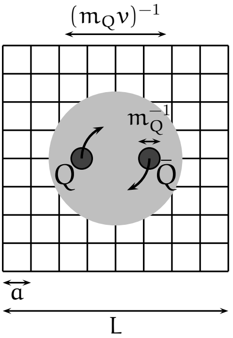

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Typical momentum scales in heavy-heavy ($Q\bar{Q}$) mesons:

- Heavy quark's mass m_Q (hard)
- Spatial momentum $\langle p \rangle \approx m_Q v$ (soft)
- Binding energy $\langle p^2 \rangle / m_Q \approx m_Q v^2$ (ultrasoft)
- Scale hierarchy, ordered via velocity $v \ll 1$: $m_Q \gg m_Q v \gg m_Q v^2$

Effective Lagrangian for heavy quarks

Non-pert. QCD dynamics gets important only in low-energy regime ($\sim\Lambda_{QCD}$)

- idea: remove / separate dominant scale m_Q from low-energy DOFs
- sufficient to work with an *effective Lagrangian*, which only treats the low-energy part in the lattice simulation
- high-energy part may be reliably treated in perturbation theory

Consider a heavy hadron in its rest frame ($v^\mu = (1,0)$) and decompose

$$p_Q^\mu = (m_Q + k^0, \mathbf{k})$$

$\Rightarrow$ Decomposition of the 4–component heavy quark Dirac spinor into

$$\psi_b = e^{-i m_Q t} \begin{pmatrix} \psi_h \\ \Psi_H \end{pmatrix} \qquad \left.\begin{matrix} \psi_h \\ \Psi_H \end{matrix}\right\} = \left\{\begin{matrix} e^{i m_Q t} P_+ \psi_b(x) \\ e^{i m_Q t} P_- \psi_b(x) \end{matrix}\right.$$

in terms of 2–component ”large” and ”small” spinors ψ_h and Ψ_H

- separates the phase factor with the trivial dependence on m_Q due to the heavy quark's free motion with $m_Q v^\mu$

Effective Lagrangian for heavy quarks

⇒ Dirac equation $(i\gamma^\mu D_\mu - m_Q)\psi_b = 0$ splits into two parts:

$$\begin{aligned} i D_0 \psi_h &= i\,\boldsymbol{\sigma}\cdot\mathbf{D}\,\Psi_H \\ (2m_Q + i D_0)\,\Psi_H &= i\,\boldsymbol{\sigma}\cdot\mathbf{D}\,\psi_h \end{aligned}$$

⇒ small component field Ψ_H is suppressed w.r.t. ψ_h by a factor $\propto \frac{1}{m_Q}$

- upon neglecting the small time-dependence of Ψ_H (i.e. $i D_0 \Psi_H$) and substituting $\Psi_H = [\,i\,\boldsymbol{\sigma}\cdot\mathbf{D}/(2m_Q)\,]\,\psi_h$ into the 1st equation:

$$i D_0 \psi_h = -\left[\frac{\mathbf{D}^2}{2m_Q} + \frac{\boldsymbol{\sigma}\cdot g\,\mathbf{B}}{2m_Q}\right]\psi_h \qquad \text{(NR Schrödinger eq. \& Pauli term)}$$

  g: QCD coupling, $B^i = \frac{1}{2}\epsilon^{ijk}F^{jk}$: magnetic QCD field strength components

- repeating this, finally translates into a (classical) Lagrangian as:

$$\begin{aligned} \mathcal{L}_{heavy} = \bar{\psi}_h \Bigg[& i D_0 + \frac{\mathbf{D}^2}{2m_Q} + \frac{\boldsymbol{\sigma}\cdot g\mathbf{B}}{2m_Q} + \frac{\mathbf{D}\cdot g\mathbf{E} - g\mathbf{E}\cdot\mathbf{D}}{8m_Q^2} \\ & + \frac{\boldsymbol{\sigma}\cdot(i\,\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times i\,\mathbf{D})}{8m_Q^2} + \frac{(\mathbf{D}^2)^2}{8m_Q^3} + \cdots \Bigg]\psi_h \end{aligned}$$

The NRQCD Lagrangian

In a heavy-light hadron:

Light DOFs (light quarks and gluons) have $k \sim O(\Lambda_{QCD})$, while exchange of spatial momenta with Q occurs through $1/m_Q$– and higher-order terms

$\Rightarrow$ motion of Q suppressed by powers of Λ_{QCD}/m_Q $\quad\rightarrow$ HQET: 2nd lecture

In a heavy-heavy hadron:

- balance between potential and kinetic energy determines the momentum of the heavy (anti-)quark
- to satisfy the Pauli equation, one must have

$$\langle i D_0 \psi_h \rangle \sim \left\langle \frac{\mathbf{D}^2}{2m_Q} \psi_h \right\rangle$$

- the heavy quark potential, described by the Coulomb form, implies

$$\left\langle \frac{\alpha_s}{r} \right\rangle \sim \langle \alpha_s p \rangle \sim \frac{\langle p^2 \rangle}{m_Q} \quad \Rightarrow \quad \langle p \rangle \sim \alpha_s m_Q \;, \quad v \sim \frac{\langle p \rangle}{m_Q} \sim O(\alpha_s)$$

NRQCD is an effective theory constructed such as to explicitly separate the dominant energy scales $m_Q \gg m_Q v \gg m_Q v^2$

The NRQCD Lagrangian

Separation of the energy scales is provided by the *velocity* $v \ll 1$:

(heavy quark mass m_Q) $\gg$ (momentum $m_Q v$) $\gg$ (binding energy $m_Q v^2$)

$\Rightarrow$ In NRQCD the counting of terms in $\mathcal{L}_{\text{heavy}}$ is done by powers of v:

- power counting of the various operators

$$\psi_h \sim (m_Q v)^{3/2} \qquad \mathbf{D} \sim m_Q v \qquad D_0 \sim m_Q v^2$$
$$g\mathbf{E} \sim m_Q^2 v^3 \qquad g\mathbf{B} \sim m_Q^2 v^4$$

- therefore, the leading contributions in $\mathcal{L}_{\text{NRQCD}}$ are

$$\overline{\psi}_h \, (i\, D_0)\, \psi_h \qquad \overline{\psi}_h \left(-\frac{\mathbf{D}^2}{2m_Q}\right) \psi_h$$

  other terms are suppressed by a relative power v^2, and higher-order ones can also be included systematically

- α_s to be evaluated at the momentum scale of the exchanged gluons, i.e., at $m_Q v \sim \alpha_s m_Q$

$\Rightarrow$ v^2– expansion effective for $b\bar{b}$ ($v^2 \sim 0.1$), but marginal for $c\bar{c}$ ($v^2 \sim 0.3$)

The NRQCD Lagrangian

Remarks

- Previous NR expansion of the heavy quark's Dirac Lagrangian corresponds to a *Foldy-Wouthuysen-Tani transformation (FWT)*
  - $\Rightarrow$ decoupling of heavy quarks & anti-quarks in 2–comp. spinors $\psi_h, \psi_{\bar{h}}$
    - ◇ in QFT / functional integral language, the FWT trafo is just a change of variables (corresponding to integrating out the small component field Ψ_H)
    - ◇ physics described by theories before & after this trafo is the same up to neglected higher-order terms

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  - ◇ physics described by theories before & after this trafo is the same up to neglected higher-order terms
- NRCD $=$ Expansion of the heavy quark action around its NR limit
  $\Rightarrow$ infinitely massive (static) quark is only a colour source, carries no spin
- $\mathcal{L}^{(n)}_{\text{HQET}}$ consists of all interactions of dimension $n+4$,
  $\mathcal{L}^{(n)}_{\text{NRQCD}}$ consists of all terms that scale like v^{2+n}
  $\Rightarrow$ kinetic energy term $\overline{\psi}_h\left(-\frac{\mathbf{D}^2}{2m_Q}\right)\psi_h$ is essential effect in NRQCD, but sub-leading in HQET
  - ◇ leading $O(m_Q v^2)$ terms: e.g., spin-independent splittings in quarkonia
  - ◇ relativistic $O(m_Q v^4)$ corrections:
    spin-independent & spin-dependent contributions
    $\rightarrow$ spin-splittings are $O(v^2)$ smaller than spin-independent splittings

The NRQCD Lagrangian

Remarks

- Power-counting rules
  - $\Rightarrow$ truncate # operators included in $\mathcal{L}_{\text{NRQCD}}$ at fixed order in v^2/c^2 ($\ll 1$)
- On the quantum level, ultraviolet divergences appear through loops, which render EFTs such as HQET and NRQCD non-renormalizable
  - $\Rightarrow$ In practice, $\mathcal{L}_{\text{heavy}}$ is truncated at some finite order (in $1/m_Q$ or v^2) s.th.
    - ◇ # renormalization conditions is finite & calculations are feasible
    - ◇ # EFT parameters is still finite and its predictivity thus remains

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  - ◇ # renormalization conditions is finite & calculations are feasible
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- Relevant effect of the integrated-out high-energy modes on the low-energy physics is encoded in *matching coefficients* c_i (multiplying the operators in $\mathcal{L}_{\text{heavy}}$) and new local interactions

  $\Rightarrow$ to obtain useful results: restrict momenta to $p < \Lambda < m_Q$ (Λ: cutoff)

  - ◇ $c_i = 1$ at tree-level
  - ◇ excluded momenta (e.g., in gluon loops) reappear in renormalization of the coefficients c_i
  - ◇ c_i are dominated by ultraviolet scales for $\Lambda \gg \Lambda_{\text{QCD}}$ and can thus be computed in perturbation theory by matching low-energy scattering amplitudes to full QCD to some order in α_s and p/m_Q

NRQCD on the lattice

The lattice discretization of $\mathcal{L}_{\text{NRQCD}}$ is straightforward

- Wick rotation to Euclidean space-time ($x^0 \equiv i\, x_4$)
- Covariant (forward and backward) derivatives:

$$\Delta_\mu^{(+)}\psi_h(x) \equiv U_\mu(x)\psi_h(x+\hat{\mu}) - \psi_h(x) \qquad \Delta_\mu^{(-)}\psi_h(x) \equiv \ldots$$

$$\Delta_\mu^{(\pm)} \equiv \frac{1}{2}\left(\Delta^{(+)} + \Delta^{(-)}\right) \qquad \Delta^{(2)} \equiv \sum_i \Delta_i^{(+)}\Delta_i^{(-)}$$

- Hamiltonian $H = H_0 + \delta H$ ordered in powers of Q's squared velocity:

$$H_0 = -\frac{\Delta^{(2)}}{2m_Q}$$

(leading NR & spin-independent, kinetic term)

$$\delta H = -c_1\frac{\left(\Delta^{(2)}\right)^2}{8m_Q^3} - c_2\frac{\boldsymbol{\Delta}^{(\pm)}\cdot g\mathbf{E} - g\mathbf{E}\cdot\boldsymbol{\Delta}^{(\pm)}}{8m_Q^2}$$

(spin-independent relativistic corrections)

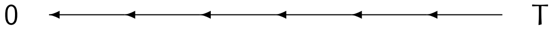
$$-c_3\frac{\boldsymbol{\sigma}\cdot\left(\boldsymbol{\Delta}^{(\pm)}\times g\mathbf{E} - g\mathbf{E}\times\boldsymbol{\Delta}^{(\pm)}\right)}{8m_Q^2} - c_4\frac{\boldsymbol{\sigma}\cdot g\mathbf{B}}{2m_Q}$$

(leading spin-dependent corrections)

$$+ \cdots$$

NRQCD on the lattice

- In the $m_Q \to \infty$ limit the heavy quark Q is static:
  world line $=$ string of SU(3) link matrices in time direction (Wilson line)



- The heavy quark propagator (as a function of spatial indices) on a given time slice obeys an evolution equation:

$$\begin{aligned} U_{4,t} G_{t+1} - G_t &= -a\,H\,G_t \\ \Leftrightarrow \qquad G_{t+1} &= U^{\dagger}_{4,t-1}\,(1 - aH)\,G_t \end{aligned}$$

  $\to$ calculable on one pass through the lattice in the time direction

- Explicit form of the lattice action via a time-evolution kernel K_t:

$$\begin{aligned} S &= \sum_{t,\mathbf{x}} \overline{\psi}_h(t,\mathbf{x})\,[\,\psi_h(t,\mathbf{x}) - K_t \psi_h(t-1,\mathbf{x})\,] \\ K_t &= \left(1 - \frac{aH_0}{2n}\right)^n_t \left(1 - \frac{a\delta H}{2}\right)_t U^{\dagger}_{4,t-1} \left(1 - \frac{a\delta H}{2}\right)_{t-1} \left(1 - \frac{aH_0}{2n}\right)^n_{t-1} \end{aligned}$$

  (parameter n affects only the cutoff scale and stabilizes the evolution equation by suppressing unphysical momenta s.th. reasonable $am_Q \sim O(1)$ can be reached)

NRQCD on the lattice

Remarks

- Coefficients c_i have to be determined s.th. $H = H_0 + \delta H$ matches the Hamiltonian of QCD
  $\Rightarrow$ adjustment to compensate for neglected high-momentum interactions
  - ◇ investigated in lattice perturbation theory by matching scattering amplitudes between lattice NRQCD and full QCD in the continuum
  - ◇ c_i have expansion in $\alpha_s(1/a)$, where the $\mathbf{p}^2/m_Q$–term in the heavy quark propagator gives additional explicit $1/(am_Q)$–contributions (power ultraviolet divergences) that diverge as $a \to 0$
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    $\to$ actual computations mostly employ tadpole-improved tree-level c_i
- Since NRQCD is non-renormalizable and the c_i diverge as $a \to 0$, the continuum limit cannot be taken
  $\Rightarrow$ demonstrate results to be independent of the lattice spacing a within some limited scaling window, staying at relatively large a ($am_Q \gtrsim 0.8$)
- Generically, lattice actions of EFTs contain higher-dimensional operators inducing $1/(am_Q)^n$ power divergences that spoil the CL in case of only *perturbative* matching & renormalization

Aspects of lattice NRQCD calculations

Problem of discretization errors:

Since the lattice spacing has to be kept finite in NRQCD, discretization errors must be corrected for (e.g., momenta are $\mathcal{O}(1\,\mathrm{GeV}$ in heavyonia)

- improved discretization of derivatives to include higher-order terms
- yields improvement terms to be added s.th. residual discretization errors become negligible against other sources of error:

$$\delta H \quad \to \quad \delta H + \delta H_{\mathrm{disc}} \qquad \delta H_{\mathrm{disc}} = c_5 a^2 \frac{\sum_i \Delta_i^{(4)}}{24 m_Q} - c_6 a \frac{\left(\Delta^{(2)}\right)^2}{16 n m_Q^2}$$

(only sensible to correct for discretization up to an order, which is comparable with the order of included relativistic corrections)

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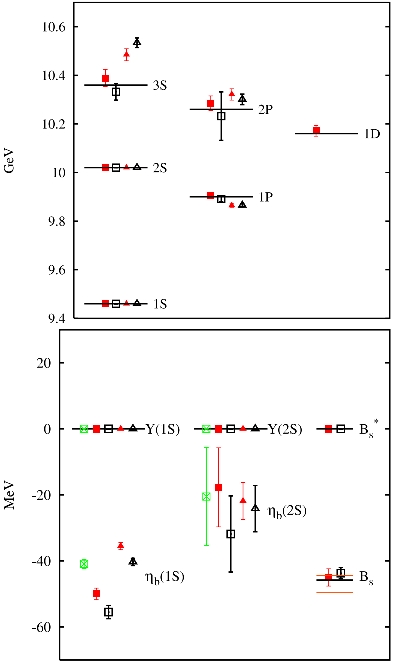
Meson correlation functions:

$\psi^\dagger, \chi^\dagger$: 2–component quark/anti-quark creation operators, Ω: (2×2) spin matrix

$$\left\langle \left(\chi \phi^\dagger \Omega^\dagger \psi\right)_T \left(\psi^\dagger \Omega \phi \chi^\dagger\right)_0 \right\rangle = \left\langle \text{Tr}\left[G \Omega^\dagger \phi^\dagger G^\dagger \phi \Omega\right] \right\rangle \stackrel{T \to \infty}{\longrightarrow} \Phi_1 e^{-E_1 T} + \Phi_2 e^{-E_2 T} + \cdots$$

$\rightarrow$ smearing techniques to optimize computation of radial & orbital excitations

Typical applications of lattice NRQCD include . . .



- Heavy quarkonium spectra; radial, orbital & spin splittings, ...
- Left: radial & orbital levels of the bottomium (Υ), hyperfine splittings in Υ, B_s in $(2+1)$–flavour QCD
  [from HPQCD, Gray et al., PRD(2005)094507)]
- Heavy hybrid mesons
- Heavy-light (resp. B-meson) systems
  - ▶ different dynamics and power counting rules (i.e., in Λ_{QCD}/m_Q)
  - ▶ properties of states determined by the light quarks & glue
  - ▶ Quantities under study:
    - ◇ B-meson masses & splittings (B, B_s, B_c, B_c^*)
    - ◇ decay constants F_B, F_{B_s}; B-meson mixing parameters

[see recent work of HPQCD Collab., Davies et al.]

"Fermilab" approach [El-Khadra, Kronfeld & Mackenzie, PRD55(1997)3933]

Adapts standard Wilson light fermion action plus higher-dim. operators (spatial derivatives) to reduce $(ap_Q)^n$ – errors & better match contin. QCD

- Different split of Symanzik's expansion into "large" + "small" suitable for $am_Q > 1$, rearrangement absorbed in short-distance coefficients of $\mathcal{L}_{\rm int}$

$$\mathcal{L}_{\rm Symanzik} = \mathcal{L}_{\rm gauge} + \overline{\psi}_b \left(\gamma_4 D_4 + \sqrt{\frac{m_1}{m_2}}\, \boldsymbol{\gamma} \cdot \mathbf{D} + m_1 \right) \psi_b + \mathcal{L}'_{\rm int}$$

  - ▶ Coefficients are perturbative series in α_s, but to all orders in am_Q
  - ▶ rest mass & kinetic mass m_1, m_2 in $E(\mathbf{p}) = m_1 + \mathbf{p}^2/(2m_2) + O(\mathbf{p}^4)$ differ sizably, lattice parameters to be NP'ly adjusted s.th. $m_1 = m_2$
  - ▶ finite – a effects estimated via Symanzik effective theory interpretation

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- Formalism covers entire mass regions (from light to heavy) with a single fermion action, achievable accuracy varies depending on am_Q:
  - ▶ at *large* am_Q (heavy quark regime) the Lagrangian becomes EFT-like, (i.e.,its accuracy is estimated in terms of HQET / NRQCD counting rules)
  - ▶ at *small* am_Q it reduces to Symanzik improvement of light quarks

  $\Rightarrow$ Fermilab action "smoothly interpolates" the static & light quark

- As discretization & perturbative errors depend on am_Q, continuum limit of B-meson observables is non-trivial and source of systematic uncertainty

Overview of lattice heavy quark formalisms

Lattice heavy quark physics has to deal with the presence of

strong lattice artefacts : $am_c \lesssim 1 \qquad am_b > 1$

Heavy quarks introduced as valence quarks $=$ ”Partially quenched” setting

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Relativistic formulations $\rightarrow$ mainly for D-physics applications

- Wilson-like (clover or twisted mass) quarks
  - $\mathrm{O}\left[(am_c)^2\right]$ discretization effects
  - $am_c \leqslant 1/2 \ll 1$ desirable — ALPHA, ETMC
- Fermilab approach & its variants $=$ RHQ actions
  - relativistic clover actions with HQET interpretation
  - adopted for charm & beauty — FNAL & MILC, PACS-CS, RBC & UKQCD
- *H*ighly *I*mproved *S*taggered *Q*uarks $=$ HISQ — [HPQCD, Follana et al., 2007]
  - perturbative Symanzik-improvement / smearing of the gauge fields
    $\Rightarrow$ no tree-level $\mathrm{O}(a^2)$, $\mathrm{O}\left[(am_Q)^4, \alpha_s(am_Q)^2\right]$ errors to LO in v/c
  - 1–loop taste-changing interactions reduced by a factor ~ 3
  - e.g., allows in principle a direct computation of heavy-to-light quark mass ratios ($\overline{m}_c/\overline{m}_s$), as both fermion discretizations are the same
  - now also being tried towards the bottom region — HPQCD

Overview of lattice heavy quark formalisms

Lattice heavy quark physics has to deal with the presence of

strong lattice artefacts : $am_c \lesssim 1 \qquad am_b > 1$

Heavy quarks introduced as valence quarks = "Partially quenched" setting

Non-relativistic / effective field theory strategies → B-physics applications

- NRQCD = Discretized, non-relativistic expansion of continuum $\mathcal{L}_D$
  - improved through $O(1/m_Q^2, a^2)$ and leading relativistic $O(1/m_Q^3)$
  - $O\left[\alpha_s^n/(am_Q)\right]$ divergences

  HPQCD
- Static approximation = Leading-order HQET
  - HQET-guided extrapolations (HQ scaling laws) of relativistic simulations in the charm regime, turning into interpolations if the static limit known
  - also in conjunction with finite-volume / finite-size scaling techniques

  INFN-TOV, ALPHA, ETMC
- HQET for the b-quark = Systematic expansion in Λ_{QCD}/m_b
  - NP fine-tuning of parameters to $O(1/m_b)$ & impr. statistical precision
  - connect different volumes iteratively with "step scaling functions"

  ALPHA

Overview of light sea quark configurations in use

[in current studies of lattice heavy quark physics]

Quenched approximation ($N_f = 0$)

- No dynamical fermions, not suitable for phenomenology
- Still useful test laboratory, e.g., to understand methodologies etc.

Two-flavour QCD ($N_f = 2$)

- NP'ly $O(a)$ improved Wilson ($=$ clover) fermions — ALPHA, QCDSF
  - theoretically sound and "simple"
  - algorithmic progress (e.g., "Hasenbusch trick" and M. Lüscher's DD-HMC) render simulations competitive in the chiral regime
- Twisted mass Wilson fermions — ETMC
  - tree-level Symanzik-improved gluon action
  - $O(a)$ improved by tuning to maximal twist; keep an exact χ-symmetry at the price of breaking part of the flavour symmetries and parity
- Stout-smeared, chirally improved fermions — BGR
  - 1–loop improved Lüscher-Weisz gluon action

Overview of light sea quark configurations in use

[in current studies of lattice heavy quark physics]

Three-flavour QCD ($N_f = 2 + 1$)

- AsqTad-improved staggered quarks — MILC & FNAL, HPQCD
  - Lüscher-Weisz-improved gluon action
  - computationally "cheap", permit simulations within the chiral regime
  - debated rooting prescription $\left[\det^{(4)}(D_{st} + m)\right]^{\frac{1}{4}} \equiv \det^{(1)}(\gamma_\mu D_\mu + m)$, but effects seem to disappear in the CL; results agree with experiment
- Domain wall fermions — RBC & UKQCD
  - Iwasaki gauge action
  - chirality preserving (realized as 5th dimension $L_s = \infty$)
- NP'ly $O(a)$ improved Wilson fermions — PACS-CS
  - Iwasaki gauge action

Four-flavour QCD ($N_f = 2 + 1 + 1$) — in progress, e.g., by ETMC

Light *valence* quarks usually discretized in the same way as the sea

Summary of current LHQP calculations

group	$a\,[\,\mathrm{fm}\,]$	$m_\pi^{(\mathrm{min})}\,[\,\mathrm{MeV}\,]$	q	Q
		$N_f = 2$		
ETMC	0.05, 0.065, 0.085, 0.10	270	TM	static / TM
Regensburg	0.08	170	clover	clover
ALPHA	0.08, 0.07, 0.05	250	clover	static $+ 1/m$
		$N_f = 2 + 1$		
FNAL & MILC I	0.09, 0.12, 0.15	230	AsqTad	Fermilab
FNAL & MILC II	0.06, 0.09, 0.12, 0.15	230	AsqTad	Fermilab
HPQCD I	0.09, 0.12	260	AsqTad	NRQCD
HPQCD II	0.09, 0.12, 0.15	320	HISQ	NRQCD
HPQCD III	0.045, 0.06, 0.09, ...	320	HISQ	HISQ
RBC & UKQCD	0.08, 0.11	330 (300)	DW	static / RHQ
PACS-CS	0.09	200	clover	RHQ
		$N_f = 2 + 1 + 1$		
ETMC	0.06, 0.079, 0.09	270 (230)	TM	Osterw.-Seiler

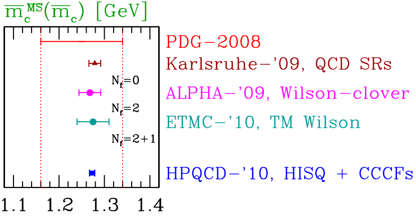
static $\equiv$ smeared static (HYP, APE)

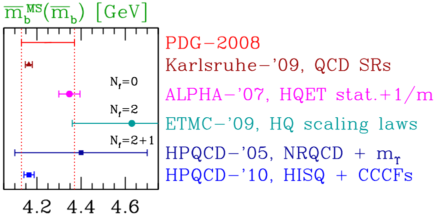
[Status: Lattice Conference 2010]

A glimpse of the status of B-physics parameters

Heavy quark masses

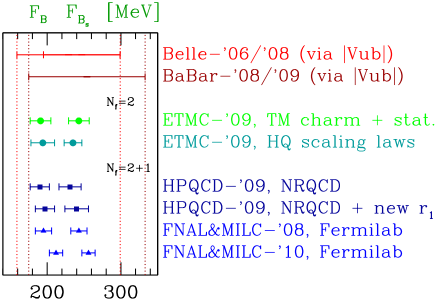
= Inputs to many (B)SM calculations





Hadronic weak matrix elements F_B & F_{B_s}

→ Extract $|V_{ub}|$ via $\underbrace{\mathcal{B}(B^- \to \tau^- \bar{\nu}_\tau)}_{\text{experiment}} \propto |V_{ub}|^2 \underbrace{F_B^2}_{\text{lattice}}$



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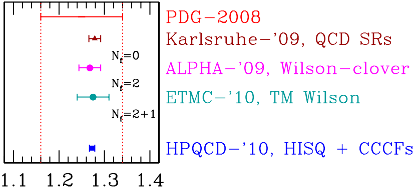
$\overline{m}_c^{\overline{MS}}(\overline{m}_c)$ [GeV]



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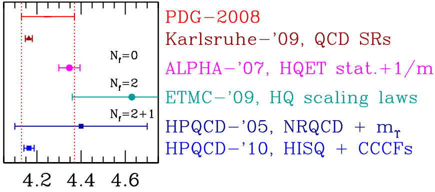

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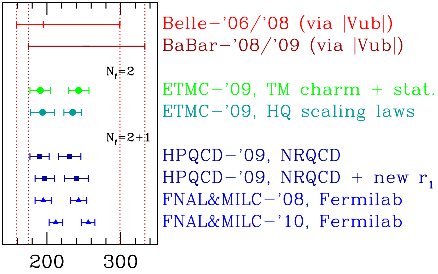

Caveat:

Lattice computations based on NRQCD, Fermilab and HQ scaling laws are standard, however, they all involve *perturbative* renormalization / matching

$\Rightarrow$ Is this accurate enough for precision flavour physics?

A glimpse of the status of B-physics parameters

Heavy quark masses

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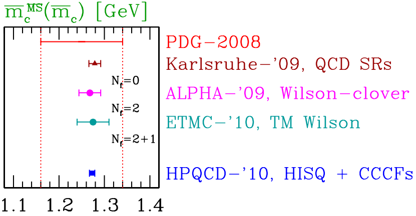

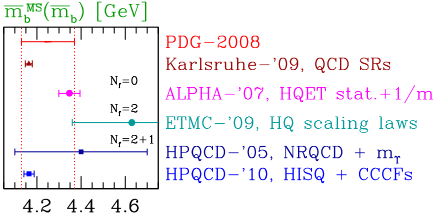

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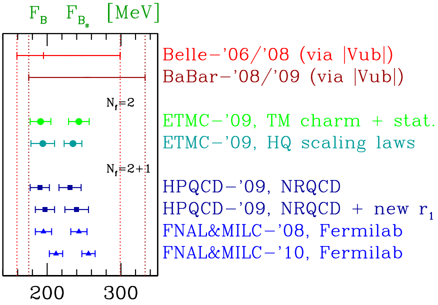


Caveat:

Lattice computations based on NRQCD, Fermilab and HQ scaling laws are standard, however, they all involve *perturbative* renormalization / matching

$\Rightarrow$ Are the claimed small (particularly systematic) errors too optimistic ?

Lecture 2

Non-perturbative Heavy Quark Effective Theory

- Introduction to HQET
- Non-perturbative formulation of HQET
- Mass dependence at leading order in $1/m$
- Strategy to determine HQET parameters at $O(1/m)$
- First physical results in two-flavour QCD

$\rightarrow$ PoS LATTICE2010 (2010) 308 & in progress by

B. Blossier, J. Bulava, M. Della Morte,
M. Donnellan, P. Fritzsch, N. Garron,
J. H., G.M. von Hippel, N. Tantalo,
H. Simma, R. Sommer

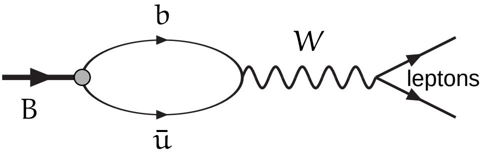
Scale & Quark masses from light sector:

F. Knechtli, B. Leder, S. Schaefer, F. Virotta

CLS based

Motivation — Precision CKM physics

- F_B
  - $\underbrace{\mathcal{B}(B^- \to \tau^- \bar{\nu}_\tau)}_{\text{experiment}} \propto |V_{ub}|^2 \underbrace{F_B^2}_{\text{lattice}}$
  - Process is sensitive probe of charged Higgs boson effects
  - 1.9σ deviation of exp. determ. from LQCD (when using $|V_{ub}|$ exclusive from the lattice)
- F_{B_s}
  - Relevant for CKM analysis & BSM effects in $B_s \to \mu^+\mu^-$ (decay will be measured at LHCb)



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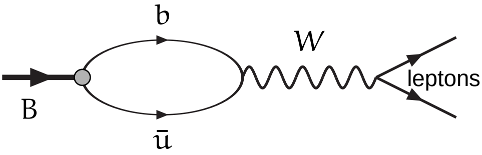

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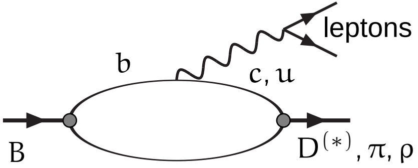

- Semi-leptonic decay form factor $B \to D^* \ell \nu_\ell$
  - Determination of $|V_{cb}|$, which normalizes the whole UT
  - $\sim 2.3\sigma$ tension between inclusive and exclusive $|V_{cb}|$ (latter relying on $B \to D^* \ell \nu_\ell$ from FNAL & MILC 2008)

Introduction to HQET

HQET is constructed to provide a simplified description of processes, in which a heavy quark (Q) strongly interacts with light DOFs by exchange of soft gluons that can only resolve distances $\gg 1/m_Q$ $\Leftrightarrow$ *EFT*

$$
\begin{aligned}
m_Q \gg \Lambda_{QCD} &= \text{high-energy scale} \\
\Lambda_{QCD} \sim 1/R_{had} &= \text{low-energy scale of hadronic physics of interest} \\
\text{Lagrangian} &= \text{systematic expansion in powers of } \Lambda_{QCD}/m_Q
\end{aligned}
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$\lambda_Q \sim 1/m_Q \ll R_{had} \sim 1\,\text{fm} \Rightarrow m_Q$ unimportant for low$-E$ properties of $Q\bar{q}$

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- Light DOFs are blind to flavour & spin of Q and only experience its colour field extending over large distances because of confinement
- *Heavy quark symmetry:* invariance under changes of flavour & spin orientation of Q (leading symmetry breaking corrections at $O(1/m_Q)$)

Introduction to HQET

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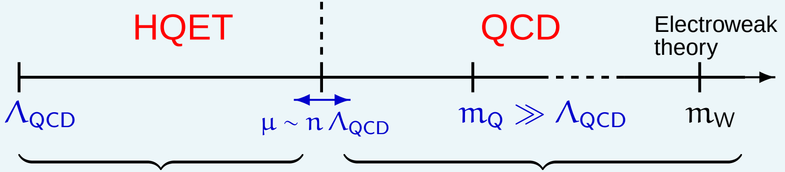
Heavy Quark Effective Theory



Long distance physics:
non-perturbative techniques

Short distance physics:
perturbation theory, renormalization group

Derivation of the HQET Lagrangian :

Start from the Euclidean Dirac-Lagrangian in the continuum

$$\begin{aligned} \mathcal{L} &= \overline{\psi}(D_\mu \gamma_\mu + m)\psi = \psi^\dagger \mathcal{D} \psi \\ \mathcal{D} &\equiv m\gamma_0 + D_0 + \gamma_0 D_k \gamma_k \end{aligned}$$

At the classical level:

One can assume smooth fields and thus can perform an expansion in D_μ, counting $D_\mu = O([1/m]^0)$

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Kinematical situation

- Dynamics of a hadron at rest containing one heavy quark
- $m = \infty$: heavy quark propagates only in time

$$\Rightarrow \qquad D_0/m = \mathrm{O}(1) \qquad D_k/m = \mathrm{O}(\epsilon)$$

when the derivatives act on heavy fields — "power counting scheme"
$(\mathrm{O}(\epsilon) \sim \mathrm{O}(1/m)$; in the quantum theory: $\epsilon = \Lambda_{QCD}/m)$

Introduction to HQET

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  when the derivatives act on heavy fields — ”power counting scheme”
  ($O(\epsilon) \sim O(1/m)$; in the quantum theory: $\epsilon = \Lambda_{QCD}/m$)
- At lowest order, the ”large components” (anti-)quark field propagates forward (backward) in time:

$$P_+\psi_h = \psi_h \qquad \overline{\psi}_h P_+ = \overline{\psi}_h \qquad P_\pm = \frac{1 \pm \gamma_0}{2}$$
$$P_-\psi_{\bar{h}} = \psi_{\bar{h}} \qquad \overline{\psi}_{\bar{h}} P_- = \overline{\psi}_{\bar{h}}$$

Introduction to HQET

Quark and anti-quark fields are connected by the $O(1/m)$ terms in $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_{h}^{\text{stat}} + \mathcal{L}_{\bar{h}}^{\text{stat}} + O(1/m)$$

$$\mathcal{L}_{h}^{\text{stat}} = \overline{\psi}_{h}(D_0 + m)\psi_{h} \qquad \mathcal{L}_{\bar{h}}^{\text{stat}} = \overline{\psi}_{\bar{h}}(-D_0 + m)\psi_{\bar{h}}$$

but they can be decoupled through a field rotation (Foldy-Wouthuysen transformation):

$$\psi \to \phi = e^{S}\psi \qquad \psi^{\dagger} \to \phi^{\dagger} = \psi^{\dagger}e^{-S}$$

$$\Rightarrow \qquad \mathcal{L} = \phi^{\dagger}\mathcal{D}'\phi$$

$$\text{with} \qquad \mathcal{D}' = e^{S}\mathcal{D}e^{-S}\ , \ S \equiv \frac{1}{2m}D_k\gamma_k = -S^{\dagger} = O(1/m)$$

(Recall that $\mathcal{D} = O(m)$ and that in this way the $D_k\gamma_k$–term is rotated away)

Introduction to HQET

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Explicitly:

$$\begin{aligned}
\mathcal{D}' &= \mathcal{D} + \frac{1}{2m}\left[D_k\gamma_k, \mathcal{D}\right] + \frac{1}{8m^2}\left[D_l\gamma_l, \left[D_k\gamma_k, \mathcal{D}\right]\right] + O\left(1/m^2\right) \\
&= \mathcal{D} + \frac{1}{2m}\left[D_k\gamma_k, \mathcal{D}\right] - \frac{1}{4m}\left[D_l\gamma_l, \gamma_0 D_k\gamma_k\right] + O\left(1/m^2\right) \\
&= \gamma_0\left\{\gamma_0 D_0 + m + \frac{1}{2m}\left(-D_kD_k - \frac{1}{2i}F_{kl}\sigma_{kl}\right) + \frac{1}{2m}F_{k0}\gamma_0\gamma_k\right\} \\
&\quad + O\left(1/m^2\right)
\end{aligned}$$

$$\mathcal{L} = \mathcal{L}_{h}^{\mathrm{stat}} + \mathcal{L}_{\bar{h}}^{\mathrm{stat}} + \frac{1}{2m}\left\{\mathcal{L}_{h}^{(1)} + \mathcal{L}_{\bar{h}}^{(1)} + \mathcal{L}_{h\bar{h}}^{(1)}\right\} + O\left(1/m^2\right)$$

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with

$$\begin{aligned}
\mathcal{L}_h^{(1)} &= \overline{\psi}_h\left(-D_k D_k - \frac{1}{2i}F_{kl}\,\sigma_{kl}\right)\psi_h \\
&= -\frac{1}{2m}\,\overline{\psi}_h\left(\mathbf{D}^2 + \boldsymbol{\sigma}\cdot\mathbf{B}\right)\psi_h \;\equiv\; -\frac{1}{2m}\left(\mathcal{O}_{kin} + \mathcal{O}_{spin}\right)
\end{aligned}$$

$$\sigma_{\mu\nu} = \tfrac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \qquad F_{kl} = \left[D_k, D_l\right]$$

Introduction to HQET

- Only double insertions of $\mathcal{L}_{h\bar{h}}^{(1)}$ contribute for heavy-light hadrons
  $\Rightarrow$ $\mathrm{O}(1/m^2)$ and may be dropped in $\mathcal{L}$ here
- $\mathcal{L} \equiv \mathcal{L}_{\mathrm{eff}} =$ low-energy effective Lagrangian
  - describes long wave length modes of the fields accurately and has truncation errors of increasing relevance for shorter wave lengths
  - removal of the mass terms
    $\Leftrightarrow$ energy shift by m of all states containing a single heavy quark

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- Quantum fluctuations:
  - are not smooth, but rather than modifying the structure of $\mathcal{L}_{\text{eff}}$, the coefficients of the various terms receive non-trivial renormalizations due to these *short-distance* fluctuations
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    $\Leftrightarrow$ local interaction terms, renormalized as in (effective) local QFT
- Therefore:
  Prefactors of the various operators to be determined by a non-trivial (— ideally non-perturbative —) matching of HQET to QCD in the quantum theory

The effective quantum field theory

$\mathcal{L}_{\mathrm{h}}^{\mathrm{stat}}$ contains local fields of a mass dimension $\mathrm{d} \leqslant 4$

$\Rightarrow$ power-counting renormalizable, counterterms restricted by symmetries

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  - heavy quark spin-symmetry
  - local conservation of heavy quark flavour number
  - gauge invariance, parity & cubic symmetry

$\Rightarrow$ Only one invariant counterterm that is $\propto \overline{\psi}_{\mathrm{h}}\psi_{\mathrm{h}}$:

$$\mathcal{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(D_0 + \delta m)\psi_{\mathrm{h}}$$ (continuum) quantum Lagrangian

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$$\mathcal{L}_{h}^{stat} = \overline{\psi}_h (D_0 + \delta m) \psi_h \qquad \text{(continuum) quantum Lagrangian}$$

Lattice formulation :

Straightforward discretization prescriptions

$D_0 \to \nabla_0^*$: backward lattice derivative , $D_k D_k \to \nabla_k^* \nabla_k$, $F_{kl} \to \widehat{F}_{ij}$

The effective quantum field theory

Lattice formulation :

- Static quark lattice action

$$S_h[W,\overline{\psi}_h,\psi_h] = a^4 \frac{1}{1+a\,\delta m} \sum_x \overline{\psi}_h(x)\,(\nabla_0^* + \delta m)\,\psi_h(x)$$

$$\nabla_0^*\,\psi_h(x) = \frac{1}{a}\left[\psi_h(x) - W^\dagger(x-a\hat{0},0)\psi_h(x-a\hat{0})\right]$$

$W(x,0) = U(x,0)$: Eichten-Hill action, but more clever choices for parallel transporters are possible [ALPHA Collaboration, 2004 & 2005]

- Static quarks propagate only forward in time

$$\Rightarrow \quad \triangle_h(x,y) = W(x-a\hat{0},0)^{-1}\,W(x-2a\hat{0},0)^{-1}\cdots W(y,0)^{-1} \times \theta(x_0-y_0)\delta(\mathbf{x}-\mathbf{y})(1+a\,\delta m)^{-(x_0-y_0)/a}\,P_+$$

(timelike Wilson line, δm cancels divergence in the static quark self-energy)

The effective quantum field theory

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$$S_h[W, \overline{\psi}_h, \psi_h] = a^4 \frac{1}{1 + a\,\delta m} \sum_x \overline{\psi}_h(x)\,(\nabla_0^* + \delta m)\,\psi_h(x)$$

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$W(x,0) = U(x,0)$: Eichten-Hill action, but more clever choices for parallel transporters are possible [ALPHA Collaboration , 2004 & 2005]

- $O(a)$ improvement:

Preserving on the lattice the above symmetries of the static theory guarantees that both universality class and $O(a)$ improvement are unchanged w.r.t. the static action, i.e. the static-light action is already improved if the light quark sector is [Kurth & Sommer, 2001]

The effective quantum field theory

Lattice formulation :

- Static quark lattice action

$$S_h[W, \overline{\psi}_h, \psi_h] = a^4 \frac{1}{1 + a\,\delta m} \sum_x \overline{\psi}_h(x)\,(\nabla_0^* + \delta m)\,\psi_h(x)$$

$$\nabla_0^* \psi_h(x) = \frac{1}{a}\left[\psi_h(x) - W^\dagger(x - a\hat{0}, 0)\psi_h(x - a\hat{0})\right]$$

  $W(x, 0) = U(x, 0)$: Eichten-Hill action, but more clever choices for parallel transporters are possible [ALPHA Collaboration, 2004 & 2005]

- $O(a)$ improvement:
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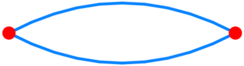
- Renormalization:
  energy shift δm included but $\delta m \propto 1/a$ for dimensional reasons (!)

The effective quantum field theory

Composite fields involving b-quarks also translate to the effective theory:

$$A_0(x) = \overline{\psi}_{\mathrm{l}}(x)\gamma_0\gamma_5\psi_{\mathrm{b}}(x) \quad \longrightarrow \quad A_0^{\mathrm{stat}} = \overline{\psi}_{\mathrm{l}}(x)\gamma_0\gamma_5\psi_{\mathrm{h}}(x)$$

A_0  A_0 : Correlation function of the axial current

$$\int \mathrm{d}^3x \left\langle A_0(x) A_0^{\dagger}(0) \right\rangle_{\mathrm{QCD}} \overset{x_0 \gg 1/M_{\mathrm{b}}}{\sim} \left[C_{\mathrm{PS}}\left(M_{\mathrm{b}}/\Lambda\right)\right]^2 \int \mathrm{d}^3x \left\langle A_0^{\mathrm{stat}}(x) \left(A_0^{\mathrm{stat}}\right)^{\dagger}(0) \right\rangle_{\mathrm{stat}}$$

$$+\,\mathrm{O}\left(1/M_{\mathrm{b}}\right) \qquad \Lambda \equiv \Lambda_{\mathrm{QCD}}$$

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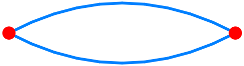
A_0  A_0 : Correlation function of the axial current

$$\int d^3x \left\langle A_0(x) A_0^\dagger(0) \right\rangle_{\rm QCD} \overset{x_0 \gg 1/M_b}{\sim} [C_{\rm PS}(M_b/\Lambda)]^2 \int d^3x \left\langle A_0^{\rm stat}(x) \left(A_0^{\rm stat}\right)^\dagger(0) \right\rangle_{\rm stat} + O(1/M_b) \qquad \Lambda \equiv \Lambda_{\rm QCD}$$

Generic structure of the HQET-expansion of QCD matrix elements

$$\Phi = \langle\, B \,|\, A_0 \,|\, 0 \,\rangle : \quad \Phi^{\rm QCD} \equiv F_B\sqrt{m_B} = \underbrace{C_{\rm PS}(M_b/\Lambda)} \times \underbrace{\Phi^{\rm stat}_{\rm RGI}} + O(1/M_b)$$

conversion function ⇐ renormalization

RGI matrix element in effective theory

- In HQET: Absence of chiral symmetry as it is met in (massless) QCD implies a scale dependence $\Phi^{\rm stat}(\mu) \equiv Z_A^{\rm stat}(\mu)\langle\, B \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle$
- M_b = scale & scheme independent (RG-invariant) quark mass

Non-perturbative formulation of HQET

Action: $S_{\mathrm{HQET}}(x) = a^4 \sum_x \mathcal{L}_{\mathrm{HQET}}(x)$ for the b-quark (zero velocity HQET)

[Eichten, 1988; Eichten & Hill, 1990]

$$\begin{aligned}
\mathcal{L}_{\mathrm{HQET}}(x) &= \mathcal{L}_{\mathrm{stat}}(x) - \omega_{\mathrm{kin}}\mathcal{O}_{\mathrm{kin}}(x) - \omega_{\mathrm{spin}}\mathcal{O}_{\mathrm{spin}}(x) \\
\mathcal{L}_{\mathrm{stat}}(x) &= \overline{\psi}_{\mathrm{h}}(x)\left[\,D_0 + m_{\mathrm{bare}}\,\right]\psi_{\mathrm{h}}(x) \qquad \tfrac{1}{2}(1+\gamma_0)\psi_{\mathrm{h}}(x) = \psi_{\mathrm{h}}(x) \\
\mathcal{O}_{\mathrm{kin}}(x) &= \overline{\psi}_{\mathrm{h}}(x)\,\mathbf{D}^2\,\psi_{\mathrm{h}}(x)
\end{aligned}$$

$\rightarrow$ kinetic energy from heavy quark's residual motion

$$\mathcal{O}_{\mathrm{spin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\psi_{\mathrm{h}}(x)$$

$\rightarrow$ chromomagnetic interaction with the gluon field

Composite fields: axial current, related to the B-meson decay constant

$F_B\sqrt{m_B} = \langle\, B(\mathbf{p}=0)\,|\,A_0(0)\,|\,0\,\rangle$, where $A_0 = \overline{\psi}_{\mathrm{l}}\gamma_0\gamma_5\psi_{\mathrm{b}} \;\rightarrow\; A_0^{\mathrm{HQET}}$

$$\begin{aligned}
A_0^{\mathrm{HQET}}(x) &= Z_A^{\mathrm{HQET}}\left[A_0^{\mathrm{stat}}(x) + c_A^{\mathrm{HQET}}\,\delta A_0^{\mathrm{stat}}(x)\right] \\
A_0^{\mathrm{stat}}(x) &= \overline{\psi}_{\mathrm{l}}(x)\gamma_0\gamma_5\psi_{\mathrm{h}}(x) \\
\delta A_0^{\mathrm{stat}}(x) &= \overline{\psi}_{\mathrm{l}}(x)\,\tfrac{1}{2}\left(\overleftarrow{\nabla}_i + \overleftarrow{\nabla}_i^{*}\right)\gamma_i\gamma_5\,\psi_{\mathrm{h}}(x)
\end{aligned}$$

EVs $=$ Functional integral representation at the quantum level :

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, O[\varphi]\, e^{-(S_{\rm rel}+S_{\rm HQET})} \qquad \mathcal{Z} = \int \mathcal{D}[\varphi]\, e^{-(S_{\rm rel}+S_{\rm HQET})}$$

Instead of including the NLO term in $1/m$ of $\mathcal{L}_{\rm HQET}$ in the action (as this theory wouldn't be renormalizable), the *FI weight* is expanded in a *power series* in $1/m$

$$\exp\{-S_{\rm HQET}\} =$$

$$\exp\left\{-a^4 \textstyle\sum_x \mathcal{L}_{\rm stat}(x)\right\}$$
$$\times \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2}\left[a^4 \sum_x \mathcal{L}^{(1)}(x)\right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots\right\}$$

$$\Rightarrow \ \langle O \rangle \ = \ \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, e^{-S_{\rm rel} - a^4 \sum_x \mathcal{L}_{\rm stat}(x)}\, O \, \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) \ + \ \ldots\right\}$$

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Explicitly:

$$\begin{aligned} \langle O \rangle &= \langle O \rangle_{\rm stat} + \omega_{\rm kin}\, a^4 \sum_x \langle O\, \mathcal{O}_{\rm kin}(x) \rangle_{\rm stat} + \omega_{\rm spin}\, a^4 \sum_x \langle O\, \mathcal{O}_{\rm spin}(x) \rangle_{\rm stat} \\ &\equiv \langle O \rangle_{\rm stat} + \omega_{\rm kin} \langle O \rangle_{\rm kin} + \omega_{\rm spin} \langle O \rangle_{\rm spin} \\ \langle O \rangle_{\rm stat} &= \tfrac{1}{\mathcal{Z}} \int_{\rm fields} O \exp\left\{ -a^4 \sum_x \left[\mathcal{L}_{\rm light}(x) + \mathcal{L}_{\rm h}^{\rm stat}(x) \right] \right\} \end{aligned}$$

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Important implications of this definition of HQET

- $1/m$ – terms appear only as *insertions* of local operators in CFs
  $\Rightarrow$ Power counting: Renormalizability at any given order in $1/m$
- $\Leftrightarrow$ Existence of the continuum limit with universality
- Effective theory $=$ Continuum asymptotic expansion in $1/m$ of QCD

Renormalization & Matching

Renormalization

- The mixing of operators of different dimension in $\mathcal{L}_{\rm HQET}$ induces power divergences [Maiani, Martinelli & Sachrajda, 1992]

  $\rightarrow$ $\mathcal{L}_{\rm stat}$: linearly divergent additive mass renormalization δm originates from mixing of $\overline{\psi}_h D_0 \psi_h$ with $\overline{\psi}_h \psi_h$ $\Rightarrow$ $E^{\rm QCD}_{h,\bar{h}} = E^{\rm stat}_{h,\bar{h}}\big|_{\delta m=0} + m_{\rm bare}$

$$m_{\rm bare} = \delta m + m \,, \quad \delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0 g_0^2)} \times \{c_1 g_0^2 + c_2 g_0^4 + \ldots\}$$

  $\rightarrow$ PT: uncertainty $=$ truncation error $\sim e^{1/(2b_0 g_0^2)}\, c_{n+1}\, g_0^{2n+2} \stackrel{g_0 \to 0}{\longrightarrow} \infty$!

  $\Rightarrow$ Non-perturbative $c(g_0)$ needed, i.e., *NP renormalization of HQET (resp. fixing of its parameters) required for the continuum limit to exist*

- Power-law divergences even worse at the level of $1/m$ – corrections: $a^{-1} \rightarrow a^{-2}$ (e.g., δm picks up a contribution $a^{-2}\omega_{\rm kin}$)

Matching

- The finite parts of renormalization constants must be fixed s.th. the effective theory describes the underlying theory, QCD
- Proper conditions for these must be imposed from QCD with finite m_b

Mass dependence at leading order in $1/m$

The rôle of perturbative anomalous dimensions

Consider matrix elements of composite fields involving b-quarks as, e.g., obtained from a QCD correlation function of the heavy-light axial current

$$
\begin{aligned}
C_{AA}^{QCD}(x_0) &= Z_A^2 a^3 \sum_{\mathbf{x}} \langle A_0(x)(A_0)^\dagger(0)\rangle_{QCD} \\
\left[\,\Phi^{QCD}\,\right]^2 &\equiv F_B^2\, m_B \;=\; \left|\,\langle\, B\,|\,Z_A A_0\,|\,0\,\rangle\,\right|^2 \\
&= \lim_{x_0\to\infty} \left[\,2\exp\left\{\,x_0\, m_B^{eff}(x_0)\,\right\} C_{AA}^{QCD}(x_0)\,\right]
\end{aligned}
$$

- B-meson state dominates spectral representation of C_{AA}^{QCD} at large x_0
- $Z_A(g_0)$ fixed by chiral Ward identities, renormalization scale independent

In the static approximation this translates into

$$
\left[\,\Phi(\mu)\,\right]^2 = \left|\,\langle\, B\,|\,Z_A^{stat} A_0^{stat}\,|\,0\,\rangle\,\right|^2 = \lim_{x_0\to\infty} \left[\,2\exp\left\{\,x_0\, E_{stat}^{eff}(x_0)\,\right\} C_{AA}^{stat}(x_0)\,\right]
$$

- Absence of chiral symmetry in HQET implies a scale dependence
  $\to\ \mu-$dependence in $Z_A^{stat}(g_0, a\mu) = 1 + g_0^2\,[\,B_0 - \gamma_0 \ln(a\mu)\,] + O(g_0^4)$
- Better alternative: work with the RGI opertator $(A_{RGI}^{stat})_0$

How does one get from $\Phi_{\rm RGI} = Z_{\rm A,RGI}^{\rm stat} \langle\, B \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle$ to F_B ?

<u>QCD</u>	<u>LO HQET</u>
$Z_A \langle\, B \,|\, A_0(0) \,|\, 0 \,\rangle_{\rm QCD}$	$C_{\rm PS}(M_b/\Lambda)\, Z_{\rm A,RGI}^{\rm stat} \langle\, B \,|\, A_0^{\rm stat}(0) \,|\, 0 \,\rangle_{\rm stat}$
$F_B \sqrt{m_B}$	$F_B \sqrt{m_B} + O(1/m_b)$

- Renormalization problem solved non-perturbatively (via interm. SF scheme)
  $\Rightarrow$ $Z_{\rm A,RGI}^{\rm stat}$: NP'ly known (to $\approx 1\%$ accuracy)
  [$N_f = 0$: H., Kurth & Sommer, 2003; $N_f = 2$: Della Morte, Fritzsch & H., 2007]
- $\langle\, B_{(s)} \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle$: known for $N_f = 0$ and in progress for $N_f = 2$
  [ALPHA Collaboration, Blossier et al., arXiv:1006.5816]
  $\Rightarrow$ $\langle\, B_{(s)} \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle_{\rm RGI} \longrightarrow F_B, F_{B_s}$ by multiplying with $C_{\rm PS}$

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QCD

$Z_A \langle\, B \,|\, A_0(0) \,|\, 0 \,\rangle_{\rm QCD}$

$F_B \sqrt{m_B}$

LO HQET

$C_{\rm PS}(M_b/\Lambda)\, Z^{\rm stat}_{\rm A,RGI} \,\langle\, B \,|\, A_0^{\rm stat}(0) \,|\, 0 \,\rangle_{\rm stat}$

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A closer look at the "conversion function" $C_{\rm PS}$ and $\gamma^{\rm match}$:

$$\text{Matching} \quad \Leftrightarrow \quad \Phi^{\rm QCD}(m) = \widetilde{C}_{\rm match}(m,\mu) \times \Phi(\mu) + O(1/m)$$

$$\widetilde{C}_{\rm match}(m,\mu) = 1 + c_1(m/\mu)\,\bar{g}^2(\mu) + \ldots$$

- $m \leftrightarrow$ heavy (b) quark mass dependence on the QCD side
- $\mu \leftrightarrow$ (arbitrary) renormalization scale dependence in the effective theory
- this fixes the (finite) renormalization $\widetilde{C}_{\rm match} \longleftrightarrow$ "matching scheme"

QCD observables F_B, F_{B_s}: independent of renormalization scheme & scale

$\Rightarrow$ the $\mu-$dependence is artificial, only the mass dependence is for real

$\Rightarrow$ choose a convenient and common scale:

$$\mu = m_\star = \overline{m}(m_\star) \qquad g_\star = \bar{g}(m_\star)$$

$$\widetilde{C}_{\rm match}(m_\star, m_\star) = C_{\rm match}(g_\star) = 1 + c_1(1)\, g_\star^2 + \ldots$$

Eliminate the scheme dependence by passing to the RGI matrix element:

$$\Phi_{\rm RGI} = \exp\left\{ -\int^{\bar{g}(\mu)} dx\, \frac{\gamma(x)}{\beta(x)} \right\} \Phi(\mu)$$

$$\Rightarrow \quad \Phi^{\rm QCD} = C_{\rm match}(g_\star)\, \Phi(\mu) = C_{\rm match}(g_\star) \exp\left\{ \int^{g_\star} dx\, \frac{\gamma(x)}{\beta(x)} \right\} \Phi_{\rm RGI}$$

$$\equiv \exp\left\{ \int^{g_\star} dx\, \frac{\gamma^{\rm match}(x)}{\beta(x)} \right\} \Phi_{\rm RGI} \qquad \text{defines } \gamma^{\rm match}$$

- $\gamma^{\rm match}(g_\star) = \frac{m_\star}{\Phi^{\rm QCD}} \frac{\partial \Phi^{\rm QCD}}{\partial m_\star}$ describes the full physical mass dependence . . .
- . . . but there is still a scheme dependence through the choice of $\overline{m}$, $\bar{g}$

Remove this renormalization scheme dependence by reparametrization in terms of renormalization group invariants Λ, M (= RGI heavy quark mass) :

$$\Phi^{\rm QCD} = C_{\rm PS}\,(M/\Lambda) \times \Phi_{\rm RGI} \;, \quad C_{\rm PS}\,(M/\Lambda) = \exp\left\{ \int^{g_\star(\frac{M}{\Lambda})} dx \, \frac{\gamma^{\rm match}(x)}{\beta(x)} \right\}$$

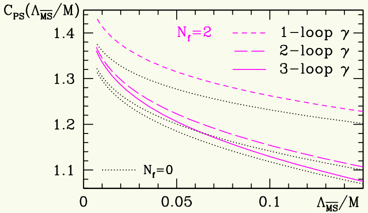
To evaluate $C_{\rm PS}$, insert $\gamma^{\rm match}(g_\star) \overset{g_\star \to 0}{\sim} -\gamma_0 g_\star^2 - \gamma_1^{\rm match} g_\star^4 - \gamma_2^{\rm match} g_\star^6 + \ldots$

$\Rightarrow$ leading large-mass behaviour via $\frac{M}{\Phi}\frac{\partial \Phi}{\partial M}\Big|_\Lambda = \frac{M}{C_{\rm PS}}\frac{\partial C_{\rm PS}}{\partial M}\Big|_\Lambda = \frac{\gamma^{\rm match}(g_\star)}{1-\tau(g_\star)}$:

$$C_{\rm PS} \overset{M\to\infty}{\sim} \left(2b_0 g_\star^2\right)^{-\gamma_0/(2b_0)} \sim \left[\log(M/\Lambda)\right]^{\gamma_0/(2b_0)}$$

$C_{\rm PS}$ perturbatively under control ?

[3-loop AD by Chetyrkin & Grozin, 2003]



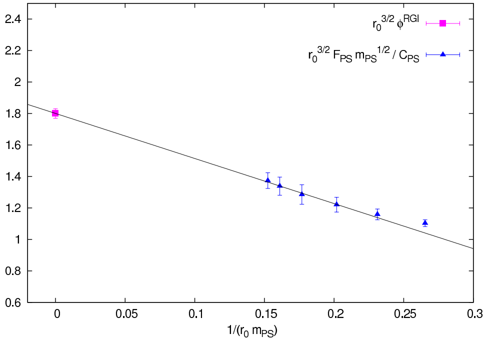
$N_f = 0, 2$

- RGI-*ratio* M/Λ : can be fixed in numerical simulations without perturbative errors
- Full (logarithmic) mass dependence $\in C_{\rm PS}$
- Fig. seems to indicate that the remaining $O\left(\bar{g}^6(m_b)\right)$ errors are relatively small
  $\rightarrow$ however: a premature conclusion . . .
- For B-Physics: $\Lambda_{\overline{\rm MS}}/M_b \approx 0.04$

An application ($N_f = 0$)

Interpolation between the static limit and the charge region

Della Morte, Dürr, Guazzini, H., Jüttner & Sommer, JHEP0802(2008)078



Looks good: under a reasonable smoothness assumption, *interpolate* the mass dependence (linearly) in the inverse PS mass to the physical point

- F_{PS} follows the heavy quark scaling law, no $1/(r_0 m_{PS})^2$ – effects are visible
  $\rightarrow$ $1/m$ – expansion appears to work very well even for charm quarks
  $\leftarrow$ surprising; needs further confirmation, as the perturbative C_{PS} is used
- Question: What is the accuracy of perturbation theory involved in this ?

Accuracy of perturbation theory in the matching

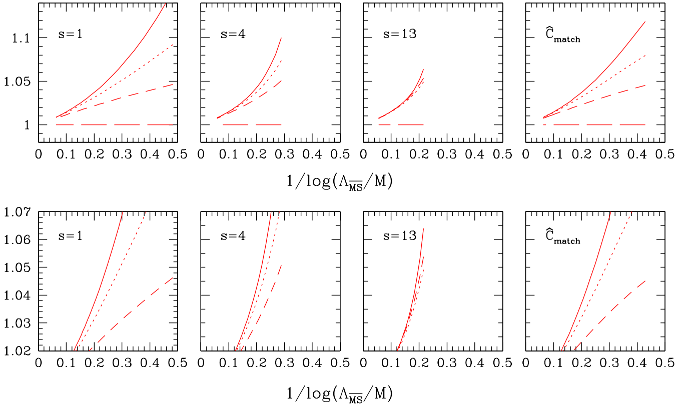
Bekavac, Grozin, Marquard, Piclum, Seidel & Steinhauser, NPB833(2010)46

$C_{\rm match}(g_\star)$ now known to N^{3}LO for various bilinears $\mathcal{O}_\Gamma = \overline{\psi}_l(x)\,\Gamma\,\psi_h(x)$

$\rightarrow$ $\gamma_\Gamma^{\rm match}$: 3-loop, $\gamma_\Gamma^{\rm match} - \gamma_{\Gamma'}^{\rm match}$: 4-loop (unknown 4-lp AD in HQET cancels)

$\Rightarrow$ *Ratios* of conversion functions reflect perturbative 4-loop precision:

$$C_{\Gamma/\Gamma'} = C_{\rm match}^{\Gamma}(m,\mu)/C_{\rm match}^{\Gamma'}(m,\mu)$$



Example

$C_{\rm PS/V} = C_{\rm PS}/C_{\rm V}$

x-axis $\propto g_\star^2(M/\Lambda)$

For B-physics:

$\Lambda_{\overline{\rm MS}}/M_b \approx 0.04$

$1/\ln(\Lambda_{\overline{\rm MS}}/M_b) \approx 0.3$

PT is badly behaved for beauty and even worse for charm

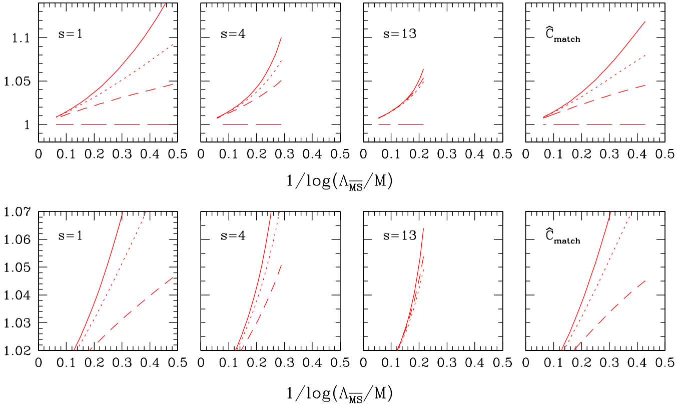
"We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best."

[quote from Bekavac at al., 2010]

Freedom to "optimize" the scale

[R. Sommer, private communication]

$$\mu = s^{-1} m_\star = \overline{m}(m_\star)\ ,\ \hat{g} = \bar{g}\left(s^{-1} m_\star\right) \qquad C_\Gamma\left(M/\Lambda\right) = \exp\left\{\int^{\hat{g}} dx\, \frac{\hat{\gamma}^{\mathrm{match}}_\Gamma(x)}{\beta(x)}\right\}$$



Matching below $m_\star$, expect $s > 1$ is better

Decrease of terms in perturbative series improved, once $s \gtrsim 4$

However:

$\alpha(m_b/4)$ is not small, series unreliable again

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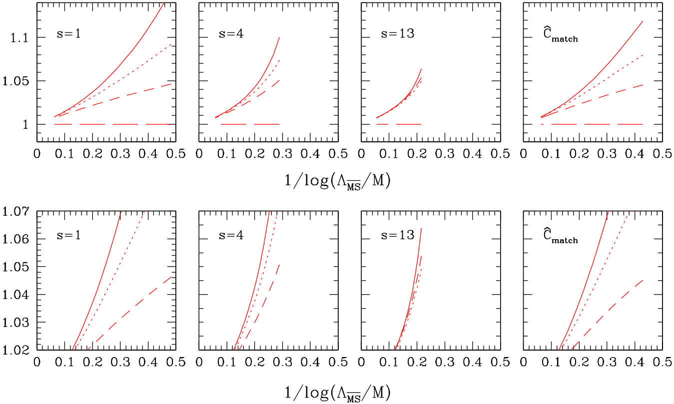

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→ Effective scale is well below $\mu = m_b$; asymptotic convergence of PT only improved far beyond m_b, where it is of limited use for us

⇒ Accuracy of perturbative matching is hard to assess for b- and c-physics

”We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best.” [quote from Bekavac at al., 2010]

Freedom to ”optimize” the scale

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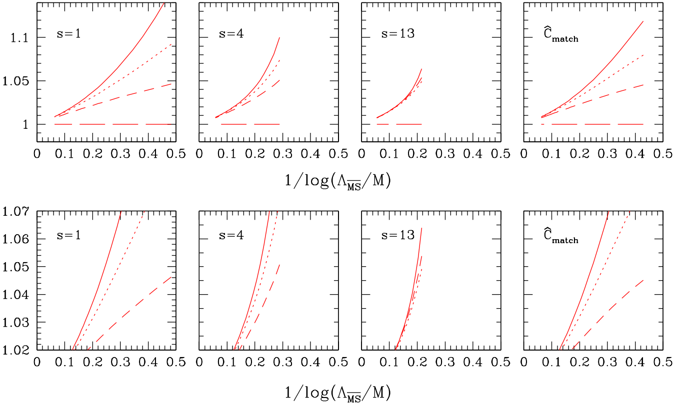

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⇒ Error estimates in the literature seem much too optimistic . . .

"We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best." [quote from Bekavac at al., 2010]

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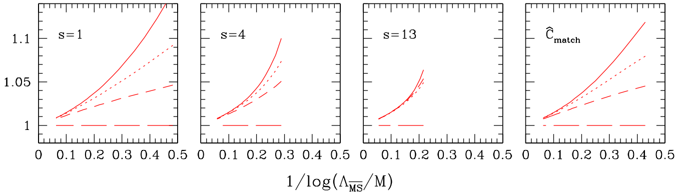
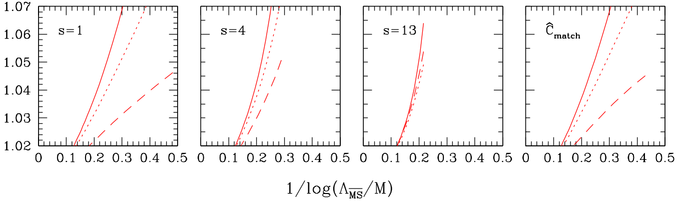
[R. Sommer, private communication]

$$\mu = s^{-1} m_\star = \overline{m}(m_\star)\,, \quad \hat{g} = \bar{g}\left(s^{-1} m_\star\right) \qquad C_\Gamma\left(M/\Lambda\right) = \exp\left\{\int^{\hat{g}} dx\, \frac{\hat{\gamma}^{\rm match}_\Gamma(x)}{\beta(x)}\right\}$$





Matching below $m_\star$, expect $s > 1$ is better

Decrease of terms in perturbative series improved, once $s \gtrsim 4$

However:

$\alpha(m_b/4)$ is not small, series unreliable again

$\Rightarrow \quad \bar{g}^{2l}(m_b) \propto \left[2b_0 \ln\left(m_b/\Lambda_{QCD}\right)\right]^{-l} \overset{m_b\to\infty}{\gg} \Lambda_{QCD}/m_b$: Pert. matching theor. consistent only at LO in $1/m_b$, a few–% error budget requires NP matching

Mass dependence in finite-volume QCD ($N_f = 2$)

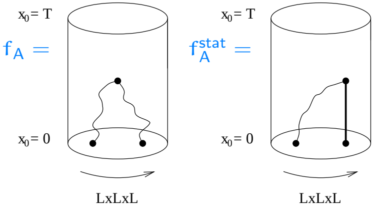
Della Morte, Fritzsch, H. & Sommer, PoS LATTICE2008(2008)226
Fritzsch & H., in progress

Non-perturbative computation of the *heavy quark mass dependence* of heavy-light meson observables in the continuum limit of finite-volume QCD

$\rightarrow$ Explicit pure theory tests that HQET is an *effective* theory of QCD

$\rightarrow$ Constraining the large-mass behaviour of QCD by the static limit

- QCD with Schrödinger Functional boundary conditions (T, L, θ):



- Renormalization [ALPHA Collaboration, 2005-2008]
  - Fix $\bar{g}^2(L_1) = 4.484$ s.th. $L_1 \approx 0.5\,\mathrm{fm}$, $L_1/a = 20, 24, 32, 40$, $L_2 = 2L_1$
  - Fix RGI (heavy) quark masses via its NP relation to bare parameters:

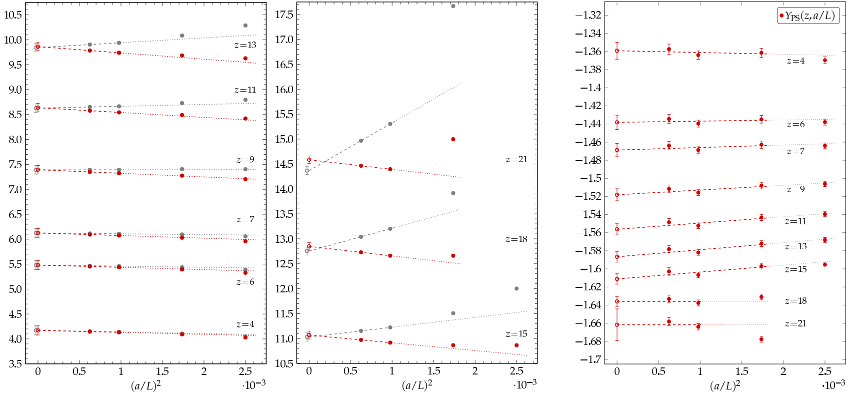
$$z \equiv L_1 M = Z_m \frac{M}{\overline{m}(\mu_0)} (1 + b_m a m_q) \times L_1 m_q \qquad Z_m = \frac{Z(g_0)\, Z_A(g_0)}{Z_P(g_0, a\mu_0)}$$

[Fritzsch, H. & Tantalo, arXiv:1004.3978]

Mass dependence in finite-volume QCD $(N_f = 2)$

The B-system in finite-volume QCD $(L = L_1)$

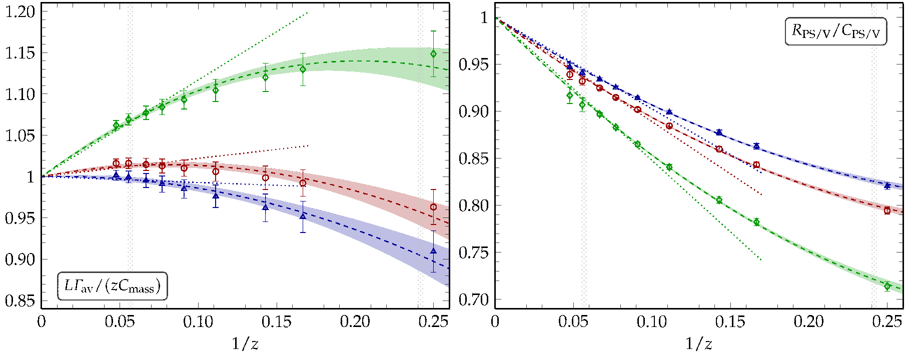
- $L_1 = 0.5\,\mathrm{fm}$, $z-$values covering the b-quark down to the charm quark region
- Removal of all $O\left(\left(\frac{a}{L}\right)^n\right)$ effects at tree-level: $O \rightarrow O_{\mathrm{impr}}(a/L) = \frac{O(a/L)}{1+\delta(a/L)}$
- Examples of continuum extrapolations (B-meson mass & decay constant):



Mass dependence in finite-volume QCD ($N_f = 2$)

The B-system in finite-volume QCD ($L = L_1$)

- Tests of HQET: validating and demonstrating the applicability of HQET
- Verification of the approach to the spin-symmetric limit:
  (B-meson mass & ratio of PS to V decay constants)



⇒ Large-mass asymptotics ($1/z \to 0$) confirms HQET predictions

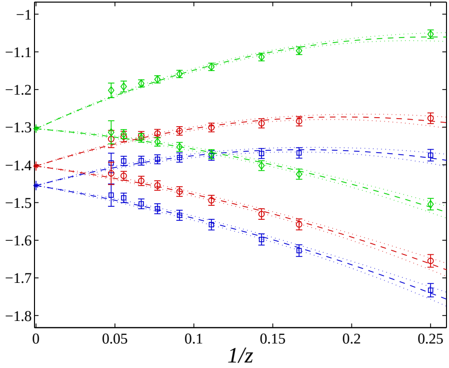
Mass dependence in finite-volume QCD ($N_f = 2$)

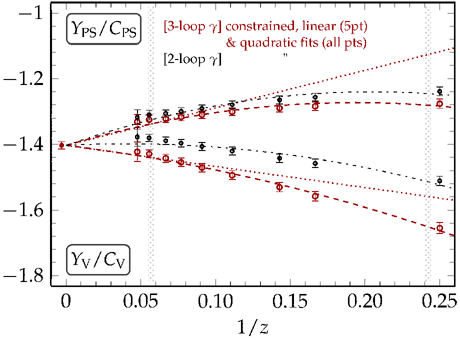
The B-system in finite-volume QCD ($L = L_1$)

- But: some numerical evidence for the previous doubts in the reliability of PT in the b-quark region is found with Y_{PS}, Y_V and its effective theory predictions

$$Y_{PS}(L,z)/C_{PS}(M/\Lambda) \;=\; X_{RGI}(L) \;+\; O(1/z)$$

$$Y_{PS}(L,z;\theta) \propto Z_A \frac{f_A(L/2,\theta)}{\sqrt{f_1(\theta)}} \qquad X_{RGI}(L;\theta) \propto \underbrace{Z^{stat}_{A,RGI} \frac{f^{stat}_A(L/2,\theta)}{\sqrt{f^{stat}_1(\theta)}}}_{=X^{stat}(\theta)}$$





Mass dependence in finite-volume QCD ($N_f = 2$)

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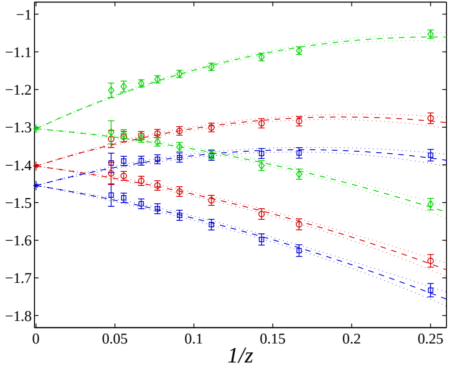
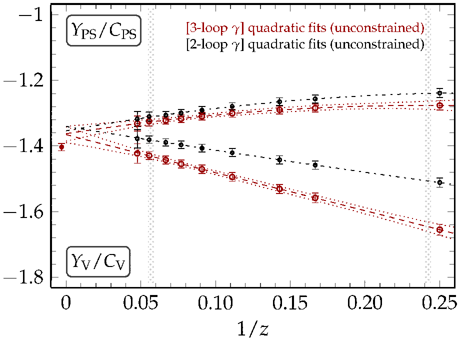


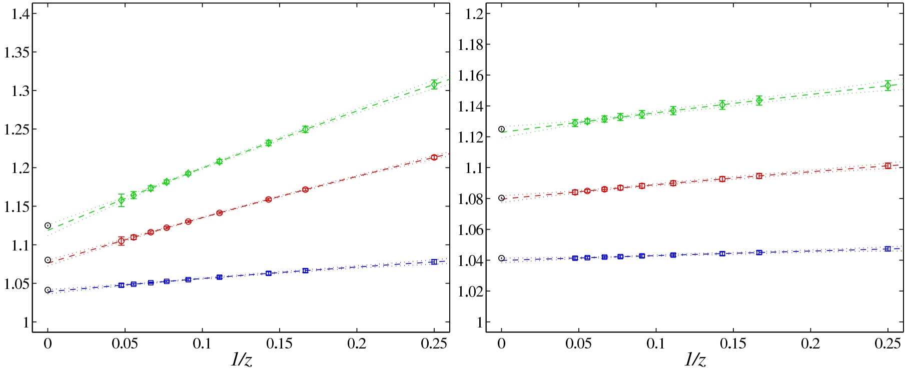


Mass dependence in finite-volume QCD ($N_f = 2$)

The B-system in finite-volume QCD ($L = L_1$)

- Consider *ratios* instead, where C_{PS} cancels completely:

$$\frac{Y_{PS}(z;\theta_1)}{Y_{PS}(z;\theta_2)} = \frac{X^{stat}(\theta_1)}{X^{stat}(\theta_2)} + O(1/z)$$



$\Rightarrow$ These turn smoothly & unconstrained into effective theory predictions

Determination of HQET parameters at $\mathrm{O}(1/m)$

Blossier, Della Morte, Garron & Sommer, arXiv:1001.4783

Vector of the $N_{\mathrm{HQET}} = 5$ parameters in $S_{\mathrm{HQET}}, A_0^{\mathrm{HQET}}$ up to $\mathrm{O}(1/m_b)$:

$$\omega = \begin{pmatrix} \omega^{\mathrm{stat}} \\ \omega^{(1/m)} \end{pmatrix}$$

$$\omega^{\mathrm{stat}} = \left(m_{\mathrm{bare}} \, , \, \ln(Z_A^{\mathrm{HQET}}) \right)^t$$

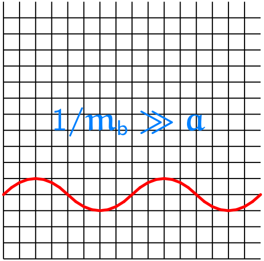
$$\omega^{(1/m)} = \left(c_A^{\mathrm{HQET}} \, , \, \omega_{\mathrm{kin}} \, , \, \omega_{\mathrm{spin}} \right)^t$$

ω_i	classical value	static value
m_{bare}	m_b	$m_{\mathrm{bare}}^{\mathrm{stat}}$
$\ln(Z_A^{\mathrm{HQET}})$	0	$\ln(Z_{A,\mathrm{RGI}}^{\mathrm{stat}} C_{\mathrm{PS}})$
c_A^{HQET}	$-1/(2m_b)$	$a c_A^{\mathrm{stat}}$
ω_{kin}	$1/(2m_b)$	0
ω_{spin}	$1/(2m_b)$	0

$\Rightarrow$ Trick: *non-perturbative* matching of HQET to QCD in a *finite* volume

[H. & Sommer, JHEP0402(2004)022]

QCD

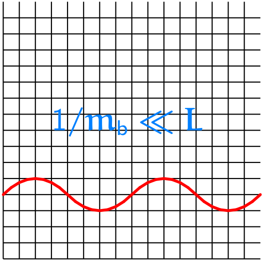


Matching conditions

$$\Phi_i^{\mathrm{QCD}} = \Phi_i^{\mathrm{HQET}}$$

for observables Φ_i

(renormal. quantities, computable for $a \to 0$)



HQET

NP matching in $L = L_1$

Suitable observables in the Schrödinger functional, $L = T = L_1 \approx 0.5\,\mathrm{fm}$

$$\Phi_i(L_1, M, a) \qquad i = 1, \ldots, N_{\mathrm{HQET}}$$

Matching conditions for $i = 1, \ldots, N_{\mathrm{HQET}}$ (note: $a \leftrightarrow g_0$)

$$\lim_{a\to 0} \Phi_i^{\mathrm{QCD}}(L_1, M, a) = \Phi_i^{\mathrm{QCD}}(L_1, M, 0) = \Phi_i^{\mathrm{HQET}}(L_1, M, a)$$

Conveniently, one chooses observables linear in ω_i, e.g.

$$\Phi(L, M, a) = \eta(L, a) + \phi(L, a)\,\omega(M, a)$$

$$\Phi_1 = L\,\langle\, B(L)\,|\,\mathbb{H}\,|\,B(L)\,\rangle \overset{L\to\infty}{\sim} L m_B$$

$$\Phi_2 = \ln\left(L^{3/2}\,\langle\, \Omega(L)\,|\,A_0\,|\,B(L)\,\rangle \right) \overset{L\to\infty}{\sim} \ln\left(L^{3/2} F_B \sqrt{m_B/2} \right)$$

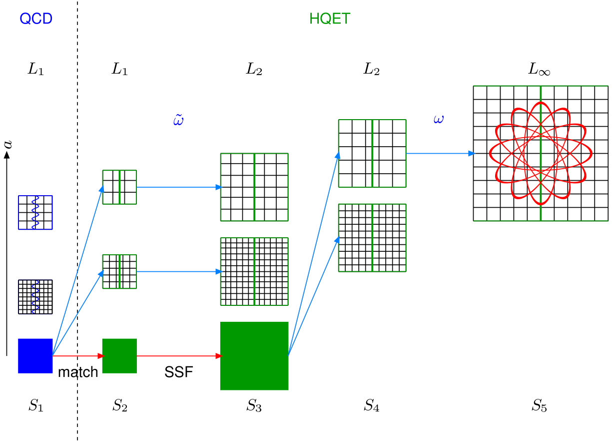
$$\ldots$$

$$\eta = \begin{pmatrix} \Gamma^{\mathrm{stat}} = \langle\, B(L)\,|\,\mathbb{H}\,|\,B(L)\,\rangle_{\mathrm{stat}} \\ \zeta_A = \ln\left(L^{3/2}\,\langle\, \Omega(L)\,|\,A_0\,|\,B(L)\,\rangle_{\mathrm{stat}} \right) \\ \ldots \end{pmatrix} \qquad \phi = \begin{pmatrix} L & 0 & \cdots \\ 0 & 1 & \cdots \\ \cdots & & \end{pmatrix}$$

Step scaling to $L = L_2$

Matching volume $L_1 \approx 0.5\,\text{fm}$ has very small a, but larger a are needed

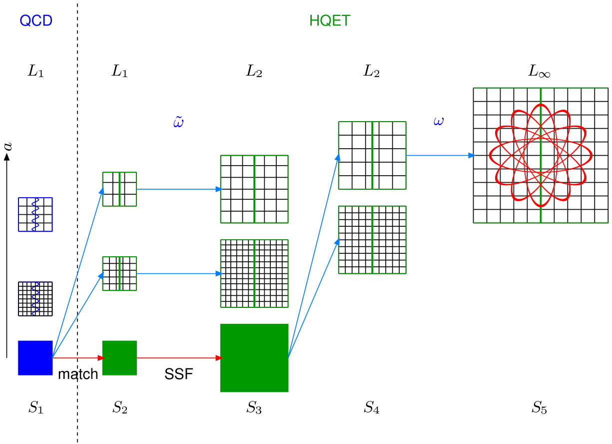
$\Rightarrow$ Gap to large volume & practicable lattice spacings, where physical quantities (m_B, F_B) are extracted, bridged by finite-size scaling steps



Fully NP, CL can be taken everywhere, $L \to 2L$ via Step Scaling Functions

$$\Phi_i^{\text{HQET}}(2L) = \sigma_i\left(\left\{\Phi_j^{\text{HQET}}(L), j = 1, \ldots, N_{\text{HQET}}\right\}\right) \qquad 2L = 2L_1 \approx 1.0\,\text{fm}$$

Step scaling to $\mathrm{L} = \mathrm{L}_2$



Finite-size scaling to $\mathrm{L}_2 = 2\mathrm{L}_1$:

- Amounts to solve a matrix equation to obtain the HQET parameters at larger lattice spacings . . .
- . . . corresponding to β–values for simulations in large volume, "L_∞", where a B-meson in HQET fits comfortably

1.) Continuum limit $a = 0.025\,\text{fm}, \ldots, 0.012\,\text{fm}$

$$\Phi_i(L_1, M, 0) = \lim_{a/L_1 \to 0} \Phi_i^{\rm QCD}(L_1, M, a)$$

2.) HQET parameters for $a = 0.05\,\text{fm}, \ldots, 0.025\,\text{fm}$

$$\omega(M, a) \equiv \phi^{-1}(L_1, a)\,[\,\Phi(L_1, M, 0) - \eta(L_1, a)\,] = \begin{pmatrix} L_1^{-1}\Phi_1(L_1, M, 0) - \Gamma^{\rm stat}(L_1, a) \\ \Phi_2(L_1, M, 0) - \zeta_A(L_1, a) \\ \ldots \end{pmatrix}$$

3.) Insert into $\Phi(L_2, M, a)$

$$\Phi(L_2, M, 0) = \lim_{a/L_2 \to 0} [\,\eta(L_2, a) + \phi(L_2, a)\,\omega(M, a)\,]$$

$$= \lim_{a/L_2 \to 0} \underbrace{\begin{pmatrix} L_2\,[\,\Gamma^{\rm stat}(L_2, a) - \Gamma^{\rm stat}(L_1, a)\,] \\ \zeta_A(L_2, a) - \zeta_A(L_1, a) \\ \ldots \end{pmatrix}}_{\text{finite SSFs}} + \underbrace{\begin{pmatrix} \frac{L_2}{L_1}\,\Phi_1(L_1, M, 0) \\ \Phi_2(L_1, M, 0) \\ \ldots \end{pmatrix}}_{\text{QCD mass dependence}}$$

4.) Repeat 2.) for $L_1 \to L_2$ to obtain $\omega(M, a)$ for $a = 0.1\,\text{fm}, \ldots, 0.05\,\text{fm}$

$$\omega(M, a) \equiv \phi^{-1}(L_2, a)\,[\,\Phi(L_2, M, 0) - \eta(L_2, a)\,]$$

Use of the HQET parameters

These HQET parameters can finally be exploited for phenomenological applications in the $B_{(s)}$–meson system, e.g. to

- calculate the b-quark mass and the $B_{(s)}$–meson decay constant:

$$m_B = m_{bare} + E_{stat} + \omega_{kin} E_{kin} + \omega_{spin} E_{spin}$$

$$\frac{\Phi}{\sqrt{2}} \equiv F_B \sqrt{m_B/2} = Z_A^{HQET} \left(1 + b_A^{stat} a m_q\right) p_{stat} \times \left(1 + c_A^{HQET} p_{\delta A} + \omega_{kin} p_{kin} + \omega_{spin} p_{spin}\right)$$

- Mass splittings, such as (radial) excitation energies of $B_{(s)}$–states and the $B_{(s)} - B^*_{(s)}$ mass difference to $O(1/m_b)$:

$$\Delta E_{n,1}^{HQET} = \left(E_{stat}^n - E_{stat}^1\right) + \omega_{kin}\left(E_{kin}^n - E_{kin}^1\right) + \omega_{spin}\left(E_{spin}^n - E_{spin}^1\right)$$

$$\Delta E_{P-V} = \tfrac{4}{3}\, \omega_{spin} E_{spin}^1$$

E_y^i, p_y : plateau averages of (bare) effective HQET energies and matrix elements in large volume

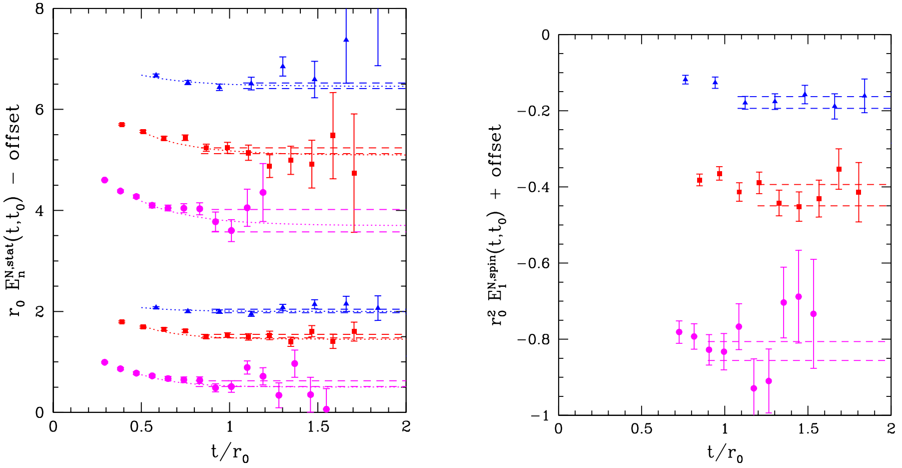
- Note: The power-divergent δm drops out in energy *differences*

Some examples of $N_f = 0$ results

Blossier, Della Morte, Garron, von Hippel, Mendes, Simma & Sommer, arXiv:1004.2661

Excited state energy levels, $a \approx (0.1, 0.08, 0.05)$ fm, $L \approx 1.5$ fm, $T = 2L$

- CF matrices $C^{\mathrm{stat}}_{ij}(t) = \sum_{x,\mathbf{y}} \langle O_i(x_0 + t, \mathbf{y})\, O^*_j(x) \rangle_{\mathrm{stat}}$ & $\mathcal{O}_{\mathrm{spin/kin}}$ insertions
- GEVP: all-to-all propagators, $t-$dilution, Gaussian smeared variational basis



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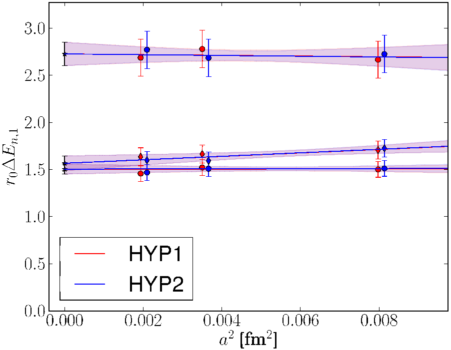
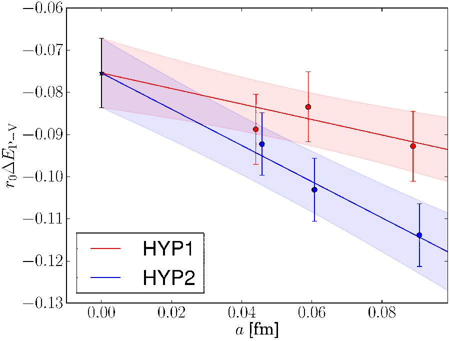


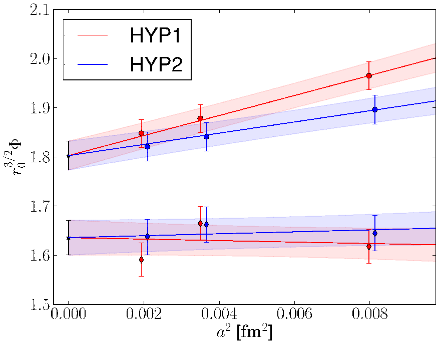


- Linear a–term suppressed by $1/m_b$, physical $O(1/m_b)$ corrections are small
- Divergences cancel after proper NP renormalization
  $\Rightarrow$ Strong *numerical* evidence for the renormalizability of HQET

Some examples of $N_f = 0$ results

Blossier, Della Morte, Garron, von Hippel, Mendes, Simma & Sommer, arXiv:1006.5816

Matrix elements in the B-meson system via applying the same techniques

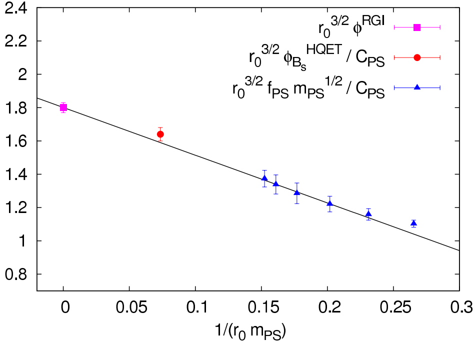


- Important remark:
  Here, the full factor $Z_A^{\rm stat} = Z_{A,RGI}^{\rm stat} C_{PS}(M_b/\Lambda)$ is implicitly evaluated non-perturbatively, i.e., C_{PS} *irrelevant* in the context of NP matching !
- HYP & GEVP lead to (2 – 3)% precision for F_{B_s} in the continuum limit

$r_0 = 0.5$ fm: $F_{B_s}^{\rm stat} = 229(3)$ MeV , $F_{B_s}^{{\rm stat}+1/m} = 212(5)$ MeV
(using $r_0 = 0.45$ fm leads to $\simeq 15\%$ increase, but $O(1/m_b^2)$ corrections are small)

Some examples of $N_f = 0$ results

Computation of F_{B_s} in HQET matches at m_{B_s} with a straight interpolation between the QCD charm sector (around F_{D_s}) and $F_{B_s}^{stat}$

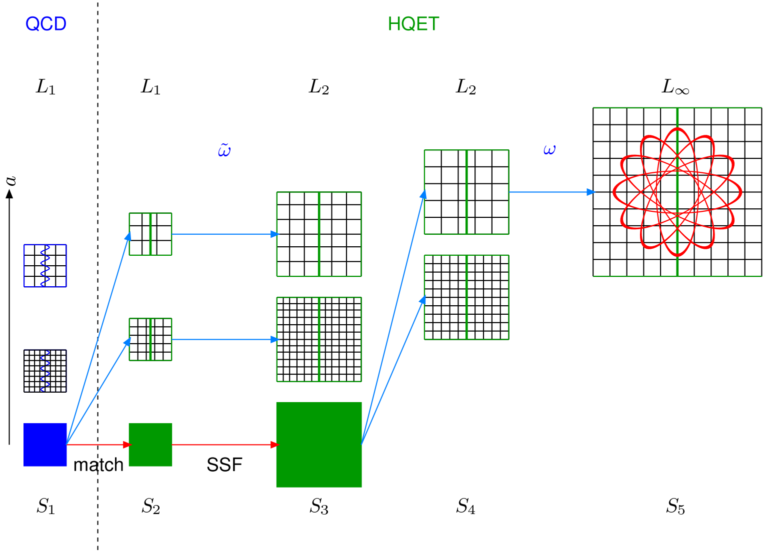


- In this comparison, C_{PS} just enters to compensate for the logarithmic scaling of Φ with m_b, i.e., $C_{PS} =$ perturbative "relic" in interpolation strategies
- Given the unclear precision of PT, interpolation methods to be taken with care, as the inherent perturbative $[\alpha_s(m_b)]^3$– errors are difficult to estimate
- Anyway, data points beyond charm computationally challenging for $N_f > 0$

First physical results in two-flavour QCD

Which ingredients are needed ?

Recall the strategy . . .



First physical results in two-flavour QCD

Which ingredients are needed ?

S_1 NP matching of HQET to QCD in finite volume with a relativistic b, to perform the power-divergent subtractions

- Crucial element of this step:
  Calculation of the *heavy quark mass dependence* of heavy-light meson observables in the continuum limit of finite-volume QCD (L_1)
- . . . already discussed above

$S_{2,3,4}$ HQET computations in small & intermediate volumes

- Evaluation of the HQET step scaling functions to connect the small matching ($L_1 \approx 0.5\,\mathrm{fm}$) to the intermediate volume ($L_2 = 2L_1 \approx 1\,\mathrm{fm}$)
- Interpolation of the resulting HQET parameters to the large-volume "L_∞" lattice spacings ($\beta = 5.2, 5.3, 5.5$)

S_5 HQET computations in large volume

- Extract HQET energies & matrix elements, using $N_f = 2$ dynamical configurations in large volume ("L_∞", periodic b.c.'s) produced by CLS
- Action: NP'ly $O(a)$ improved $N_f = 2$ Wilson ; algorithm: DD-HMC
- Problem of slowed sampling of topological modes with decreasing a less relevant, because HQET can afford to work with coarser lattices

HQET energies & matrix elements (preliminary)

Preliminary $N_f = 2$ HQET results in large volume

ALPHA Collaboration, in progress
CLS based

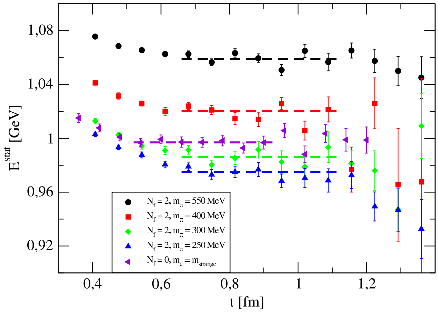
- Gauge configuration ensembles with $N_f = 2$ NP'ly $O(a)$ improved Wilson fermions generated within Coordinated Lattice Simulations
  (= community European team effort, employing Lüscher's DD-HMC)

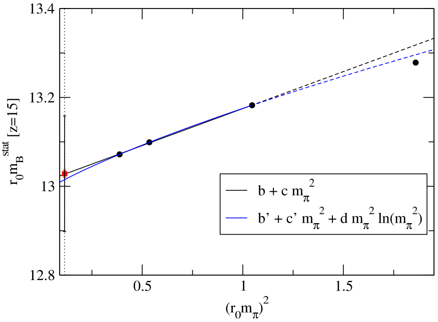
β	a [fm]	$L^3 \times T$	m_π [MeV]	#	traj. sep.
5.2	0.08	$32^3 \times 64$	700	110	16
		$32^3 \times 64$	370	160	16
5.3	0.07	$32^3 \times 64$	550	152	32
		$32^3 \times 64$	400	600	32
		$48^3 \times 96$	300	192	16
		$48^3 \times 96$	250	350	16
5.5	0.05	$32^3 \times 64$	430	250	20
		$48^3 \times 96$	430	30	16

- High numerical accuracy of lattice HQET thanks to technical advances:
  [Hasenfratz & Knechtli, 2001; Lüscher & Wolff, 1990; Foley et al., 2005; ALPHA Collaboration 2004-2009]
  - ◇ HYP-smeared static actions, giving improved statistical precision
  - ◇ solve the Generalized EigenValue Problem for a correlator matrix to cleanly quantify systematic errors from excited state contaminations
  - ◇ Variant of the stochastic all-to-all propagator method for light quarks

HQET energies & matrix elements (preliminary)

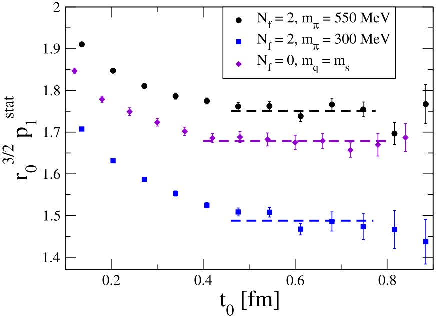
Static energies ($\beta = 5.3$, $a \approx 0.07\,\text{fm}$) & extrapolation to the chiral limit, where the uncertainty due to r_0/a is still large [Scale setting preliminary]

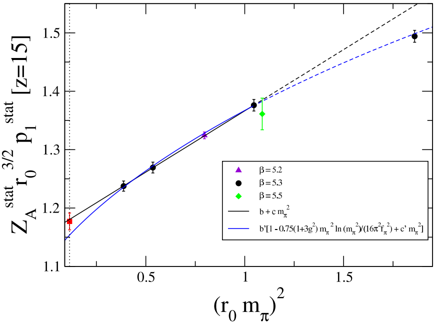




HQET energies & matrix elements (preliminary)

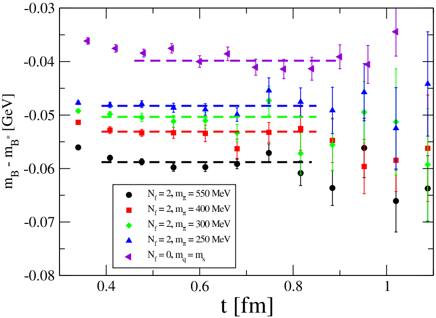
B-meson decay constant (F_B): renormalized (not $O(a)$ improved) matrix element of A_0^{stat}, data well described by HMχPT

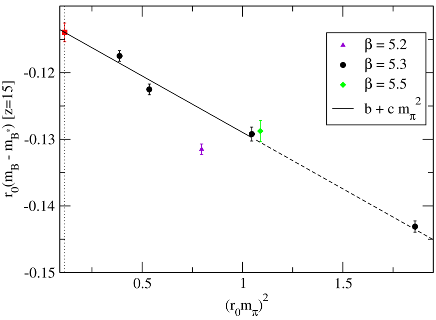




HQET energies & matrix elements (preliminary)

Spin-splitting: situation for $O(1/m)$ terms of energies is encouraging





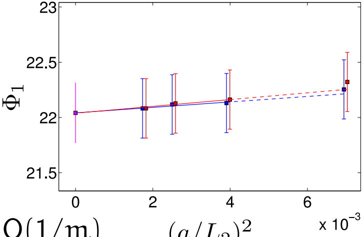
HQET parameters (preliminary)

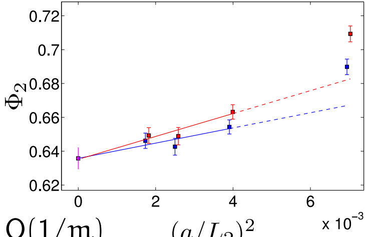
ALPHA Collaboration, in progress

After evolution to L_2 where $5.3 \lesssim \beta \lesssim 5.8$

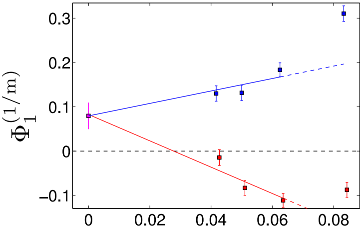
$\Phi_1 = L \langle B(L) | \mathbb{H} | B(L) \rangle$ $\qquad$ $\Phi_2 = \ln \left(L^{3/2} \langle \Omega(L) | A_0 | B(L) \rangle \right)$

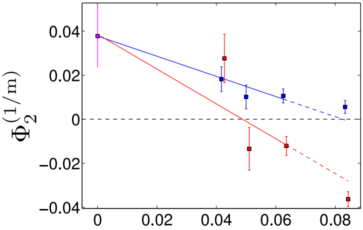
$O(m)$ $\qquad$ $O(1)$





$O(1/m)$ $\qquad$ $O(1/m)$





(a finer lattice resolution is still running)

b-quark mass interpolation (preliminary)

ALPHA Collaboration, in progress

Now insert $\omega_1 \in \omega(M, a)$ for $N_f = 2$:

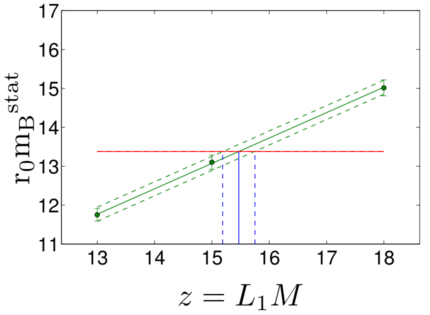
$$m_B = \omega_1 + E_{stat} = m_{bare} + E_{stat} = \omega_1 + E_{stat}$$

$$= \lim_{a\to 0} \left[E_{stat} - \Gamma^{stat}(L_2, a) \right] \qquad a = (0.1 - 0.05)\,\text{fm}$$

$$+ \lim_{a\to 0} \left[\Gamma^{stat}(L_2, a) - \Gamma^{stat}(L_1, a) \right] \qquad a = (0.05 - 0.025)\,\text{fm}$$

$$+ \frac{1}{L_1} \lim_{a\to 0} \Phi_1(L_1, M_b, a) \qquad a = (0.025 - 0.012)\,\text{fm}$$

Analysis with $r_0 m_B^{(exp)}$, $r_0 = (0.475 \pm 0.025)$ fm [Scale setting preliminary]



- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat} = 4.255(25)_{r_0}(50)_{stat+renorm}(?)_a$ GeV
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
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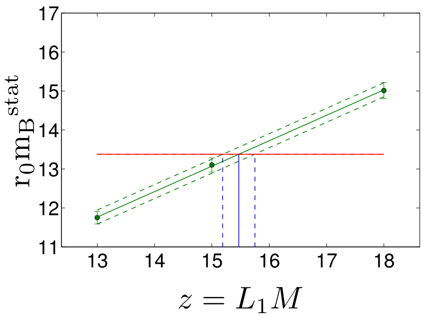

- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat+1/m} =$ $4.276(25)_{r_0}(50)_{stat+renorm}(?)_a$ GeV
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
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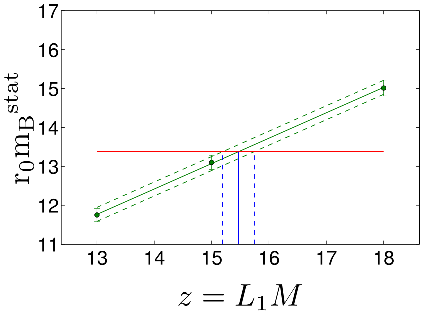

- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat+1/m} = 4.347(40)_{r_0}(48)$ GeV ($N_f = 0$!)
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
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Unquenching effect is presently not significant

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ALPHA Collaboration, in progress

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$$
\begin{aligned}
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&\quad + \lim_{a\to 0}\left[\Gamma^{stat}(L_2, a) - \Gamma^{stat}(L_1, a)\right] \qquad a = (0.05 - 0.025)\,\text{fm} \\
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$$

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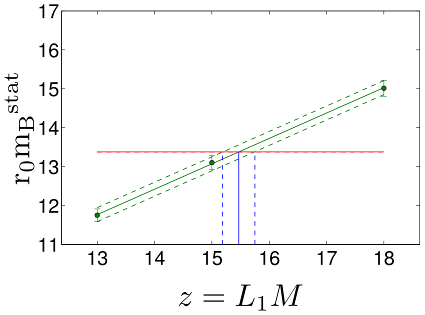

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Conclusions & Outlook

- Lattice heavy flavour physics is becoming a precision field
- Lattice QCD inputs have to be pushed to few-% level (incl. a reliable assessment of all systematics), to contribute to uncovering signals for BSM physics in CKM analyses and resolve / support current tensions
- Dynamical quark simulations ($N_f = 2, 2+1, 2+1+1$) are routine: $m_\pi \sim 500\,\text{MeV}$ (2001) $\rightarrow$ $m_\pi \lesssim 250\,\text{MeV}$ (2010), but the behaviour of algorithms at small lattices spacings needs to be understood

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- An entirely non-perturbative renormalization & matching in HQET is doable with considerable accuracy
  - ▶ Pert. functions C_X not needed altogether within our NP HQET strategy
  - ▶ Physics goals of lattice HQET with $1/m$–corrections: b-quark mass, decay constants $F_{B_{(s)}}$ (1st $O(1/m)$ computation ever !), mass splittings, semi-leptonic form factors

    [ALPHA Collaboration , in progress; tree-level matching: Della Morte, Dooling, H.]
  - ▶ Continuum limit of the large volume part for $N_f = 2$ finished soon
  - ▶ $N_f = 4$ in the longer run: add also strange & charm sea quark flavours