Topology and the QCD vacuum

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1.Topological effect in quantum mechanics

Assume:

- Infinitely long electric coil in z-direction with radius $R \to 0$.
- Constant magnetic flux Φ inside the coil.
- Particle to move around the coil along circle in $x - y$-plane.

Action: $S_0 = \int_0^T dt \left(\frac{\dot{\varphi}^2}{2}\right)$, $\varphi(T) - \varphi(0) = 2\pi n$, n "winding number".

Solution: $\varphi(t) = \varphi(0) + \omega_n t$, $\omega_n = \frac{2\pi n}{T}$.

Interaction with magnetic field described by "topological term":

$$S_{top} = \frac{e_0}{c} \int_0^T dt \; \dot{\vec{x}} \cdot \vec{A}(\vec{x}(t)) = \hbar \frac{\Phi}{\Phi_0} \int_0^T dt \; \dot{\varphi} = \hbar \frac{\Phi}{\Phi_0} \; 2\pi n \,, \qquad \Phi_0 \equiv \frac{2\pi \hbar c}{e_0} \,,$$

because $A_i = \frac{\Phi}{2\pi} \frac{\partial}{\partial x_i} \arctan\left(\frac{x_2}{x_1}\right) = \frac{\Phi}{2\pi} \frac{1}{r^2}(-x_2, x_1) = \frac{\Phi}{2\pi} \frac{1}{r}(-\sin\varphi, \cos\varphi)$

$$\to \; B_3 = \Phi \delta^{(2)}(\vec{x}) \quad \to \quad \int d^2 x B_3 = \Phi.$$

S_{top} does not contribute to the classical equation of motion !

Situation changes in the quantum case: **Aharonov-Bohm effect**.

Quantum mechanical scattering states:

$$\psi_k(x) \to e^{i\vec{k}\vec{x}} + \frac{e^{ikr}}{r} T(\varphi), \quad \varphi = \arctan\left(\frac{x_2}{x_1}\right).$$

Diff. cross section:

$$\begin{aligned} \frac{d\sigma}{d\varphi} = |T(\varphi)|^2 \quad &= \quad \frac{1}{2\pi} \sin^2\left(\pi \frac{\Phi}{\Phi_0}\right) \frac{1}{\cos^2(\varphi/2)}, \\ &\neq \quad 0 \;\; \text{for} \;\; \Phi \neq m\Phi_0, \quad m \in \mathbf{N}. \end{aligned}$$

$\Longrightarrow$ Topology causes effect not existing in classical physics.

2. Topology: non-linear $O(3)$ σ-model in 2D

Interesting play model for QCD:

– perturbation theory shows asymptotic freedom,

– existence of topological solutions - instantons,

– lattice simulations easy - cluster algorithm available,

– model generalizable to larger number of degrees of freedom, with local $U(1)$ invariance and allowing $1/n$-expansion ($CP(n-1)$ model),

– interaction with fermion fields can be implemented.

Studied also in condensed matter theory (in D=2+1 or 3+1).

Consider 2D Euclidean space $\mathbf{R}^2$:

$$S[\Phi] = \int d^2x \frac{1}{2} \left(\partial_i \Phi^a(x) \cdot \partial_i \Phi^a(x)\right), \quad i = 1, 2, \; a = 1, 2, 3,$$

with condition: $\sum_a \Phi^a \Phi^a = 1 \; \rightarrow \; \vec{\Phi} \in S^2_{int.sym.}$.
Model has global $O(3)$ symmetry.

Field equations by varying with Lagrange parameter $\lambda(x)$

$$\widetilde{S} = \int d^2x \left[\frac{1}{2} \partial_i \vec{\Phi} \cdot \partial_i \vec{\Phi} + \lambda(x)(\vec{\Phi}^2 - 1) \right].$$

$$\begin{aligned} -\Delta\Phi^a + 2\lambda\Phi^a &= 0 \quad \rightarrow \quad \lambda = \frac{1}{2}\Phi^a \Delta\Phi^a, \\ \Delta\vec{\Phi} - (\vec{\Phi}\cdot\Delta\vec{\Phi})\vec{\Phi} &= 0. \end{aligned}$$

Search for fields with $S[\Phi] < \infty$,

$r|\text{grad } \Phi_i| \rightarrow 0$ for $r \rightarrow \infty$, $\quad \lim_{r\rightarrow\infty} \vec{\Phi} = \vec{\Phi}^{(vac)} = \text{const.}$,

$\Longrightarrow$ vacuum field breaks $O(3)$ symmetry.

$\Longrightarrow$ $\mathbf{R}^2$ gets compactified $\equiv$ S^2.

Homotopy: Mapping $x \in \mathbf{R}^2(S^2) \rightsquigarrow \vec{\Phi}(x) \in S^2_{int.sym.}$ called $\pi_2(S^2)$.

Mapping decays into equivalence classes of continuously deformable mappings with fixed integer winding number or topological charge Q_t.

$Q_t = 0, \pm 1, \pm 2, \ldots$ corresponds to oriented surface on $S^2_{int.sym.}$, when covering $\mathbf{R}^2(S^2)$ once.

Illustration for homotopy: $\pi_1(S^1)$ mapping circle onto circle.

$$\theta \in [0,\ 2\pi] \quad \rightsquigarrow \quad f(\theta) \in \mathrm{R}$$

with $f(\theta)$ continuous function
satisfying b. c. $f(0) = 0$ and $f(\theta = 2\pi) = 2\pi\, n$ with $n \in \mathbf{Z}$.

Examples:

- zero winding: $f_0(\theta) = 0$ for all θ,

$$\tilde{f}_0(\theta) = \begin{cases} t\ \theta & \text{for } 0 \leq \theta < \pi \\ t\ (2\pi - \theta) & \text{for } \pi \leq \theta < 2\pi \end{cases}$$

with $t \in [0, 1]$ for deformation.

- unit winding: $f_1(\theta) = \theta$ for all θ.

Winding number: $Q_t = \frac{1}{2\pi} \int_0^{2\pi} d\theta\ (df/d\theta) = n,$
thus $\pi_1(S^1) \equiv \mathbf{Z}$ for arbitrary mapping $f(\theta)$.

Back to $O(3)$ σ-model:

$$
\begin{aligned}
Q_t &= \frac{1}{4\pi}\int dS^{(int.sym.)} = \frac{1}{4\pi}\int dS^a \cdot \Phi^a \in \mathbf{Z}, \\
&= \frac{1}{8\pi}\int d^2x \; \epsilon_{\mu\nu}\epsilon_{abc}\frac{\partial \Phi^b}{\partial x_\mu}\frac{\partial \Phi^c}{\partial x_\nu}\cdot \Phi^a, \\
&= \frac{1}{8\pi}\int d^2x \; \epsilon_{\mu\nu}\vec{\Phi}\cdot(\partial_\mu\vec{\Phi}\times\partial_\mu\vec{\Phi}) \equiv \int d^2x \; \rho_t(x).
\end{aligned}
$$

Q_t invariant against continuous deformations of $x \to \vec{\Phi}(x)$.

Q_t defines lower bound for $S[\Phi]$:

$$\int d^2x \; [(\partial_\mu\vec{\Phi} \pm \epsilon_{\mu\nu}\vec{\Phi}\times\partial_\nu\vec{\Phi})(\partial_\mu\vec{\Phi} \pm \epsilon_{\mu\sigma}\vec{\Phi}\times\partial_\sigma\vec{\Phi})] \geq 0$$

then

$$
\begin{aligned}
\frac{1}{2}\int d^2x \; (\partial_\mu\vec{\Phi}\cdot\partial_\mu\vec{\Phi}) &\geq \pm\frac{1}{2}\int d^2x \; \epsilon_{\mu\nu}\vec{\Phi}\cdot(\partial_\mu\vec{\Phi}\times\partial_\nu\vec{\Phi}), \\
S[\Phi] &\geq 4\pi \; |Q_t|.
\end{aligned}
$$

$S[\Phi] = 4\pi \; |Q_t|$ if $\partial_\mu\vec{\Phi} = \pm\epsilon_{\mu\sigma}\vec{\Phi}\times\partial_\sigma\vec{\Phi}$, "(anti)selfduality".

2D lattice discretization (spacing a):

$$S \to S_L = a^2 \sum_{n,i} \frac{1}{2a^2} (\vec{\Phi}_{n+\hat{i}} - \vec{\Phi}_n) \cdot (\vec{\Phi}_{n+\hat{i}} - \vec{\Phi}_n),$$
$$= \sum_{n,i} (1 - \vec{\Phi}_{n+\hat{i}} \cdot \vec{\Phi}_n).$$

represents $O(3)$ spin model. Considered on finite lattice with p.b.c. (T^2).

$$Q_t \to Q_L = \frac{1}{4\pi} \sum_{\sigma \in T^2} A_\sigma \equiv \sum_{\sigma \in T^2} \rho_\sigma$$

with $\sigma \equiv (l, m, n)$ simplex of adjacent lattice sites,
and A_σ oriented surface spanned by 3-leg $(\vec{\Phi}_l, \vec{\Phi}_m, \vec{\Phi}_n)$.
Spherical triangle with angles $(\alpha_l, \alpha_m, \alpha_n)$, then

$$A_\sigma = \mathrm{sign}[\vec{\Phi}_l \cdot (\vec{\Phi}_m \times \vec{\Phi}_n)]\ (\alpha_l + \alpha_m + \alpha_n - \pi).$$

Q_L alternatively computable by counting how often a reference point on S^2 is covered by Φ simplices taking the orientation $\mathrm{sign}[\vec{\Phi}_l \cdot (\vec{\Phi}_m \times \vec{\Phi}_n)]$ into account.

Properties of Q_L:

- $Q_L \in \mathrm{Z}$ by definition;
- local topological density defined $\rho_t \equiv \rho_\sigma$;
- smoothness condition for configurations $\{\vec{\Phi}_n\}$:

$1 - \vec{\Phi}_m \cdot \vec{\Phi}_n < \epsilon$ for all neighbour pairs $\langle m, n \rangle$
keeping $\vec{\Phi}_l \cdot (\vec{\Phi}_m \times \vec{\Phi}_n) \neq 0$,

$\Longrightarrow$ Q_L can be uniquely defined if ϵ sufficiently small .

Solution of selfduality equation: [Belavin, Polyakov, '75]

Complex formulation via stereographic projection:

$$\omega(z) = 2\,\frac{\Phi^1 + i\Phi^2}{1-\Phi^3}, \quad z \;=\; x_1 + i x_2\,.$$

$$S = \int d^2x \frac{|d\omega/dz|^2}{(1+|\omega|^2/4)^2} = 4\pi Q_t\,.$$

Selfduality equation = Cauchy-Riemann eqs. for analytic functions:

$$\partial_1\omega = \mp i\;\partial_2\omega$$

Multi- (anti-) instanton solutions:

'instantons'= 'pseudo-particles' localized in (Euclidean) space-time

$$\omega_{multiinst}(z) = \frac{P_1(z)}{P_2(z)} \quad \text{or} \quad \frac{P_1(\overline{z})}{P_2(\overline{z})} \quad \text{, where} \;\; P_i, \quad i=1,2 \quad \text{polynoms.}$$

$$Q_t[\omega_{inst}] = \begin{cases} +\max(\deg\, P_1(z), \deg\, P_2(z)) > 0 & \text{'instantons'} \\ -\max(\deg\, P_1(\overline{z}), \deg\, P_2(\overline{z})) < 0 & \text{'antiinstantons'} \end{cases}$$

Example: $\omega_{inst}(z) = (z - z_0)/\lambda$, z_0 'position', $\lambda =$ 'width'

$\Longrightarrow \quad Q_t = +1, \quad S = 4\pi \quad$ independent of z_0 and λ !

Path integral quantization:

compute Euclidean vacuum transition amplitudes or correlation functions
(β inverse coupling)

$$\langle \Omega(\Phi) \rangle = \frac{1}{Z} \int D\Phi(x)\ \Omega(\Phi) \exp(-\beta S[\Phi])$$

$$Z = \int D\Phi(x)\ \exp(-\beta S[\Phi]), \quad D\Phi(x) = \prod_{x,a} d\Phi^a(x)\ \delta(\Phi^2 - 1)$$

Interesting non-perturbative observables $< \Omega >$:

– topological susceptibility: $\chi_t = \frac{1}{V^{(2)}} \langle Q_t^2 \rangle$, $V^{(2)}$ 2D volume,
  χ_t diverges for one-instanton contribution (see below),

– correlation length ξ from correlator:
  $C^{ab}(x,y) = \langle \Phi^a(x) \Phi^b(y) \rangle \propto \delta^{ab}\ (C \exp(-|x-y|/\xi) + \ldots)$ for $|x-y| \to \infty$,

– dimensionless combination of both: $\chi_t\ \xi^2$.

Semiclassical approximation in general – the limit $\hbar \to 0$:

Taylor expansion 'around' classical solutions (multi-instantons): $\Phi = \Phi_{cl} + \eta$

$$S(\Phi) = S(\Phi_{cl}) + \frac{1}{2!} \int d^2x \;\; \eta \, \frac{\delta^2 S}{\delta \eta^2}\bigg|_{\Phi=\Phi_{cl}} \eta + \ldots$$

$$Z \simeq \sum_{\Phi_{cl}} \; \exp(-\beta S[\Phi_{cl}]) \; \cdot \; \mathrm{Det} \left(\frac{\delta^2 S}{\delta \eta^2}\bigg|_{\Phi=\Phi_{cl}} \right)^{-\frac{1}{2}} + \ldots$$

Difficulties:

– zero-modes of $\frac{\delta^2 S}{\delta \eta^2}\Big|_{\Phi=\Phi_{cl}}$,

$\Rightarrow$ method of collective coordinates (instanton positions, scales, etc.),

– treat determinant of non-zero modes (UV divergencies, renormalization).

Concrete for non-linear $O(3)$ σ model: [Fateev, Folov, Schwarz, 1978]

– one-instanton amplitude is IR divergent in the zero-mode scale integration,

– multi-instanton contributions to vacuum amplitude well-defined,
dominate in the form

$$\omega_{multiinst}(z) = c\frac{(z-a_1)\cdot\ldots\cdot(z-a_q)}{(z-b_1)\cdot\ldots\cdot(z-b_q)} \quad \to \quad Q_t = q > 0.$$

Result: Partition function

$$Z \simeq \sum_q Z_q, \quad Z_q \propto \frac{1}{(q!)^2}\int \prod_{i=1}^{q} d^2a_i \prod_{j=1}^{q} d^2b_j \int \frac{d^2c}{(1+|c|^2)^2}\exp(-\epsilon_q(a,b))$$

with 'energy'

$\epsilon_q(a,b) = -\sum_{i<j}^q \log|a_i-a_j|^2 - \sum_{i<j}^q \log|b_i-b_j|^2 + \sum_{i,j}^q \log|a_i-b_j|^2$

Corresponds to 2D Coulomb gas of positively (negatively) charged constituents

$$a_i \ (b_j) \quad \Rightarrow \quad \text{\textit{“instanton quarks”}}.$$

Known to have phase transition:

$T > T_c \quad \to \quad$ molecular phase of bound constituents,

$T < T_c \quad \to \quad$ plasma phase of unbound constituents (realized for $O(3)$ σ m.).

Hope, that similar mechanism works also in QCD $\quad\Rightarrow\quad$ confinement (??).

3. Topology and instantons in 4D Yang-Mills theory

[Belavin, Polyakov, Schwarz, Tyupkin, '75; 't Hooft, '76; Callan, Dashen, Gross, '78 -'79]

Mostly talk about $SU(2)$ for simplicity.

Potentials: $A_\mu \equiv \sum_a g \frac{\sigma^a}{2i} A^a_\mu \in su(2)$,

σ^a Pauli matrices with $[\sigma^a/2, \sigma^b/2] = i\epsilon^{abc}\sigma^c/2$.

Field strength: $G_{\mu\nu} = \sum_a g \frac{\sigma^a}{2i} G^a_{\mu\nu}$, $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

Gauge transformation: $U(x) = e^{-i\omega^a\sigma^a/2} \in SU(2)$

$$
\begin{aligned}
A_\mu(x) \quad &\to \quad A^U_\mu(x) = U^\dagger(x) A_\mu(x) U(x) + U^\dagger(x)\partial_\mu U(x) \\
G_{\mu\nu}(x) \quad &\to \quad G^U_{\mu\nu}(x) = U^\dagger(x) G_{\mu\nu}(x) U(x)
\end{aligned}
$$

Euclidean action: $S[A] = -\frac{1}{2g^2}\int d^4x \; \mathrm{tr}\,(G_{\mu\nu}G_{\mu\nu}) = \frac{1}{4}\int d^4x \; G^a_{\mu\nu}G^a_{\mu\nu}$.

Field equation: $\frac{\delta S}{\delta A^a_\mu} = 0 \;\Rightarrow\; D_\mu G_{\mu\nu} = \partial_\mu G_{\mu\nu} + [A_\mu, G_{\mu\nu}] = 0$.

Want to compute Euclidean vacuum-to-vacuum amplitude with path integral:

$$Z = \langle vac|\exp(-\frac{1}{\hbar}\hat{H}(\tau - \tau_0))|vac\rangle = C\int DA_\mu(x)\exp\left(-\frac{1}{\hbar}S[A]\right)$$

. $|vac\rangle \implies$ boundary condition for "field trajectories":

$$A_\mu(x) \to A_\mu^{vac}(x) \equiv U^\dagger(x)\partial_\mu U(x) \quad \text{for} \quad |x| \to \infty.$$

Show that "pure gauge" contribution $A_\mu^{vac}(x)$ is characterized by integer "winding number" or "Pontryagin index".

Introduce topological charge:

$Q_t[A] = \int d^4x\ \rho_t(x), \quad \rho_t(x) = -\frac{1}{16\pi^2}\ \mathrm{tr}\,(G_{\mu\nu}\tilde{G}_{\mu\nu}),$ gauge invariant,

dual field strength $\tilde{G}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}$.

Topological density can be rewritten as

$\rho_t(x) = \partial_\mu K_\mu, \qquad K_\mu = -\frac{1}{8\pi^2}\ \epsilon_{\mu\nu\rho\sigma}\ \mathrm{tr}\,[A_\nu(\partial_\rho A_\sigma + \frac{2}{3}A_\rho A_\sigma)],$

"Chern-Simons density", gauge variant current.

Winding at $|x| \to \infty$:

$$w_\infty = \oint_{S^3(R\to\infty)} d\sigma_\mu K_\mu = \oint d^3\sigma \; n_\mu K_\mu = \frac{1}{24\pi^2} \oint d^3\sigma \; n_\mu \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\,[A_\nu A_\rho A_\sigma].$$

Used vanishing of $G_{\mu\nu} = 0$ for $|x| \to \infty$, thus $\epsilon_{\mu\nu\rho\sigma}\partial_\rho A_\sigma = -\epsilon_{\mu\nu\rho\sigma} A_\rho A_\sigma$.

$$w_\infty = \frac{1}{24\pi^2} \oint_{S^3(R\to\infty)} d^3\sigma \; n_\mu \; \epsilon_{\mu\nu\rho\sigma} \mathrm{tr} \left[(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U) \right],$$

$\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\,[\ldots]$ represents Jacobian for mapping $S^3(R) \to SU(2)$.

Indeed, for $SU(2)$ $U = B_0 + i\vec{\sigma}\cdot\vec{B}$, $\sum_{i=0}^{3} B_i B_i = 1$, thus $SU(2) \equiv S^3$.

$$w_\infty = \frac{1}{2\pi^2} \oint_{S^3(R\to\infty)} d^3\sigma \, \det(\nabla_i B_j)$$

w_∞ counts how often $S^3(R)$ is continuously mapped onto S^3-sphere of $SU(2)$.

Thus, (gauge-variant) vacuum fields $A_\mu^{(vac)}(x)$ characterized by classes of non-trivial Pontryagin index $w_\infty \in \mathbf{Z}$.

Notice: $S^3(R) \to SU(N_c)$ continuously deformable to $S^3(R) \to SU(2)$, i.e. same homotopy group.

Now additional assumption:

$$A_\mu(x) \to U^\dagger(x)\partial_\mu U(x) \quad \text{for} \quad x \to x^{(i)}, \quad i = 1, 2, \ldots, q.$$

Then, due to Gauss' law:

$$\begin{aligned} Q_t[A] \quad &= \quad w_\infty + \sum_{i=1}^{q} w_i = w_\infty + \sum_i \lim_{R^{(i)} \to 0} \oint_{\sigma^{(i)}} d^3\sigma \; n_\mu K_\mu \\ &\equiv \quad \text{sum of 'Pontryagin indices' at singular points } x_i \text{ and } |x| \to \infty \,. \end{aligned}$$

Example for (pure gauge) vacuum field with $w_\infty = +1$:

$$A_\mu^{(+1)}(x) = U^\dagger(x)\partial_\mu U(x) \quad \text{with} \quad U \equiv U_1 = \frac{\mathbf{1}x_4 - i\vec{\sigma}\cdot\vec{x}}{\sqrt{x^2}} \in SU(2).$$

$$A_{a,\mu}^{(+1)}(x) = \frac{2}{g}\eta^{(+)}_{a\mu\nu}\frac{x_\nu}{x^2},$$

't Hooft symbols:

$\eta^{(\pm)}_{a\mu\nu} = \epsilon_{a\mu\nu}$ for $\mu,\nu = 1,2,3$, $\eta^{(\pm)}_{a4\nu} = -\eta^{(\pm)}_{a\nu 4} = \pm\delta_{a\nu}$, $\eta^{(\pm)}_{a44} = 0$.

U_1 can be used to characterize homotopy equivalence classes by

$$U_n = (U_1)^{\pm n} \quad \rightarrow \quad w_\infty[U_n] = \pm n$$

"Little" gauge transformations ($w_\infty[U] = 0$) deform U_n, but leave winding invariant.

Comment: $S[A^{(+1)}] = 0$ and $Q_t[A^{(+1)}] = w_\infty + w_{x=0} = 1 - 1 = 0$.

Quantum case: vacuum state(s) classified by winding

$$|n\rangle \quad \leftrightarrow \quad w_\infty = n \quad (\text{“prevacua”})$$

“Large” gauge transformation (with $w = 1$) represented by unitary operator

$$\hat{T}(U_1): \quad \hat{T}\,|n\rangle = |n+1\rangle$$

Hamiltonian $\hat{H}$ invariant: $[\hat{T}, \hat{H}] = 0.$

Physical gauge invariant ground state – “θ-vacuum:”

$$\begin{aligned} \hat{H}|\theta\rangle &= E_0|\theta\rangle, \\ \Rightarrow \quad \hat{H}\hat{T}|\theta\rangle &= E_0\hat{T}|\theta\rangle, \\ \Rightarrow \quad \hat{T}|\theta\rangle &= \exp(-i\theta)|\theta\rangle. \end{aligned}$$

Realized from “prevacua” $|n\rangle$ as “Bloch states” for periodic potentials:

$$|vac\rangle \equiv |\theta\rangle \equiv \sum_{n=-\infty}^{n=+\infty} e^{in\theta}\,|n\rangle$$

Vacuum transition amplitude $(\hbar = 1, \quad \tau \to \infty)$

$$Z(\theta,\theta') = \langle\theta'|\exp(-\hat{H}\tau)|\theta\rangle = \sum_{n,n'} e^{i(n\theta - n'\theta')}\langle n'|\exp(-\hat{H}\tau)|n\rangle$$

$$= \sum_{n,n'} e^{i(n\theta - n'\theta')} \int DA_\mu(x)\big|_{n,n'} \exp\left(-S[A]\right)$$

with b.c.'s $\quad A_\mu \to \begin{cases} A_\mu^{(n')} & \text{for} \quad \tau' \to +\infty \\ A_\mu^{(n)} & \text{for} \quad \tau' \to -\infty \end{cases}$

put $\quad \nu \equiv n' - n \equiv Q_t$

$$= \sum_{n,n'} e^{i(n\theta - n'\theta')} f(n' - n)$$

$$= \sum_{n} e^{i\ n\ (\theta - \theta')} \sum_{\nu} e^{-i\ \nu\ \theta'} f(\nu)$$

$$= \delta(\theta - \theta') \sum_{\nu} \int DA_\mu(x)\big|_\nu \exp\left(-S[A] - i\ Q_t[A]\ \theta'\right)$$

i.e. superselection rule.

Comments:

- So far no reference to specific classical solutions of field equations;
- θ-term in the action: 4-divergence does contribute, if topologically non-trivial field configurations with $Q_t \neq 0$ exit;
- θ-term violates P-, T-, thus CP-invariance: "Strong CP-violation";
- electric dipole moment of the neutron provides bound: $\theta < O(10^{-9})$;
- θ as a variational parameter: $< Q_t^2 > \sim \frac{1}{Z(\theta)} \left. \frac{d^2}{d\theta^2} Z(\theta) \right|_{\theta=0}$;
- Open question: occurence of a phase transition, when varying θ.

Instanton solutions: [Belavin, Polyakov, Schwarz, Tyupkin, '75]

As for $O(3)$ σ-model: $S[A] \;\geq\; \frac{8\pi^2}{g^2}|Q_t[A]|$, since

$$-\int d^4x \;\mathrm{tr}\,[(G_{\mu\nu} \pm \tilde{G}_{\mu\nu})^2] \geq 0\,,$$

$$\text{iff} \quad S[A] = \frac{8\pi^2}{g^2}\;|Q_t[A]|, \qquad \text{then} \quad G_{\mu\nu} = \pm\tilde{G}_{\mu\nu}.$$

BPST one-(anti)instanton solution (singular gauge) for $SU(2)$:

$$\mathcal{A}^{(\pm)}_{a,\mu}(x-z,\rho,R) = R^{a\alpha}\eta^{(\pm)}_{\alpha\mu\nu}\frac{2\;\rho^2\;(x-z)_\nu}{(x-z)^2\;((x-z)^2+\rho^2)},$$

with free parameters ρ – scale size,
z_ν – position,
$R^{a\alpha}\;T^\alpha \;=\; U^\dagger\;T^a\;U$ – global $SU(2)$ orientation.

$\Rightarrow \quad S[\mathcal{A}^{(\pm)}] = \frac{8\pi^2}{g^2}, \quad Q_t[\mathcal{A}^{(\pm)}] = \pm 1$ independent of the 8 parameters $\Rightarrow$
fluctuations around $\mathcal{A}^{(\pm)}$ provide 8 zero modes !

Multi-Instantons exist for any $Q_t \in \mathbf{Z}$ [Atiyah, Hitchin, Manin, Drinfeld, '78],
in practice difficult to handle.

Instanton contributions to vacuum amplitude – semiclassical approximation

['t Hooft, '76; Callan, Dashen, Gross, '78, '79]

$$Z(\theta) \equiv \sum_\nu \int DA_\mu(x)|_\nu \exp\left(-S[A] - i\;\nu\;\theta\right), \quad \nu = Q_t[A],$$

approximated by (sufficiently dilute) superpositions

$$\begin{aligned}
\mathcal{A}^{[\nu]}_{a,\mu}(x) &= \sum_{\sigma=\pm}\sum_{i=1}^{N_\sigma} \mathcal{A}^{(\sigma)}_{a,\mu}(x - z^{(i)}, \rho^{(i)}, R^{(i)}), \\
&\quad \text{with} \quad \nu = N_+ - N_-, \quad \rho^{(i)}\rho^{(j)} \ll (z^{(i)} - z^{(j)})^2 \\
A_{a,\mu}(x) &= \mathcal{A}^{[\nu]}_{a,\mu}(x) + \varphi_{a,\mu}(x)
\end{aligned}$$

$$\begin{aligned}
Z(\theta = 0) &= \sum_\nu \int DA_\mu(x)|_\nu \exp\left(-S[A]\right) \\
&\simeq \sum_\nu \exp(-S[\mathcal{A}^{[\nu]}]) \int D\varphi \exp\left(-\int \left.\frac{\delta S}{\delta A}\right|_{\mathcal{A}^{[\nu]}} \varphi - \frac{1}{2}\int \varphi \left.\frac{\delta^2 S}{\delta A^2}\right|_{\mathcal{A}^{[\nu]}} \varphi\right) + \cdots \\
&\quad S[\mathcal{A}^{[\nu]}] \simeq (N_+ + N_-)\frac{8\pi^2}{g^2}, \qquad \left.\frac{\delta S}{\delta A}\right|_{\mathcal{A}^{[\nu]}} \simeq 0.
\end{aligned}$$

$$Z(\theta=0) \simeq \sum_{\nu} \exp(-S[\mathcal{A}^{[\nu]}]) \, \mathrm{Det}\left(\left. \frac{\delta^2 S}{\delta A^2} \right|_{\mathcal{A}^{[\nu]}} \right)^{-\frac{1}{2}} + \cdots$$

Since single (anti-)instantons localized in space-time, expression can be factorized into one-instanton contributions.

One-instanton amplitude: ['t Hooft, '76]

$$\begin{aligned} Z_1 &= \exp(-\frac{8\pi^2}{g^2}) \, \mathrm{Det}\left(\left. \frac{\delta^2 S}{\delta A^2} \right|_{\mathcal{A}^{(\pm)}} \right)^{-\frac{1}{2}} \\ &= Z_0 \cdot \int [dR] \int_V d^4 z \int \frac{d\rho}{\rho} \, d(\rho), \\ & \quad d(\rho) = C \rho^{-4} \left(\frac{8\pi^2}{g^2} \right)^{2N_c} \exp\left(-\frac{8\pi^2}{\bar{g}(\rho)^2} \right), \\ & \quad -\frac{8\pi^2}{\bar{g}(\rho)^2} = -\frac{8\pi^2}{g^2} + b \, \ln M\rho = b \, \ln \rho\Lambda, \quad b = 11N_c/3. \end{aligned}$$

Final result: Partition function of an interacting instanton gas or liquid:

$$\frac{Z}{Z_0} = \sum_N \frac{1}{N!} \prod_{l=1}^{N} \sum_{\sigma_l=\pm 1} \int [dR_l] \int_V d^4 z_l \int \frac{d\rho_l}{\rho_l} \; d(\rho_l) \cdot \exp(-\sum_{m,n} V(m,n)).$$

"Interaction potentials" $V(m,n)$ contain all non-factorization corrections.

Problem: ρ-integration infrared divergent.
Wayout: repulsive interactions at small (anti-)instanton distances.

$$d(\rho) \to d_{eff}(\rho) = d(\rho) \exp(-a \frac{\rho^2}{<\rho>^2})$$

[Ilgenfritz, M.-P., '81; Münster, '81; Shuryak, '82; Diakonov, Petrov, '84]

To be used for computing gluonic vacuum expectation values like:

– gluon condensates like $< \mathrm{tr}\, G_{\mu\nu} G_{\mu\nu} >$ $\Rightarrow$ $\surd$
– topological susceptibility $\chi_t = (1/V)\langle Q_t^2 \rangle$ $\Rightarrow$ $\surd$
– glueball correlator $\Rightarrow$ $\surd$
– contribution to Wilson loops, i.e. potential between static $Q\bar{Q}$-pair
$\Rightarrow$ no confinement for uncorrelated dilute instanton gas.

as well as for fermionic observables: quark condensate and hadronic correlators.

First resumé and further problems:

- Useful phenomenological approach for non-perturbative quantities in pure Yang-Mills. $\Longleftrightarrow$ Confinement hard to explain.
- What about fermions: chiral symmetry breaking and $U_A(1)$ problem?

  See reviews by Schäfer, Shuryak, '98; Dyakonov, '03;...
- Instantons found in lattice YM theory by minimizing lattice gauge action with various methods like "cooling", "smoothing",...

  Teper, '86; Ilgenfritz, Laursen, M.-P., Schierholz, Schiller, '86; Polikarpov, Veselov, '88; ...
- Relation to models of confinement as proven on the lattice: monopole and vortex condensation?
- Are BPST instantons really the dominant semiclassical building blocks?
- Can the semiclassical approach be improved?

  Recent attempts:

  – interacting instanton (-meron) liquid model [Lenz, Negele, Thies, '03-'04]

  – instantons at $T > 0$ - "calorons" with non-trivial holonomy

  [Kraan, van Baal, '98; Lee, Lu, '98; Ilgenfritz, Martemyanov, MP, Shcheredin, Veselov, '03; Ilgenfritz, MP, Peschka, '05; Gerhold, Ilgenfritz, MP, '06]

  – pseudoparticle approximation of path integral [M. Wagner,.. '06-'08]

4. Topology of gauge fields and fermions

[See e.g. book R. Rajaraman, Solitons and Instantons, review by A. Smilga, arXiv:0010049 (2000)].

Full QCD Lagrangian:

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2}\int d^4x \ \mathrm{tr}\,(G_{\mu\nu}G_{\mu\nu}) + \sum_{f=1}^{N_f} \bar{\psi}_f(i\gamma^\mu \mathcal{D}_\mu - m_f)\psi_f \ ,$$

invariant under flavour transformations

$$\begin{aligned} \delta\psi_f &= i\alpha_A[t^A\psi]_f, \quad \text{if all } m_f = m \ \text{ identical,} \\ \delta\psi_f &= i\beta_A\gamma^5[t^A\psi]_f, \quad \text{for } m_f \to 0 \end{aligned}$$

t^A $(A = 0, 1, \ldots, N_f^2 - 1)$ generators of the $U(N_f)$ flavour group.

Noether currents:

$$\begin{aligned} (j^\mu)^A &= \bar{\psi}t^A\gamma^\mu\psi, \\ (j^{\mu 5})^A &= \bar{\psi}t^A\gamma^\mu\gamma^5\psi\,. \end{aligned}$$

Singlet axial anomaly ($T^A \equiv 1$): quantum amplitude not invariant under

$$\delta\psi = i\alpha\gamma^5\psi, \qquad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5 \ ,$$

Axial anomaly [Adler, '69; Bell, Jackiw, '69; Bardeen, '74]

$$\partial_\mu j^{\mu 5}(x) = D(x) + 2N_f \; \rho_t(x)$$

$$\text{with} \quad j^{\mu 5}(x) = \sum_f^{N_f} \bar{\psi}_f(x)\gamma^\mu\gamma^5\psi_f(x)$$

$$D(x) = 2im \sum_{f=1}^{N_f} \bar{\psi}_f(x)\gamma^5\psi_f(x)$$

– Related to triangle diagram for process $\pi_0 \to \gamma\gamma$.
– Occurs from proper regularization of the divergent fermionic determinant in the path integral [Fujikawa, '79].

$\rho_t \neq 0$ due to non-trivial topology $\Longrightarrow$ solution of the $U_A(1)$ problem:

η'-meson (pseudoscalar singlet) for $m \to 0$ not a Goldstone boson, $m_{\eta'} \gg m_\pi$.

Related Ward identity in full QCD:

$$\begin{aligned} 4N_f^2 \int d^4x \ \langle \rho_t(x)\rho_t(0)\rangle &= 2iN_f\langle -2m\bar{\psi}_f\psi_f\rangle + \int d^4x \ \langle D(x)D(0)\rangle \\ &= 2iN_f m_\pi^2 F_\pi^2 + O(m^2) \\ \chi_t \equiv \frac{1}{V}\langle Q_t^2\rangle\Big|_{N_f} &= \frac{i}{2N_f} m_\pi^2 F_\pi^2 + O(m_\pi^4) \end{aligned}$$

$\Longrightarrow$ vanishes in the chiral limit.

However, using $1/N_c$-expansion, i.e. fermion loops suppressed ("quenched approximation") one gets [Witten, '79, Veneziano '79]

$$\chi_t^q = \frac{1}{V}\langle Q_t^2\rangle\Big|_{N_f=0} = \frac{1}{2N_f} F_\pi^2 \ [m_{\eta'}^2 + m_\eta^2 - 2m_K^2] \simeq (180\text{MeV})^4.$$

Discuss simplified case $N_f = 1$, Euclidean space.

$$\partial_\mu j_{\mu 5}(x) = -2m\bar{\psi}\gamma_5\psi - 2iQ_t(x)$$

Integrating axial anomaly over Euclidean space, taking fermionic path integral average: Atiyah-Singer index theorem

$$Q_t[A] = n_+ - n_-,$$

$n_\pm$ number of zero modes of the Dirac operator

$$(i\gamma^\mu \mathcal{D}_\mu[A])f_r(x) = \lambda_r f_r(x), \quad \text{with} \quad \lambda_r = 0, \quad \text{chirality} \quad \gamma_5 f_r = \pm f_r\,.$$

$\Rightarrow$ Alternative definition of Q_t.
$\Rightarrow$ Important for lattice computations, when employing a chiral $i\gamma^\mu\mathcal{D}_\mu$.

Zero mode in one-instanton background:

$$f_0(x-z,\rho) = \frac{\rho}{(\rho^2 + (x-z)^2)^{3/2}} u_0, \quad \text{with} \quad u_0 \quad \text{fixed spinor.}$$

Consequence:
Transition amplitude including massless dynamical fermions $\sim \mathrm{Det}(i\gamma^\mu\mathcal{D}_\mu)$

$$\langle n+1|\exp(-\hat{H}\tau)|n\rangle = 0$$

More general:

$$\langle n'|\exp(-\hat{H}\tau)|n\rangle = 0 \quad \text{for} \quad n' \neq n.$$

$$\begin{aligned}
\langle\theta'|\exp(-\hat{H}\tau)|\theta\rangle &= \sum_{n,n'}\langle n'|\exp(-\hat{H}\tau)|n\rangle\; e^{i(n\theta-n'\theta')} \\
&= \sum_{n}\langle n|\exp(-\hat{H}\tau)|n\rangle\; e^{in(\theta-\theta')}
\end{aligned}$$

Since $[\hat{T},\hat{H}]=0$ and $\hat{T}\;|n\rangle = |n+1\rangle$, $\quad\langle n|\exp(-\hat{H}\tau)|n\rangle$ independent of n.

Hence,

$$\begin{aligned}
\langle\theta'|\exp(-\hat{H}\tau)|\theta\rangle &= e^{-E_o\tau}\sum_{n} e^{in(\theta-\theta')} \\
&= 2\pi\delta(\theta-\theta')e^{-E_o\tau}
\end{aligned}$$

Allows for path integral representation including fermions, summing over all Q_t-sectors with $\langle Q_t\rangle = 0$. Semiclassically approximated by superpositions (of equal number) of instantons (I) and anti-instantons ($\bar{I}$).

Notice: $I\bar{I}$-pairs also responsible for $<\bar{\psi}\psi>\neq 0$.

5. Abelian monopoles and center vortices

(A) Abelian monopoles:

Conjecture: QCD as a dual superconductor.
Confinement in QCD is due to condensation of monopoles, leading to a dual Meissner effect. ['t Hooft '75, Mandelstam '76]

Main ingredience: Abelian projection

Assume
A_μ^a – $SU(2)$ gauge field,
Φ^a – Higgs field in adjoint representation of $SU(2)$.

Georgi-Glashow model:

$$L = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{1}{2} D_\mu \vec{\Phi} \cdot D_\mu \vec{\Phi} - V(|\vec{\Phi}|)$$

(Gauge invariant) Abelian projection:

$$F_{\mu\nu} \equiv \Phi^a G^a_{\mu\nu} = \text{ e.-m., } U(1) \text{ gauge field.}$$

Monopole solutions exist (topological objects - "3d instantons"), sources of magnetic flux localized at zeros of $\Phi^a(x)$ ('t Hooft-Polyakov monop.)

Yang-Mills theory on the lattice: $A_\mu(x_n) \to U_{n,\mu} \in SU(2)$

No Higgs available, but may diagonalize any operator transforming as $\vec{\Phi}$:
e.g. Polyakov loop, some plaquette loop etc.

Alternative: maximally Abelian gauge (MAG)

[Kronfeld, Laursen, Schierholz, Wiese, '87]

$$\sum_\mu (\partial_\mu \mp i A_\mu^3) A_\mu^\pm = 0, \quad A_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \pm i A_\mu^2)$$

On the lattice suppress non-diagonal $SU(2)$ components:

$$\begin{aligned}
F_U[\Omega] &= \sum_{n,\mu} \left\{ 1 - \frac{1}{2} \mathrm{tr} \left(\sigma_3 U_{n\mu}^{(\Omega)} \sigma_3 U_{n\mu}^{(\Omega)\dagger} \right) \right\} = \text{Min.}, \\
&= \sum_{n,\mu} \left\{ 1 - \frac{1}{2} \mathrm{tr} \left(\Phi_n U_{n\mu} \Phi_{n+\hat\mu} U_{n\mu}^\dagger \right) \right\}, \quad \Phi_n \equiv \Omega_n^\dagger \sigma_3 \Omega_n = \Phi_n^a \sigma_a, \quad ||\Phi_n|| = 1, \\
&= \sum_{n,\mu} \left\{ 1 - \Phi_n^a R_{n\mu}^{ab}(U) \Phi_{n+\hat\mu}^b \right\}, \quad R_{n\mu}^{ab} = \frac{1}{2} \mathrm{tr}\, (\sigma_a U_{n\mu} \sigma_b U_{n\mu}^\dagger), \\
&= \frac{1}{2} \sum_{na;mb} \Phi_n^a \{ -\Box_{nm}^{ab}(U) \} \Phi_m^b \equiv \hat F_U[\Phi], \quad \Box_{nm}^{ab}(U) = \text{lattice Laplacian.}
\end{aligned}$$

Problem with Gribov copies. Careful gauge fixing required (sim. annealing).

Modification: Laplacian Abelian gauge (LAG)

[Vink, Wiese; van der Sijs]

Relax the normalization condition: $||\Phi_n|| = 1$,
minimize $\hat{F}_U[\Phi]$ by finding lowest lying eigenmode of the Laplacian (e.g. with CG method).

'tHooft-Polyakov-like monopole excitations expected at zeros $\Phi_n \simeq 0$.

Finally rotate locally $U_{n\mu}$ and Φ_n such that $\Phi_n = \Phi_n^a \sigma_a \to \phi_n \sigma_3$.

Having fixed the gauge, Abelian projection = coset decomposition:

$$U_{n\mu} = C_{n\mu} \cdot V_{n\mu}, \qquad V_{n\mu} = \exp(i\theta_{n\mu}\sigma_3),$$

$C_{n\mu}$ representing charged components w. r. to residual $U(1)$.

Compute observables with Abelian fields $V_{n\mu}$ or $\theta_{n\mu}$,
e.g. Wilson loops to check confinement force (so-called Abelian dominance).

Check dual superconductor scenario by studying magnetic monopole currents:

[DeGrand, Toussaint, '80]

$$a^2 F_{\mu\nu} \quad \equiv \quad \theta_{n\mu\nu} = \theta_{n\mu} + \theta_{n+\hat{\mu},\nu} - \theta_{n+\hat{\nu},\mu} - \theta_{n\nu}.$$

Gauge invariant flux through plaquette P:

$$\overline{\theta}_P \equiv \overline{\theta}_{n,\mu\nu} = \theta_{n,\mu\nu} - 2\pi M_{n,\mu\nu}, \quad M \in \mathbf{Z}$$

such that $-\pi \leq \overline{\theta}_{n,\mu\nu} < \pi$.

Then magnetic charge of 3-cube C:

$$m_c = \frac{1}{2\pi} \sum_{P \in \partial c} \overline{\theta}_P = 0, \pm 1, \pm 2.$$

Monopole current along dual links:

$$K_{n\mu} = \frac{1}{4\pi} \epsilon_{\mu\nu\rho\sigma} \partial_\nu \overline{\theta}_{n,\rho\sigma}, \quad \sum_\mu \partial_\mu K_{n\mu} = 0.$$

Conservation law for dual currents $K_{n\mu}$ leads to closed monopole loops on the 4d dual lattice.

Consequences:

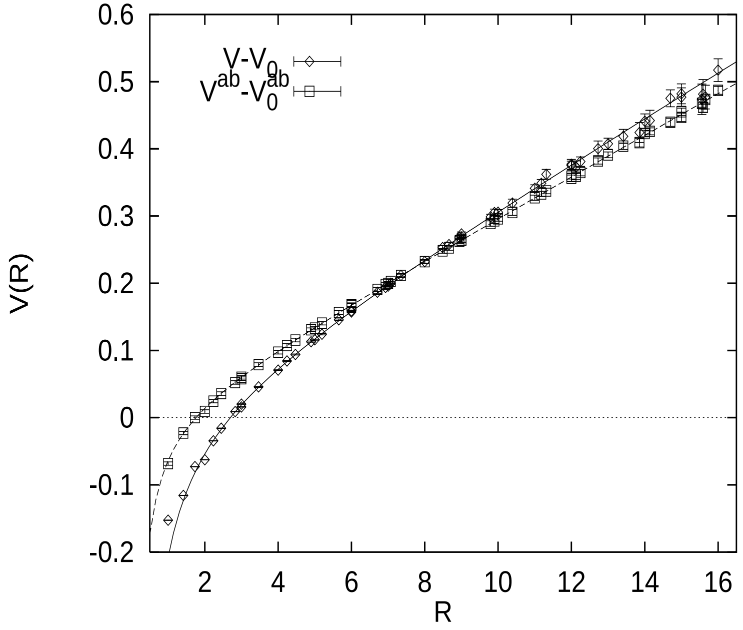
[Suzuki, Yotsuyanagi, '90; Bali, Bornyakov, M.-P., Schilling, '96; Chernodub, Polikarpov, Veselov,...; Del Debbio, DiGiacomo, Paffuti,...; ...]

- Abelian dominance $\langle O[U_{n\mu}]\rangle \simeq \langle O[V_{n\mu}]\rangle$, not surprising at least without prior gauge fixing (MAG or LAG).
- Monopole dominance, i.e. string tension reproduced from monopole contributions alone.
- Monopole condensation for $T < T_c$ from monopole creation operator with $< \mu_{mon} > \neq 0$
  [DiGiacomo, Lucini, Montesi, Paffuti, '00].
- Deconfinement transition at T_c can be viewed as bond percolation of monopole clusters. [Bornyakov,Mitrjushkin,M.-P., '92]

Abelian static potential $V(R)$

from full $SU(2)$ Wilson loop (V) and Abelian Wilson loop (V^{ab})

$\Longrightarrow$ string tension: $\sigma_{SU(2)} \simeq 0.94\, \sigma_{abelian}$.



[Bali, Bornyakov, M.-P., Schilling, '96]

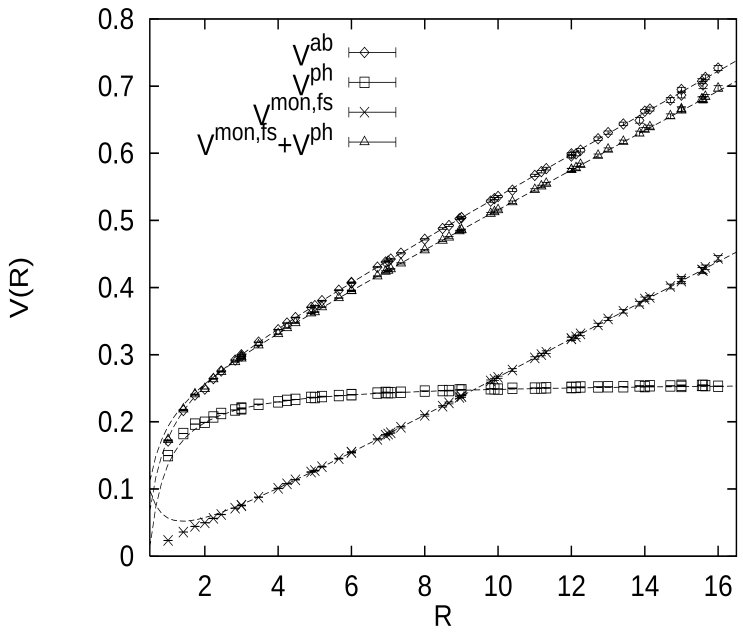
Splitting the potential into monopole and photon contributions:

$$u^{mon}_{n\mu} = \exp[i\theta^{mon}_{n\mu}], \qquad \theta^{mon}_{n\mu} = -\sum_m D(n,m)\ \partial'_\nu M_{m,\mu\nu}.$$

$D(n,m)$ – lattice Coulomb propagator, ∂' – means backward derivative.

Define photon contributions from:

$$u^{ph}_{n\mu} = \exp[i\theta^{ph}_{n\mu}], \qquad \theta^{ph}_{n\mu} = \theta_{n\mu} - \theta^{mon}_{n\mu}.$$



(B) Center vortices:

J. Greensite's criticism (see [Greensite, arXiv:hep-lat/0301023, '03]):

Abelian and monopole potentials from different group representations do not show so-called Casimir scaling at intermediate distances.

Better does another model: Center vortices.

Center vortex for $SU(2)$ in 4d:

Assume link variables taking values as center elements $z \in Z(2) \subset SU(2)$:

$$U_{n\mu} = z_{n\mu} = \pm \mathbf{1}_2 .$$

Vortex = plaquette with $\frac{1}{2} \mathrm{tr}\, U_P = -1.$

Build up closed vortex sheets (or end at world lines of center-monopoles).

Modelling Confinement: percolating vortex sheets provide area law of the Wilson loop.

Direct maximal center gauge (DMCG):

Find the gauge, which fits link variables $\{U_{n\mu}\}$ at best by

$$u_{n\mu} = \Omega_n z_{n\mu} \Omega^\dagger_{n+\hat{\mu}} .$$

Sufficient to maximize first

$$R_U[\Omega] = \sum_{n,\mu} \text{tr}_A \{\Omega_n^\dagger U_{n\mu} \Omega_{n+\hat{\mu}}\}$$

($\text{tr}_A O \equiv (\text{tr}_F O)^2 - 1$ = trace in adjoint representation).

Second minimize for fixed $\Omega_{n\mu}$ w.r. to $z_{n\mu}$:

$$\sum_{n,\mu} \text{tr}_F \left[(U_{n\mu} - \Omega_n z_{n\mu} \Omega_{n+\hat{\mu}}^\dagger) \times (h.c.) \right]$$

putting $z_{n\mu} = \text{sign tr}[\Omega_n^\dagger U_{n\mu} \Omega_{n+\hat{\mu}}]$

Other gauge fixing prescriptions have been tested (Laplacian center gauge, indirect center gauges,...).

$\Longrightarrow$ String tension σ_F similar as for Abelian monopoles.

However, center-valued field contains less information than Abelian one.

Question: How instanton models are related to monopole and vortex models?

6. Instantons at $T > 0$: calorons

Partition function

$$Z_{\mathrm{YM}}(T,V) \equiv \mathrm{Tr}\, e^{-\frac{\hat{H}}{T}} \propto \int DA\, e^{-S_{\mathrm{YM}}[A]} \;\; \text{with } A(\vec{x}, x_4{+}b) = A(\vec{x}, x_4),\;\; b = 1/T\,.$$

Old semiclassical treatment with Harrington-Shepard (HS) caloron solutions

$\equiv$ x_4-periodic instanton chains   Gross, Pisarski, Yaffe, '81

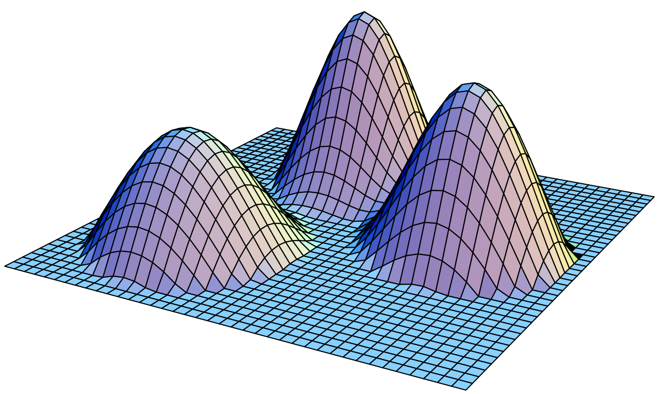
$$A^{\mathrm{HS}}_{a\mu} = \bar{\eta}_{a\mu\nu}\partial_\nu \log(\Phi(x))$$

$$\begin{aligned} \Phi(x) &= 1 + \sum_{k\in\mathbf{Z}} \frac{\rho^2}{(\vec{x}-\vec{z})^2 + (x_4 - z_4 - kb)^2} \\ &= 1 + \frac{\pi\rho^2}{b|\vec{x}-\vec{z}|}\, \frac{\sinh\left(\frac{2\pi}{b}|\vec{x}-\vec{z}|\right)}{\cosh\left(\frac{2\pi}{b}|\vec{x}-\vec{z}|\right) - \cos\left(\frac{2\pi}{b}(x_4 - z_4)\right)} \end{aligned}$$

Kraan - van Baal - Lee - Lu solutions (KvBLL)
= (multi-) calorons with non-trivial asymptotic holonomy

$$P(\vec{x}) = \mathbf{P} \exp \left(i \int_0^{b=1/T} A_4(\vec{x},t)\ dt \right) \overset{|\vec{x}|\to\infty}{\Longrightarrow} \quad \mathcal{P}_\infty \ = e^{2\pi i \omega \tau_3} \notin \ \mathbf{Z}$$

Kraan, van Baal, '98 - '99, Lee, Lu '98



Action density of an $SU(3)$ caloron (van Baal, '99)

$\Longrightarrow$ not a simple $SU(2)$ embedding into $SU(3)$!!

Calorons with non-trivial holonomy

K. Lee, Lu, '98, Kraan, van Baal, '98 - '99, Garcia-Perez et al. '99

- x_4-periodic, (anti)selfdual solutions from ADHM formalism,
- generalize Harrington-Shepard calorons (i.e. x_4 periodic BPST instantons).

For $SU(2)$: holonomy parameter $\bar{\omega} = 1/2 - \omega$, $0 \leq \omega \leq 1/2$.

$$
\begin{aligned}
A_\mu^C &= \frac{1}{2}\bar{\eta}^3_{\mu\nu}\tau_3\partial_\nu \log\phi + \frac{1}{2}\,\phi\;\mathrm{Re}\left((\bar{\eta}^1_{\mu\nu} - i\bar{\eta}^2_{\mu\nu})(\tau_1 + i\tau_2)(\partial_\nu + 4\pi i\omega\delta_{\nu,4})\tilde{\chi}\right) \\
&\qquad + \delta_{\mu,4}\; 2\pi\omega\tau_3\,, \\
\phi(x) &= \frac{\psi(x)}{\hat{\psi}(x)}\,, \qquad x = (\vec{x}, x_4 \equiv t)\,, \qquad r = |\vec{x} - \vec{x}_1|, \quad s = |\vec{x} - \vec{x}_2|\,, \\
\psi(x) &= -\cos(2\pi t) + \cosh(4\pi r\bar{\omega})\cosh(4\pi s\omega) + \frac{r^2 + s^2 + \pi^2\rho^4}{2rs}\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega) \\
&\quad + \frac{\pi\rho^2}{s}\sinh(4\pi s\omega)\cosh(4\pi r\bar{\omega}) + \frac{\pi\rho^2}{r}\sinh(4\pi r\bar{\omega})\cosh(4\pi s\omega), \\
\hat{\psi}(x) &= -\cos(2\pi t) + \cosh(4\pi r\bar{\omega})\cosh(4\pi s\omega) + \frac{r^2 + s^2 - \pi^2\rho^4}{2rs}\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega)\,, \\
\tilde{\chi}(x) &= \frac{1}{\psi}\left\{e^{-2\pi i t}\frac{\pi\rho^2}{s}\sinh(4\pi s\omega) + \frac{\pi\rho^2}{r}\sinh(4\pi r\bar{\omega})\right\}.
\end{aligned}
$$

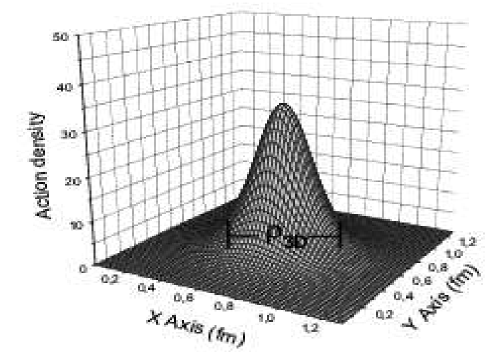
Properties:

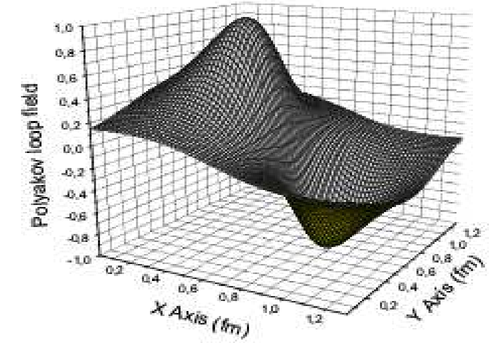
- periodicity with $b = 1/T$,
- (anti)selfdual with topological charge $Q_t = \pm 1$,
- has two centers at $\vec{x}_1, \vec{x}_2 \rightarrow$ "instanton quarks",
- scale-size versus distance: $\pi\rho^2 T = |\vec{x}_1 - \vec{x}_2| = d$,
- limiting cases:
  - $\omega \rightarrow 0 \implies$ 'old' HS caloron,
  - $|\vec{x}_1 - \vec{x}_2|$ large $\implies$ two static BPS monopoles or dyons (DD) with mass ratio $\sim \bar{\omega}/\omega$,
  - $|\vec{x}_1 - \vec{x}_2|$ small $\implies$ non-static single caloron (CAL).
- $L(\vec{x}) = \frac{1}{2} \mathrm{tr} P(\vec{x}) \rightarrow \pm 1$ close to $\vec{x} \simeq \vec{x}_{1,2} \implies$ "dipole structure"

KvBLL $SU(2)$ caloron:

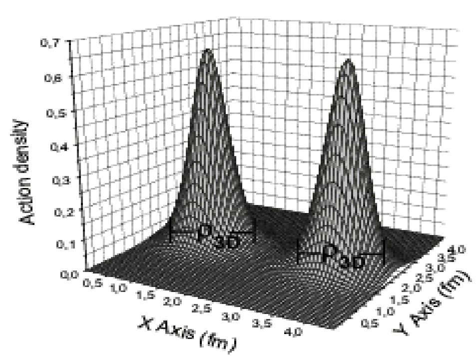
Action density

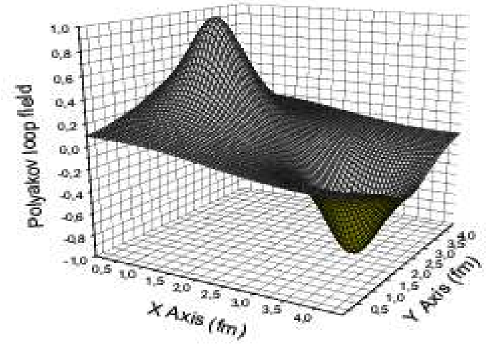
Polyakov loop





CAL





DD

- Localization of the zero-mode of the Dirac operator:

- time-antiperiodic b.c.:
  around the center with $L(\vec{x}_1) = -1$,

$$|\psi^-(x)|^2 = -\frac{1}{4\pi}\partial_\mu^2 \left[\tanh(2\pi r\bar{\omega})/r\right] \quad \text{for large} \quad d,$$

- time-periodic b.c.:
  around the center with $L(\vec{x}_2) = +1$,

$$|\psi^+(x)|^2 = -\frac{1}{4\pi}\partial_\mu^2 \left[\tanh(2\pi s\omega)/s\right] \quad \text{for large} \quad d.$$

- $SU(N_c)$ KvBLL calorons

- - consist of N_c monopole constituents becoming well-separated static BPS monopoles (dyons) in the limit of large distances or scale sizes,
  - resemble single-localized HS calorons (BPST instantons) at small distances, but are genuine $SU(N_c)$ objects - not embedded $SU(2)$.

- Eigenvalues of the (asymptotic) holonomy

$$\mathcal{P}_\infty = g \ \exp(2\pi i \, \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N)) \ g^\dagger$$

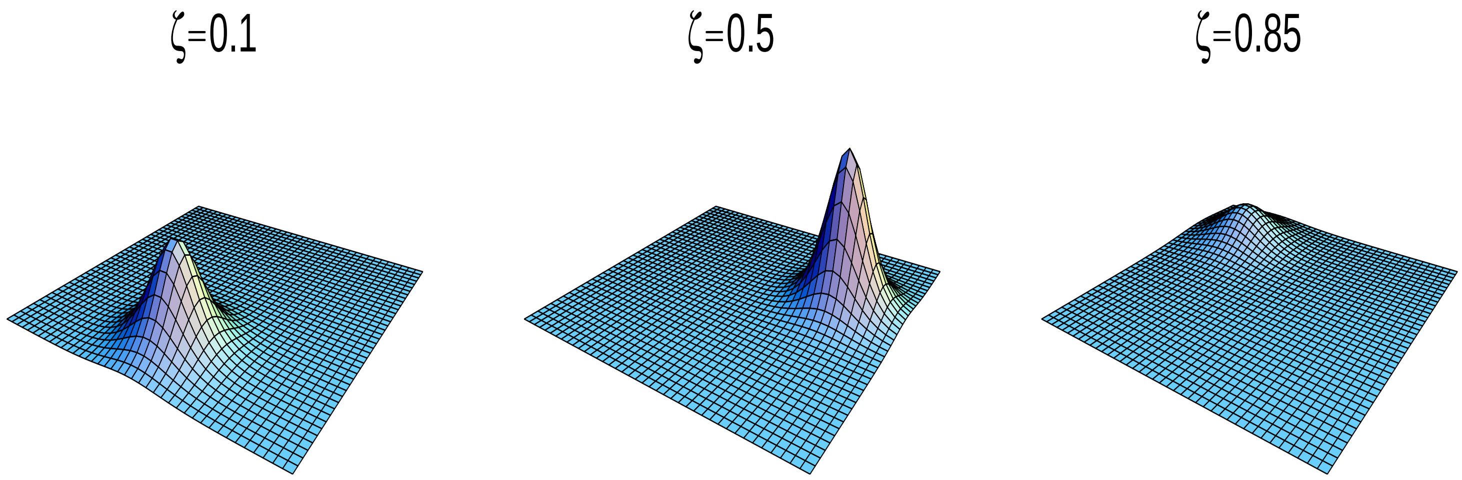
  with ordering $\mu_1 < \mu_2 < \cdots < \mu_{N+1} \equiv 1 + \mu_1$, $\mu_1 + \mu_2 + \cdots + \mu_N = 0$ determine the masses of the dyons: $M_i = 8\pi^2(\mu_{i+1} - \mu_i), \ i = 1, \cdots, N.$

- Monopole constituents are localized at positions $\vec{x}_m$, where eigenvalues of the Polyakov loop $P(\vec{x})$ degenerate.

- $SU(3)$: moving localization of the fermionic zero mode from constituent to constituent when changing the boundary condition with phase $\zeta \in [0,1]$:

$$\Psi_z(x_0 + b, \vec{x}) = e^{-2\pi i \zeta}\ \Psi_z(x_0, \vec{x})$$

(with $b = 1/T$)



Garcia Perez, et al., '99; Chernodub, Kraan, van Baal, '00

- Multi-calorons known only in very special cases

  van Baal, Bruckmann, Nogradi, '04

- Treatment of the path integral in the background of KvBLL calorons in terms of monopole constituents: free energy favours non-trivial holonomy at $T \simeq T_c$ Diakonov, '03; Diakonov, Gromov, Petrov, Slizovskiy, '04

Lattice tools for the instanton and caloron search

Gauge fields:

$$A_\mu(x_n) \Longrightarrow U_{n,\mu} \equiv P \exp i \int_{x_n}^{x_n+\hat{\mu}a} A_\mu dx_\mu \in SU(N_c)$$

Gauge action (Wilson '74):

$$S_W = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re\ Tr\ U_{x,\mu\nu}}\right) \sim a^4 \sum_{x,\mu<\nu} \mathrm{Tr\ G_{\mu\nu} G_{\mu\nu}(x)}, \quad \beta = \frac{\mathrm{2N_c}}{\mathrm{g_0^2}}$$

Path integral quantization:

$$< W >= Z^{-1} \int \prod_{n,\mu} dU_{n,\mu} W(U) \exp(-S_W(U))$$

$$Z = \int \prod_{n,\mu} dU_{n,\mu} \exp(-S_W(U))$$

Monte Carlo method: Generates ensemble of lattice fields in a Markov chain

$$\{U\}_1, \{U\}_2, \cdots, \{U\}_N$$

with resp. to probability distribution ('Importance sampling')

$$W(\{U\}) = Z^{-1} \exp(-S_W(U))\,.$$

Take x_4-periodic quantum lattice fields as "snapshots" at $T \neq 0$
in order to search for semi-classical objects

$\Longrightarrow$ calorons with non-trivial holonomy ??

- Cooling and smearing:

  Successive minimization of the (Wilson plaquette) action $S(U)$ by replacing $U_{x,\mu} \to \bar{U}_{x,\mu}$

$$\bar{U}_{x,\mu} = \mathbf{P}_{SU(N_c)} \Bigg((1-\alpha)\, U_{x,\mu} + \frac{\alpha}{6} \sum_{\nu(\neq\mu)} \left[U_{x,\nu} U_{x+\hat{\nu},\mu} U^{\dagger}_{x+\hat{\mu},\nu} + U^{\dagger}_{x-\hat{\nu},\nu} U_{x-\hat{\nu},\mu} U_{x+\hat{\mu}-\hat{\nu},\nu} \right] \Bigg)$$

  with

  - $\alpha = 1.0 \quad \to \quad$ cooling

    iteration down to action plateaus in order to search for (approximate) solutions of the classical (lattice) equations of motion $\delta S / \delta U_{x,\mu} = 0\,.$

  - $\alpha = 0.45 \quad \to \quad$ 4d APE smearing

    iteration in order to remove short-range fluctuations

    $\to$ clusters of top. charge far from being class. solutions.

- Gluonic observables
  - *action density* $\varsigma(\vec{x}) = \frac{1}{N_t}\sum_t s(\vec{x},t)$;
  - *topological density*

$$q_t(\vec{x}) = -\frac{1}{2^9\pi^2 N_t}\sum_t \left(\sum_{\mu,\nu,\rho,\sigma=\pm 1}^{\pm 4} \epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\left[U_{x,\mu\nu}U_{x,\rho\sigma}\right]\right);$$

  - *spatial Polyakov loop distribution*

$$L(\vec{x}) = \frac{1}{N_c}\mathrm{tr}\ \mathcal{P}(\vec{x}), \qquad P(\vec{x}) = \prod_{t=1}^{N_t} U_{\vec{x},t,4};$$

  in particular asymptotic holonomy

$$L_\infty = \frac{1}{N_c}\mathrm{tr}\left(\frac{1}{V_\alpha}\sum_{\vec{x}\in V_\alpha}\left[\mathcal{P}(\vec{x})\right]_{\mathrm{diagonal}}\right),$$

  where V_α region of minimal action (topological) density;
  - *Abelian magnetic fluxes and monopole charges* within MAG.
  - *Center vortices* within DMCG.

  [Bruckmann, Ilgenfritz, Martemyanov, Zhang, '10]

- Fermionic modes:

  eigenvalues and eigenmode densities of lattice Dirac operator

$$\sum_y D[U]_{x,y}\ \psi(y) = \lambda\ \psi(x)$$

(with varying x_4-boundary conditions) determined numerically by applying Arnoldi method (ARPACK code package).

Standard Wilson - badly breaking chiral invariance:

$$D_W[U]_{x,y} = \delta_{xy} - \kappa \sum_\mu \left\{ \delta_{x+\hat\mu,y}\,(\mathbf{1}-\gamma^\mu)U_{x,\mu} + \delta_{y+\hat\mu,x}\,(\mathbf{1}+\gamma^\mu)U^\dagger_{y,\mu} \right\}$$

Chiral improvement - overlap operator:

$$D_{\mathrm{ov}} = \frac{\rho}{a}\,\left(1 + D_{\mathrm{W}}/\sqrt{D_{\mathrm{W}}^\dagger\ D_{\mathrm{W}}}\right)\ , \qquad D_{\mathrm{W}} = M - \frac{\rho}{a}\ ,$$

satisfies Ginsparg-Wilson relation $\Longrightarrow$ chiral symmetry at $a \neq 0$

$$D\gamma_5 + \gamma_5 D = \frac{a}{\rho} D\gamma_5 D\ ,$$

D_{ov} guarantees index theorem $\quad Q_{\text{index}} = n_- - n_+$.

Topological charge density filtered by truncated mode expansion:

$$q_{\lambda_{\text{cut}}}(x) = -\sum_{|\lambda| \leq \lambda_{\text{cut}}} \left(1 - \frac{\lambda}{2}\right) \psi_\lambda^\dagger \gamma_5 \psi_\lambda(x) \ ,$$

Numerical evidence for equivalence of filters:

Chirally improved fermionic filter applied to equilibrium (quantum) fields reveals similar cluster structures as 4D smearing, if mode truncation is tuned to appropriate number of smearing steps:

$$\text{Small } N_{\text{smear}} \quad \Longleftrightarrow \quad \text{large } N_{\text{modes}}.$$

$\Longrightarrow$ moderate smearing of MC lattice fields seems justified.

[Bruckmann, Gattringer, Ilgenfritz, M.-P., A. Schäfer, Solbrig, '07]

Lattice filter strategies:

(A) Lowest action plateaux, i.e. extract classical solutions with various minimization or "cooling" methods:

$S \approx n\ S_0, \quad n = 1, \cdots, 6, \qquad (S_0 \equiv 8\pi^2/g^2)$

$\Longrightarrow$ KvBLL-like topological clusters seen for $SU(2)$ (and $SU(3)$)

– "dipole (triangle)" constituent structure for the Polyakov loop,

– MAG Abelian monopoles correlated with dyon constituents,

– and fermionic mode "hopping" from constituent to constituent.

[Ilgenfritz, Martemyanov, M.-P., Shcheredin,Veselov '02; Ilgenfritz,M.-P., Peschka, '05]

(B) Clusters of top. charge by 4d smearing $\quad S \approx n\ S_0, \quad n = O(30-40)$, string tension reduced but non-zero.

(C) Equilibrium lattice gauge fields:
low-lying modes of chirally improved or exact (overlap) Dirac operator in equilibrium without and in combination with smearing.

ad (B) Topological clusters from 4d smearing - SU(2) case

Ilgenfritz, Martemyanov, M.-P., Veselov, ’04 - ’05

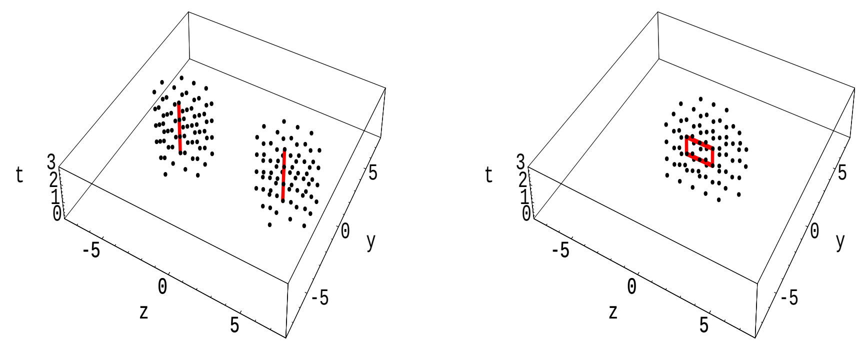
4D APE smearing:

- reduces quantum fluctuations while keeping long range physics,
- (spatial) string tension becomes slowly reduced, stop at $\sigma_{sm} \simeq 0.6\,\sigma_{full}$,
- lumps (clusters) of topological charge become visible.

We analyse top. clusters w. r. to their MAG Abelian monopole content,

select – static monopole world lines = ’distinct dyons’,

– closing monopole world lines = ’distinct calorons’.



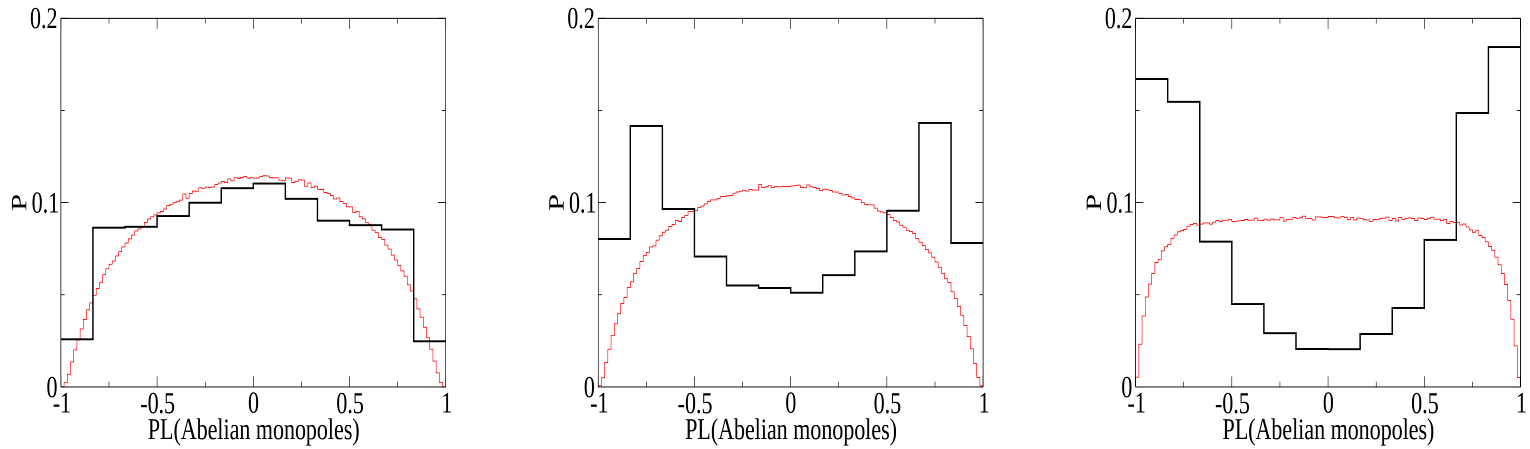
Analytic DD and CAL, both with their (MAG) Abelian monopole loops.

Estimate cluster radius from peak values of top. density $\Longrightarrow$ cluster charges.

$T < T_c$:   lattice size $24^3 \times 6$, 50 4d smearing steps

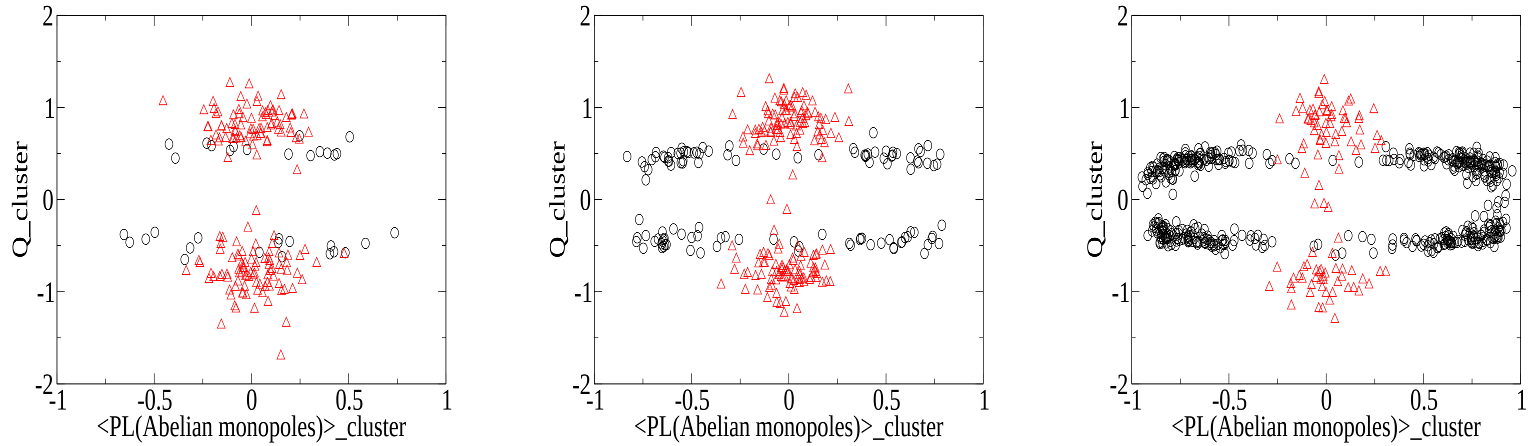
Polyakov loop distributions in lattice sites with time-like MAG Abelian monopoles.

For comparison: unbiased distribution of Polyakov loops in all sites.



$\beta = 2.2$ $\beta = 2.3$ $\beta = 2.4$

Q_{cluster} versus Pol. loop averaged over positions of time-like Abelian monopoles



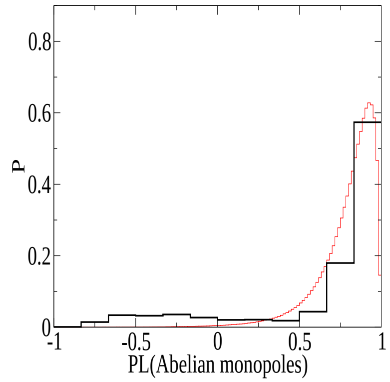
$\beta = 2.2$ $\beta = 2.3$ $\beta = 2.4$

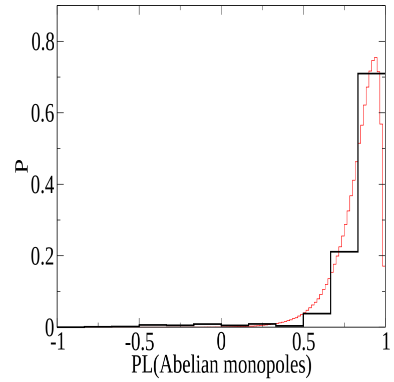
$\Rightarrow$ Topological clusters with $Q_t \simeq \pm\frac{1}{2}$ identified.

$\Rightarrow$ $N_{\mathrm{dyon}} : N_{\mathrm{caloron}}$ of identifiable single dyons and non-dissociated calorons rises with $T \to T_c$.

$T > T_c$: lattice size $24^3 \times 6$, 25 (20) smearing steps for $\beta = 2.5$ (2.6).

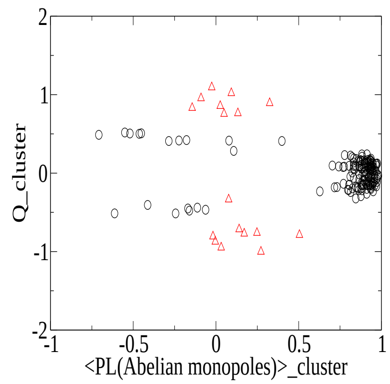
Polyakov loop distributions in lattice sites with time-like MAG Abelian monopoles.

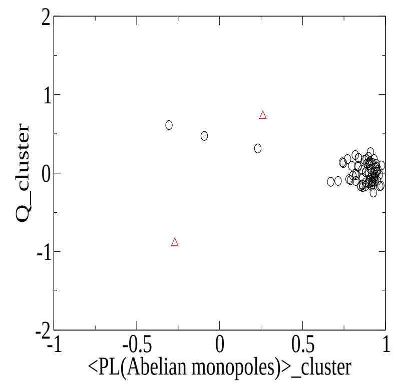




$\beta = 2.5$ $\beta = 2.6$

Q_{cluster} versus Pol. loop averaged over positions of time-like Abelian monopoles





$\Rightarrow$ dominantly light monopoles (dyons) found, calorons suppressed for $T > T_c$.

ad (C) Equilibrium fields: low-lying fermionic modes ($SU(2)$)

[Bornyakov, Ilgenfritz, Martemyanov, Morozov, M.-P., Veselov, '07; Bornyakov, Ilgenfritz, Martemyanov, M.-P., '09]

Use tadpole-improved Lüscher-Weisz action for better performance of the overlap operator.

Observables:

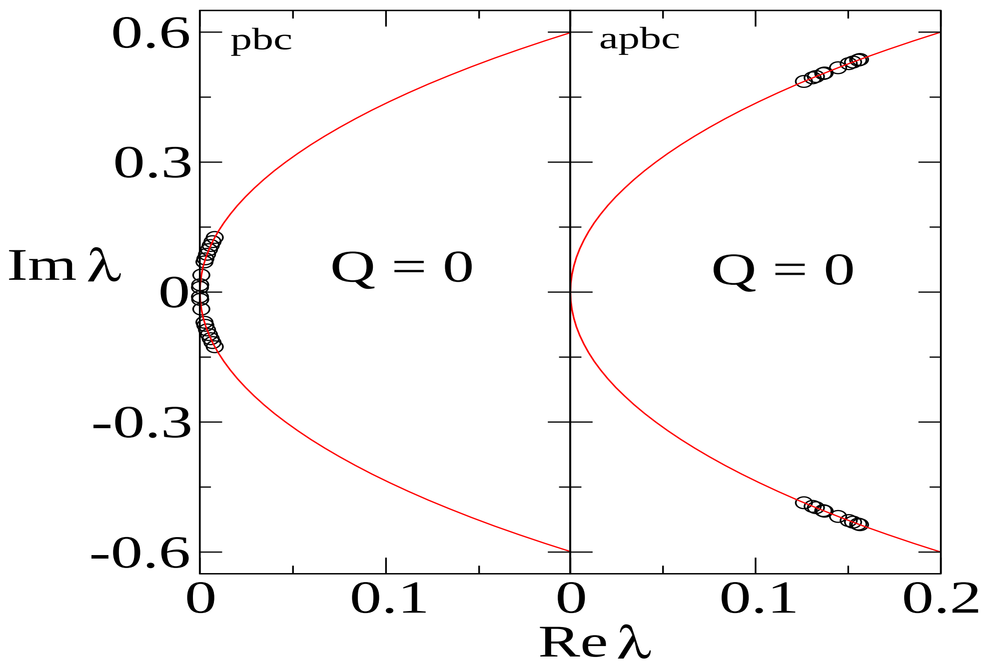
- $q_{\lambda_{cut}}(x)$ with 20 lowest-lying modes for p.b.c. and (anti-) p.b.c.,
- identify topological clusters of both sign, find $q_{\max}(cluster)$,
- Polyakov loop $P(x)$ inside top. clusters after 10 APE smearings, find $P_{\text{extr}}(cluster)$,
- identify clusters of type "CAL $\equiv$ DD" and "D"

Results support previous observations relying on top. clusters found with smearing.

Illustration at $T \simeq 1.5\ T_c$

Realized with $20^3 \times 4$, within $Z(2)$ sector with $\langle L \rangle > 0$.

$\Longrightarrow$ Overlap eigenvalues of a typical MC configuration:

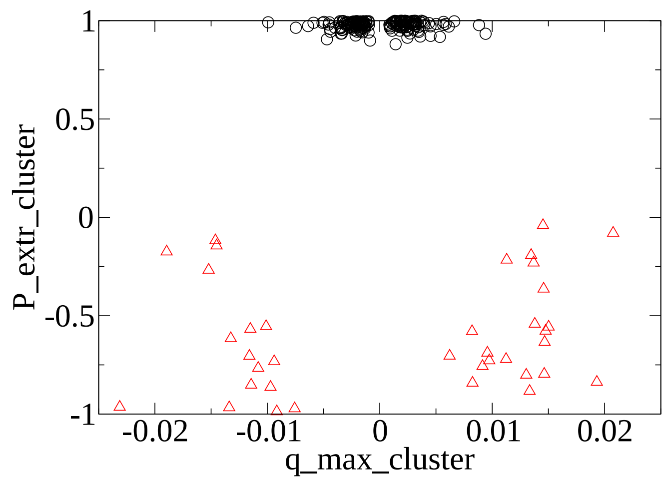


For $\langle L \rangle < 0$ Figs. for "pbc" and "apbc" would interchange!

For clusters containing static MAG Abelian monopoles show
- the extremal value of the topological charge density,
- the peak value of the local Polyakov line.

(Anti)selfduality with field strength from low-lying modes is well satisfied.

Circles $\longleftrightarrow$ clusters found with pbc (light dyons),
triangles $\longleftrightarrow$ clusters found with apbc (heavy dyons).



$\Longrightarrow$ KvBLL-like constituents again visible.

$\Longrightarrow$ But D's (not CAL's) are statistically dominant.

Simulating a caloron gas

[HU Berlin master thesis by P. Gerhold, '06; Gerhold, Ilgenfritz, M.-P., '06]

Model based on random superpositions of KvBLL calorons.

Superpositions made in the algebraic gauge – A_4-components fall off.
Gauge rotation into periodic gauge

$$A_\mu^{per}(x) = e^{-2\pi i x_4 \vec{\omega}\vec{\tau}} \cdot \sum_i A_\mu^{(i),alg}(x) \cdot e^{+2\pi i x_4 \vec{\omega}\vec{\tau}} + 2\pi\vec{\omega}\vec{\tau} \cdot \delta_{\mu,4}.$$

First important check: study the influence of the holonomy

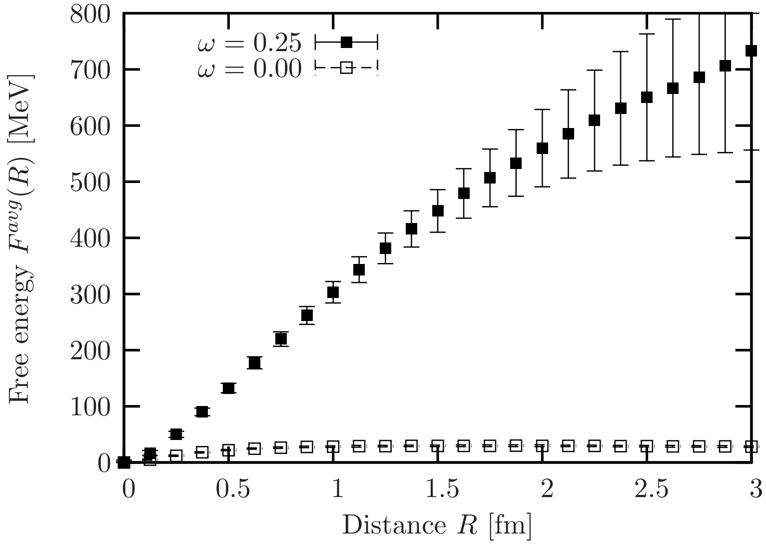
- same fixed holonomy for all (anti)calorons: $\mathcal{P}_\infty = \exp 2\pi i \omega \tau_3$
  $\omega = 0$ – trivial, $\omega = 1/4$ – maximally non-trivial,
- put equal number of calorons and anticalorons randomly but with fixed distance between monopole constituents $d = |\vec{x}_1 - \vec{x}_2| = \pi\rho^2 T$, in a 3d box with open b.c.'s,
- for measurements use a $32^3 \times 8$ lattice grid and lattice observables,
- fix parameters and lattice scale: temperature: $T = 1 \text{ fm}^{-1} \simeq T_c$,
  density: $n = 1 \text{ fm}^{-4}$, scale size: fixed $\rho = 0.33$ fm vs. distribution $D(\rho) \propto \rho^{7/3} \exp(-c\rho^2)$ such that $\bar{\rho} = 0.33$ fm.

Polyakov loop correlator $\rightarrow$ quark-antiquark free energy

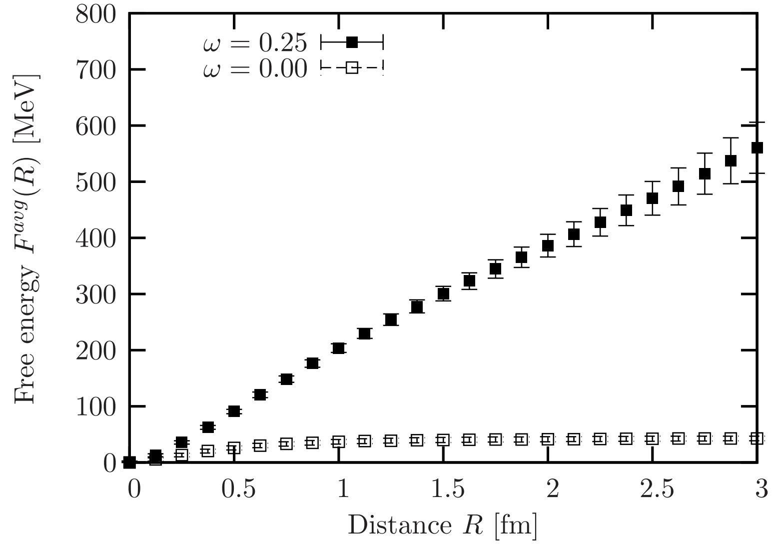
$$F(R) = -T \log\langle L(\vec{x})L(\vec{y})\rangle, \quad R = |\vec{x} - \vec{y}|$$

with trivial ($\omega = 0$) and maximally non-trivial holonomy ($\omega = 0.25$).

ρ fixed



ρ sampled with distribution



$\Longrightarrow$ Non-trivial (trivial) holonomy (de)confines

for standard instanton or caloron liquid model parameters.

Building a more realistic model for the deconfinement transition

Main ingrediences:

- Holonomy parameter: $\omega = \omega(T)$
  lattice results for the (renormalized) average Polyakov loop.
  Digal, Fortunato, Petreczky, '03; Kaczmarek, Karsch, Zantow, Petreczky, '04
  $\omega = 1/4$ for $T \leq T_c$, ω smoothly decreasing for $T > T_c$.

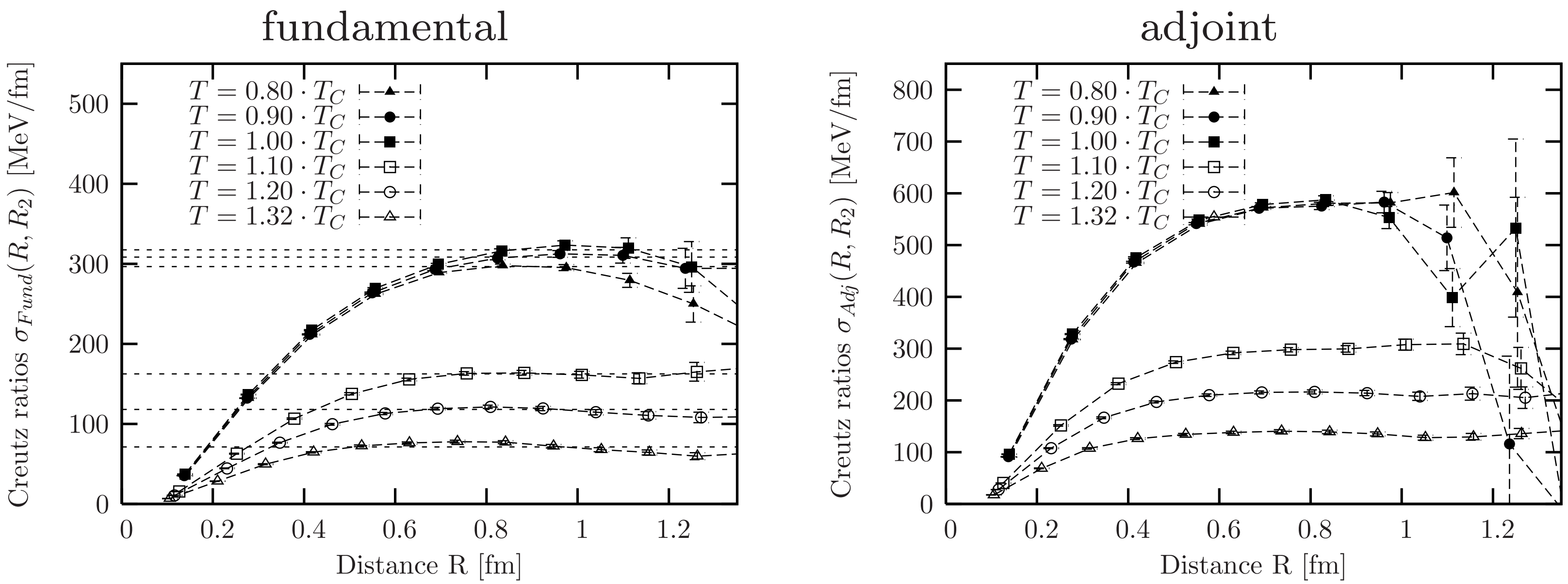
- Density parameter: $n = n(T)$ for uncorrelated caloron gas to be identified with top. susceptibility $\chi(T)$ from lattice results
  Alles, D'Elia, Di Giacomo, '97

- ρ-distribution:
  $T = 0$: Ilgenfritz, M.-P., '81; Dyakonov, Petrov, '84
  $T > 0$: Gross, Pisarski, Yaffe, '81

$T < T_c$	$D(\rho, T) = A \cdot \rho^{7/3} \cdot \exp(-c\rho^2)$	$\int D(\rho, T) d\rho = 1, \quad \bar{\rho}$ fixed
$T > T_c$	$D(\rho, T) = A \cdot \rho^{7/3} \cdot \exp(-\frac{4}{3}(\pi\rho T)^2)$	$\int D(\rho, T) d\rho = 1, \quad \bar{\rho}$ running

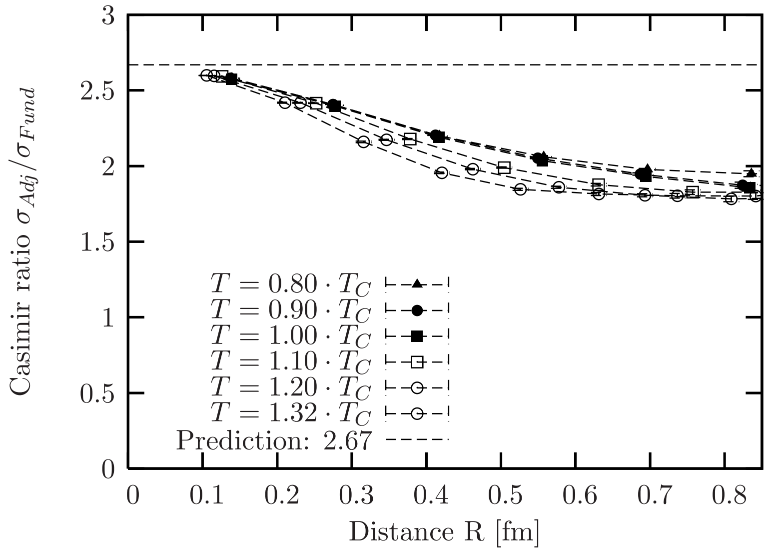
Distributions sewed together at $T_c \implies$ relates $\overline{\rho}(T = 0)$ to T_c,
then $\overline{\rho}(T = 0)$ to be fixed from known lattice space-like string tension
$T_c/\sqrt{\sigma_s(T = 0)} \simeq 0.71$: $\overline{\rho} = 0.37$ fm

Effective string tension $\sigma(R, R_2)$ from Creutz ratios of spatial Wilson loops
(with $R_2 = 2 \cdot R$) versus distance R
$T/T_c = 0.8,\ 0.9,\ 1.0$ for confined phase,
$T/T_c = 1.10,\ 1.20,\ 1.32$ for deconfined phase.

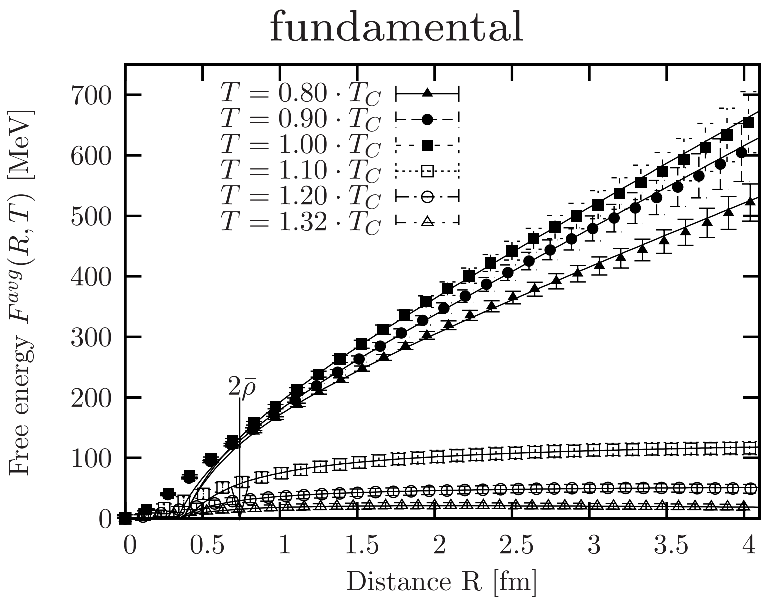


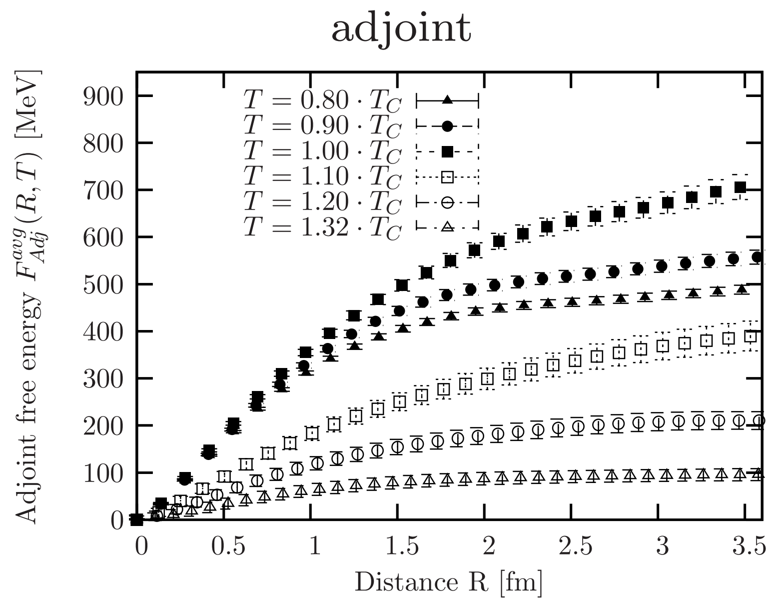
$\Longrightarrow$ Nice plateaux, but no rising $\sigma(T)$ for $T > T_c$.

Test of Casimir scaling for ratio $\sigma_{Adj}/\sigma_{Fund}$ at various T:



Color averaged free energy versus distance R at different temperatures from Polyakov loop correlators.



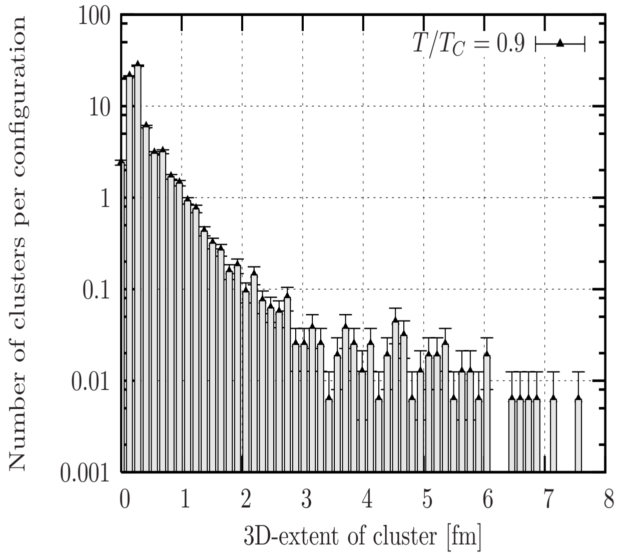


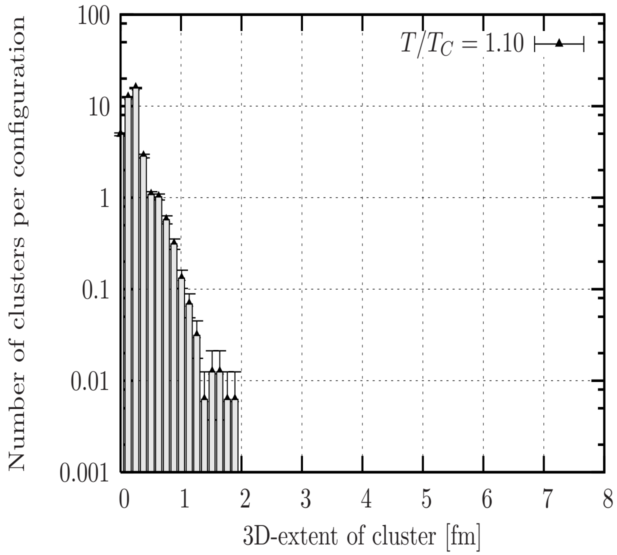
$\Longrightarrow$ successful description of the deconfinement transition,

$\Longrightarrow$ but still no realistic description of the deconf. phase.

Test of the magnetic monopole content in MAG:

histograms of 3-d extensions of dual link-connected monopole clusters





$\Longrightarrow$ Some percolation seen for $T < T_c$ as well as its disappearance for $T > T_c$

Summary

- Topological aspects in QCD occur naturally and have phenomenological impact. Instanton gas/liquid model remains phenomenologically important. Main qualitative achievements: chiral symmetry breaking, solution of $U_A(1)$, ...
- Drawback: no confinement. Alternative models: monopoles, vortices - explaining confinement.
- Check of models is possible with lattice methods.
- Basic quantity: $\chi_t =< Q_t^2 > /V$. To be computed on the lattice, too. Requires suitable lattice definition of Q (e.g. via overlap operator modes).
- KvBLL calorons with non-trivial holonomy have been identified by cooling, 4d smearing and with fermionic modes in the confinement phase.
- For $T \nearrow T_c$ calorons seem to dissociate more and more into well-separated monopoles.
- For $T > T_c$ (corresp. to trivial holonomy) light monopole pairs with opposite top. charge are dominating.
  $\Longrightarrow$ Requires more investigations.
- KvBLL caloron gas model very encouraging !!

Some literature for further reading

Books:

- R. Rajaraman, *Solitons and Instantons*,
- M. Shifman, *Instantons in Gauge Theories*,
- J. Greensite, *An Introduction to the Confinement Problem*.

Reviews:

- T. Schäfer, E. Shuryak, *Instantons in QCD*, arXiv:hep-ph/9610451v3,
- D. Diakonov, *Instantons at Work*, arXiv:hep-ph/0212026,
- J. Greensite, *The Confinement Problem in Lattice Gauge Theory*, arXiv:hep-lat/0301023.

Thank you for your attention !