1 Solution to Exercise 1

The vertices come from the hopping terms. Leaving aside the color factor, in momentum space we have

$$\begin{aligned} S_{qqg} &= -\frac{\mathrm{i}g_0}{2} a^4 \sum_{x,\mu} \Big(\overline{\psi}(x)(r-\gamma_\mu)A_\mu(x)\psi(x+a\hat{\mu}) - \overline{\psi}(x+a\hat{\mu})(r+\gamma_\mu)A_\mu(x)\psi(x) \Big) \\ &= -\frac{\mathrm{i}g_0}{2} a^4 \sum_{x,\mu} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p'}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}x(p+k-p')}\, \mathrm{e}^{\mathrm{i}ak_\mu/2} \\ &\quad \times \Big(\overline{\psi}(p')(r-\gamma_\mu)A_\mu(k)\psi(p)\mathrm{e}^{\mathrm{i}ap_\mu} - \overline{\psi}(p')\mathrm{e}^{-\mathrm{i}ap'_\mu}(r+\gamma_\mu)A_\mu(k)\psi(p) \Big). \end{aligned}$$

The first exponential gives a δ-function in momentum space, expressing the conservation of the 4-momentum at the vertex (where $p' = p + k$). Then

$$\begin{aligned} S_{qqg} &= \frac{\mathrm{i}g_0}{2} \sum_{\mu} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p'}{(2\pi)^4}\, (2\pi)^4 \delta^{(4)}(p+k-p')\, \mathrm{e}^{\mathrm{i}ak_\mu/2} \\ &\quad \times \Big(\overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p)(\mathrm{e}^{\mathrm{i}ap_\mu} + \mathrm{e}^{-\mathrm{i}ap'_\mu}) + r\overline{\psi}(p')A_\mu(k)\psi(p)(-\mathrm{e}^{\mathrm{i}ap_\mu} + \mathrm{e}^{-\mathrm{i}ap'_\mu}) \Big) \\ &= \frac{\mathrm{i}g_0}{2} \sum_{\mu} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p'}{(2\pi)^4}\, (2\pi)^4 \delta^{(4)}(p+k-p')\, \mathrm{e}^{\mathrm{i}ak_\mu/2} \\ &\quad \times \Bigg(\overline{\psi}(p')\gamma_\mu A_\mu(k)\psi(p)\, \mathrm{e}^{\mathrm{i}ap_\mu/2}\mathrm{e}^{-\mathrm{i}ap'_\mu/2} \cdot 2\cos\frac{a(p+p')_\mu}{2} \\ &\qquad + r\overline{\psi}(p')A_\mu(k)\psi(p)\, \mathrm{e}^{\mathrm{i}ap_\mu/2}\mathrm{e}^{-\mathrm{i}ap'_\mu/2} \cdot (-2\mathrm{i})\sin\frac{a(p+p')_\mu}{2} \Bigg). \end{aligned}$$

At this point all remaining exponential phases exactly cancel, thanks to the δ-function from the momentum conservation at the vertex. Thus we obtain

$$\begin{aligned} S_{qqg} &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p'}{(2\pi)^4}\, (2\pi)^4 \delta^{(4)}(p+k-p') \\ &\quad \times \mathrm{i}g_0 \sum_{\mu} \overline{\psi}(p') \left(\gamma_\mu \cos\frac{a(p+p')_\mu}{2} - \mathrm{i}r\sin\frac{a(p+p')_\mu}{2} \right) A_\mu(k)\psi(p). \end{aligned}$$

Comment: Notice that the Fourier transform of $A_\mu(x)$ is taken at the point $x + a\hat{\mu}/2$, halfway between x and the neighboring point $x + a\hat{\mu}$, that is in the middle of the link, according to the definition of the link variables. This choice turns out to be quite important for the general economy of the calculations. Had we chosen for the Fourier transform of the gauge potential the expression

$$A_\mu(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}xk}\, A_\mu(k)$$

the exponential phases would not have canceled, and the $\mathrm{e}^{\mathrm{i}ap_\mu/2}\mathrm{e}^{-\mathrm{i}ap'_\mu/2}$ terms would still be present in the final expression of the vertex. This is a general feature of lattice perturbation theory.

2 Solution to Exercise 2

It is convenient to use the shorthand notation[1]

$$c_\lambda = \cos k_\lambda .$$

Then:

(a)

$$\sum_\rho \gamma_\rho \gamma_\mu \gamma_\rho = \sum_\rho \gamma_\rho (-\gamma_\rho \gamma_\mu + 2\delta_{\rho\mu}) = \gamma_\mu \sum_\rho (-\gamma_\rho^2 + 2\delta_{\rho\mu})$$

(b)

$$\sum_\rho \gamma_\rho \gamma_\mu \gamma_\rho \, c_\rho^2 = \sum_\rho (-\gamma_\mu \gamma_\rho^2 + 2\delta_{\rho\mu}\gamma_\mu)\, c_\rho^2 = -\gamma_\mu \sum_\rho c_\rho^2 + 2\gamma_\mu c_\mu^2$$

(c)

$$\sum_\rho \gamma_\rho \gamma_\mu \gamma_\rho \, c_\mu^2 = \sum_\rho (-\gamma_\mu \gamma_\rho^2 + 2\delta_{\rho\mu}\gamma_\mu)\, c_\mu^2 = -4\gamma_\mu c_\mu^2 + 2\gamma_\mu c_\mu^2 = -2\gamma_\mu c_\mu^2$$

(d)

$$\sum_\rho \gamma_\rho \gamma_\mu \gamma_\rho \, c_\mu c_\rho = \sum_\rho (-\gamma_\mu \gamma_\rho^2 + 2\delta_{\rho\mu}\gamma_\mu)\, c_\mu c_\rho = -\gamma_\mu c_\mu \sum_\rho c_\rho + 2\gamma_\mu c_\mu^2$$

[1] In (a) we keep the Kronecker δ-symbols, to show that they are important if further terms are present.