1 Solution to Exercise 5

We use the shorthand notations

$$\Gamma_\lambda = \sin k_\lambda, \tag{1.1}$$

$$W = 2\sum_\lambda \sin^2\frac{k_\lambda}{2}, \tag{1.2}$$

$$N_\rho = \sin\frac{k_\rho}{2}, \tag{1.3}$$

$$M_\rho = \cos\frac{k_\rho}{2}. \tag{1.4}$$

We also put

$$\not{s} = \sum_\lambda \gamma_\lambda \sin k_\lambda, \tag{1.5}$$

and of course we also have

$$\Gamma^2 = \sum_\lambda \sin^2 k_\lambda. \tag{1.6}$$

It should be noted that Γ and N are odd in k, while M and W are even.

The zero-momentum part for the sunset diagram of the quark self-energy is:

$$\begin{aligned} J &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^dk}{(2\pi)^d} \sum_\rho G_{\rho\rho}(p-k)\cdot\Big[V_\rho(k,p)\cdot S(k)\cdot V_\rho(p,k)\Big]\Big|_{ap=0} \\ &= \frac{g_0^2}{a}C_F \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\sum_\rho\left(\frac{1}{2W}+\frac{2a\sum_\lambda p_\lambda\Gamma_\lambda}{(2W)^2}\right)\left(N_\rho+\mathrm{i}\gamma_\rho M_\rho+\frac{ap_\rho}{2}(M_\rho-\mathrm{i}\gamma_\rho N_\rho)\right) \\ &\times\frac{-\mathrm{i}\not{s}+W}{\Gamma^2+W^2}\left(N_\rho+\mathrm{i}\gamma_\rho M_\rho+\frac{ap_\rho}{2}(M_\rho-\mathrm{i}\gamma_\rho N_\rho)\right), \end{aligned} \tag{1.7}$$

where we have rescaled the integration variable. After combining the various factors a coming from the propagator and the vertices, as well as from the rescaling of k, we are left with an overall factor $1/a$. Then we extract the contribution to the critical mass, i.e., the $1/a$ part:

$$J = \frac{g_0^2}{a}C_F\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\sum_\rho\frac{1}{2W}(N_\rho+\mathrm{i}\gamma_\rho M_\rho)\frac{-\mathrm{i}\not{s}+W}{\Gamma^2+W^2}(N_\rho+\mathrm{i}\gamma_\rho M_\rho), \tag{1.8}$$

which gives

$$J = \frac{g_0^2}{a}C_F\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\sum_\rho\frac{1}{2W(\Gamma^2+W^2)}\Big(N_\rho^2W-\gamma_\rho^2M_\rho^2W+(\gamma_\rho\not{s}+\not{s}\gamma_\rho)N_\rho M_\rho\Big), \tag{1.9}$$

where we have dropped terms in the numerator which are odd in k (because the denominator is even in k).

After these manipulations, no Dirac matrices are left in the contribution to m_c. The corresponding integral is not divergent and is given by

$$\begin{aligned} m_c^{(a)} &= g_0^2C_F\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\sum_\rho\frac{1}{2W(\Gamma^2+W^2)}\Big((N_\rho^2-M_\rho^2)W+\Gamma_\rho^2\Big) \\ &= g_0^2C_F\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\left\{\frac{\sum_\rho\cos k_\rho}{2\Big(\sum_\lambda\sin^2k_\lambda+\Big(2\sum_\lambda\sin^2\frac{k_\lambda}{2}\Big)^2\Big)^2}\right. \end{aligned} \tag{1.10}$$

$$+\frac{\sum_\rho \sin^2 k_\rho}{4\Big(\sum_\lambda \sin^2 \frac{k_\lambda}{2}\Big)\Big(\sum_\lambda \sin^2 k_\lambda + \Big(2\sum_\lambda \sin^2 \frac{k_\lambda}{2}\Big)^2\Big)^2}\Bigg\}$$
$$= -\frac{g_0^2}{16\pi^2} C_F \cdot 2.502511.$$

This is the contribution to the critical mass coming from the sunset diagram of the self-energy.

Solution to the advanced problem:

We start again from the expansion in (1.7). Since there is an overall factor $1/a$ in front of the whole expression, in order to compute the contribution of order zero in a we have to keep all terms of order ap in the Taylor expansions of propagator and vertices. Then, multiplying everything together, we have

$$\begin{aligned} J &= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \Bigg\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \frac{-\mathrm{i}\not{s} + W}{\Gamma^2 + W^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \\ &+ \frac{1}{2W}\frac{p_\rho}{2} \Bigg[(M_\rho - \mathrm{i}\gamma_\rho N_\rho) \frac{-\mathrm{i}\not{s} + W}{\Gamma^2 + W^2} (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \\ &+ (N_\rho + \mathrm{i}\gamma_\rho M_\rho) \frac{-\mathrm{i}\not{s} + W}{\Gamma^2 + W^2} (M_\rho - \mathrm{i}\gamma_\rho N_\rho) \Bigg] \Bigg\}. \end{aligned}$$

We now do the multiplications, and in the numerator we drop all terms which are odd in k. This gives

$$\begin{aligned} J &= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \Bigg\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2 (\Gamma^2 + W^2)} \Big(-\mathrm{i}\not{s} N_\rho^2 + \mathrm{i}\gamma_\rho \not{s} \gamma_\rho M_\rho^2 + 2\mathrm{i}\gamma_\rho N_\rho M_\rho W \Big) \\ &+ \frac{1}{2W(\Gamma^2 + W^2)} \frac{p_\rho}{2} \Big(2\mathrm{i}\gamma_\rho (M_\rho^2 - N_\rho^2) W - 2\mathrm{i}\not{s} N_\rho M_\rho - 2\mathrm{i}\gamma_\rho \not{s} \gamma_\rho N_\rho M_\rho \Big) \Bigg\}. \end{aligned} \tag{1.11}$$

At this point we can do the Dirac algebra, and so we arrive at an expression which contains only one Dirac matrix in each monomial:

$$\begin{aligned} J &= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_\rho \Bigg\{ \frac{2\sum_\lambda p_\lambda \Gamma_\lambda}{(2W)^2 (\Gamma^2 + W^2)} \Big(-\mathrm{i}\not{s}(N_\rho^2 + M_\rho^2) + 2\mathrm{i}\gamma_\rho \Gamma_\rho M_\rho^2 + \mathrm{i}\gamma_\rho \Gamma_\rho W \Big) \\ &+ \frac{1}{2W(\Gamma^2 + W^2)} p_\rho \Big(\mathrm{i}\gamma_\rho (M_\rho^2 - N_\rho^2) W - \mathrm{i}\gamma_\rho \Gamma_\rho^2 \Big) \Bigg\} \end{aligned} \tag{1.12}$$

$$\begin{aligned} &= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \Bigg\{ \frac{\mathrm{i}\not{p}}{(2W)^2 (\Gamma^2 + W^2)} \Big(-2\Gamma_\nu^2 \sum_\rho (N_\rho^2 + M_\rho^2) + 4\sum_\rho \Gamma_\rho^2 M_\rho^2 + 2\Gamma_\nu^2 W \Big) \\ &+ \frac{\mathrm{i}\not{p}}{2W(\Gamma^2 + W^2)} \Big((M_\nu^2 - N_\nu^2) W - \Gamma_\nu^2 \Big) \Bigg\}. \end{aligned} \tag{1.13}$$

In the last passage we have used the substitution

$$\sum_\lambda \gamma_\lambda p_\lambda \int f_\lambda(k) = \not{p} \int f_\mu(k), \tag{1.14}$$

since this kind of integrals does not depend on the direction, with the understanding that the index μ is then fixed and must not appear again in the rest of the monomial. This reconstructs the factor $i\not{p}$, and so we finally obtain the result for the $i\not{p}$ contribution:

$$\begin{aligned} J &= g_0^2 C_F i\not{p} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{\cos k_\nu}{2\Big(\sum_\lambda \sin^2 k_\lambda + \Big(2 \sum_\lambda \sin^2 \frac{k_\lambda}{2}\Big)^2 \Big)^2} \right. \\ & \left. + \frac{-4 \sin^2 k_\nu + 2 \sum_\rho \sin^2 k_\rho \cos^2 \frac{k_\rho}{2}}{8\Big(\sum_\lambda \sin^2 \frac{k_\lambda}{2}\Big)^2 \Big(\sum_\lambda \sin^2 k_\lambda + \Big(2 \sum_\lambda \sin^2 \frac{k_\lambda}{2}\Big)^2 \Big)^2} \right\}. \end{aligned} \tag{1.15}$$

This integral is logarithmically divergent, and will need to be treated in some way.