1 Solution to Exercise 3

The vertex, when inserted in the tadpole diagram, depends only on the external momentum $p_1 = p_2 = p$.

Since the gluon is emitted and reabsorbed at the same vertex, there is a Kronecker δ-symbol in color space induced from the gluon propagator, and the color factor becomes $\sum_a \{T^a, T^a\}_{bb} = 2\sum_a (T^a)^2_{bb} = 2(N_c^2-1)/(2N_c) = 2C_F$, the quadratic Casimir invariant of $SU(N_c)$.

The calculation goes as follows:

$$
\begin{aligned}
I &= \frac{1}{2}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^dk}{(2\pi)^d} \sum_\rho G_{\rho\rho}(k)\cdot (V_2^{aa})_{\rho\rho}(p,p) \\
&= \frac{1}{2}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^dk}{(2\pi)^d}\, a^2 \frac{1}{4\sum_\lambda \sin^2 \frac{ak_\lambda}{2}} \Big(-\frac{1}{2} a g_0^2 \sum_a \{T^a,T^a\}_{cc}\Big) \sum_\rho \Big(-\mathrm{i}\gamma_\rho \sin ap_\rho + \cos ap_\rho \Big).
\end{aligned}
$$

At this point one has to rescale the integration variable:

$$
k \to k' = ak.
$$

This means

$$
\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^dk}{(2\pi)^d} f(ak, ap) = \int_{-\pi}^{\pi} \frac{d^dk'}{(2\pi)^d} \frac{1}{a^4} f(k', ap).
$$

Note that the domain of integration after the rescaling becomes independent of a.

Then, taking also the limit of small ap:

$$
\begin{aligned}
I &= -\frac{1}{2} g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{1}{4\sum_\lambda \sin^2 \frac{k_\lambda}{2}} \Big(-\mathrm{i}\not{p} + \frac{4}{a}\Big) \\
&= -\frac{1}{2} g_0^2 C_F Z_0 \Big(-\mathrm{i}\not{p} + \frac{4}{a}\Big).
\end{aligned}
$$

Note that we have used the fact that $\lim_{a\to 0}\sum_\rho \cos ap_\rho = 4$. The quantity Z_0 is an often recurring lattice integral:

$$
\int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{1}{4\sum_\lambda \sin^2 \frac{k_\lambda}{2}} = Z_0 = 0.15493339.
$$

The first term of the result contributes to the wave-function renormalization. The last term diverges like $1/a$, and therefore contributes to the shift of the mass under renormalization (critical mass), due to the breaking of chiral symmetry for Wilson fermions (indeed, it is proportional to r, and thus vanishes for naive fermions). The tadpole actually gives the dominant contribution to the critical mass:

$$
-g_0^2 C_F \cdot 2Z_0 = -\frac{g_0^2}{16\pi^2} C_F \cdot 48.932201,
$$

indeed the total 1-loop critical mass for Wilson fermions, after adding the standard sunset diagram, is

$$
-\frac{g_0^2}{16\pi^2} C_F \cdot 51.434712 = -g_0^2 C_F \cdot 0.325714.
$$

2 Solution to Exercise 4

It is convenient to use the shorthand notations[1]

$$\begin{aligned} c_\lambda &= \cos k_\lambda \\ s_\lambda &= \sin k_\lambda \\ \not{s} &= \sum_\lambda \gamma_\lambda \sin k_\lambda \\ s^2 &= \sum_\lambda \sin^2 k_\lambda . \end{aligned}$$

Then:

(e)

$$\sum_\rho \gamma_\rho \not{s} \gamma_\mu \not{s} \gamma_\rho = \sum_\rho \gamma_\rho (-\gamma_\mu s^2 + 2\not{s} s_\mu)\, \gamma_\rho = \sum_\rho (\gamma_\mu \gamma_\rho^2 s^2 - 2\delta_{\rho\mu}\gamma_\mu s^2 - 2\not{s}\gamma_\rho^2 s_\mu + 4\gamma_\rho s_\rho s_\mu)$$

(f)

$$\begin{aligned} \sum_\rho \gamma_\rho \not{s} \gamma_\mu \not{s} \gamma_\rho c_\mu c_\rho &= \sum_\rho (\gamma_\mu \gamma_\rho^2 s^2 - 2\delta_{\rho\mu}\gamma_\mu s^2 - 2\not{s}\gamma_\rho^2 s_\mu + 4\gamma_\rho s_\rho s_\mu)\, c_\mu c_\rho \\ &= (\gamma_\mu s^2 - 2\not{s} s_\mu)\, c_\mu \sum_\rho c_\rho - 2\gamma_\mu s^2 c_\mu^2 + 4 s_\mu c_\mu \sum_\rho \gamma_\rho s_\rho c_\rho \end{aligned}$$

[1] In (e) we keep the Kronecker δ-symbols, as well as factors like γ_ρ^2, to show that they are important if further terms are present.