COMPLEX ACTIONS AND THE SIGN PROBLEM: COMPLEX LANGEVIN DYNAMICS

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OUTLINE

- QCD phase diagram from the lattice ?
- sign problem at finite chemical potential

- complex Langevin dynamics: a solution ?
  - gentle introduction
  - relativistic Bose gas
  - heavy dense QCD
  - XY model
  - SU(3) spin model

- summary

ROUGH GUIDE TO THE LITERATURE

READING MATERIAL

- original suggestion: Parisi & Wu 81, Parisi, Klauder 83
- classic paper: three-dimensional SU(3) spin model at finite μ
  Karsch & Wyld PRL 85
- disasters of various degrees: Ambjørn et al NPB 86
- overview: Damgaard and Hüffel, Physics Reports 87
- renewed interest for Minkowski dynamics:
  Berges, Borsanyi, Sexty, Stamatescu 05-08

recent activity:

papers with Frank James, Erhard Seiler, Nucu Stamatescu, Kim Splittorff
from hep-lat/0807.1597 onwards

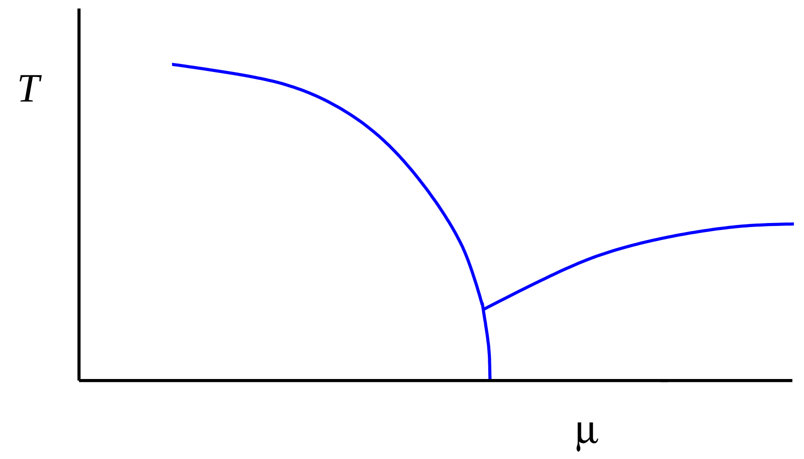
QCD PHASE DIAGRAM

NONPERTURBATIVE DETERMINATION

- QCD is confining at low temperature and chemical potential

$\Rightarrow$ nonperturbative study

lattice QCD



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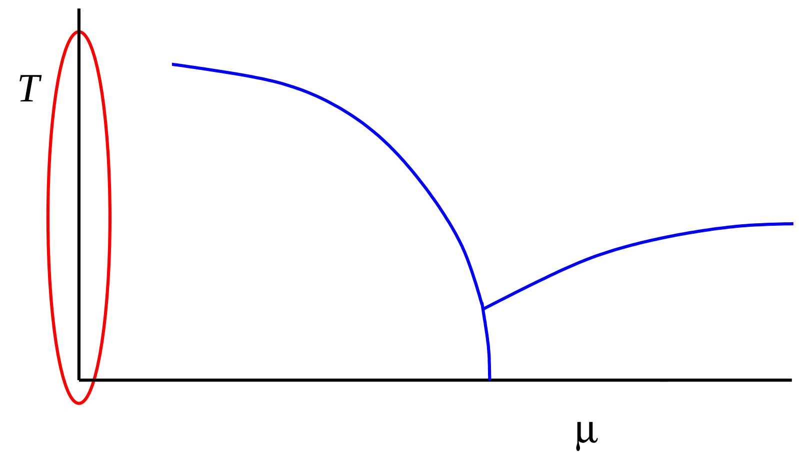

- works well at $\mu = 0$

see Edwin Laermann

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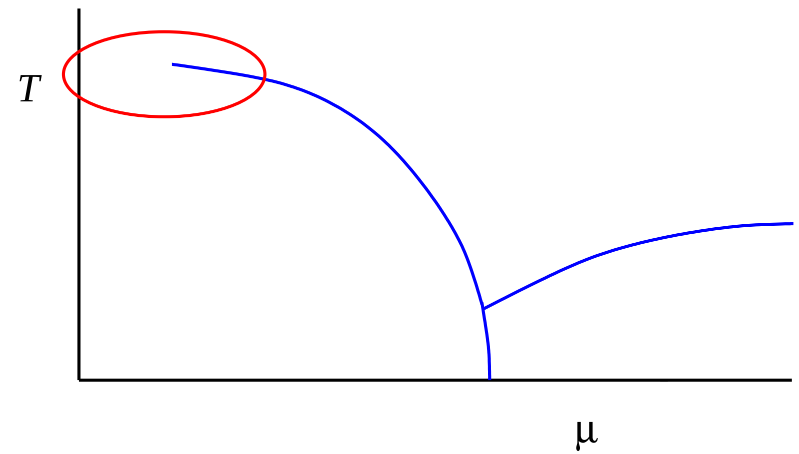

- works well at $\mu = 0$ see Edwin Laermann
- progress for $\mu \lesssim T$, $T \sim T_c$ see Christian Schmidt

QCD PHASE DIAGRAM

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- QCD is confining at low temperature and chemical potential

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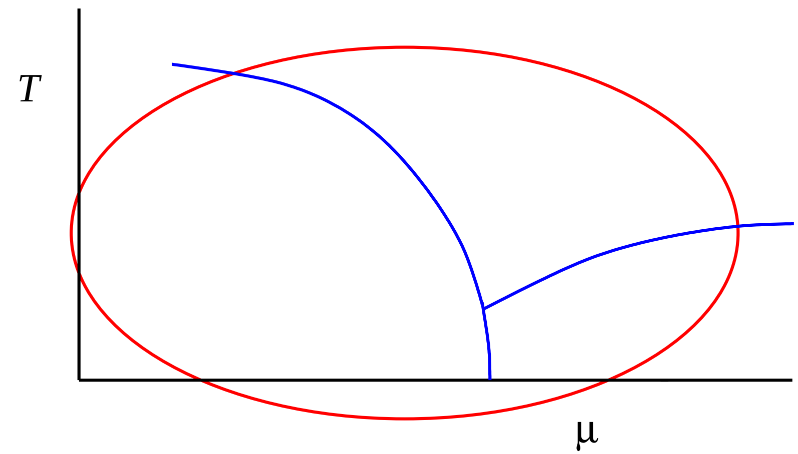

- works well at $\mu = 0$ — see Edwin Laermann
- progress for $\mu \lesssim T$, $T \sim T_c$ — see Christian Schmidt
- importance sampling breaks down at $\mu > 0$

LATTICE QCD

IMPORTANCE SAMPLING

partition function: $$Z = \int DU\, D\bar{\psi} D\psi\, e^{-S} = \int DU\, e^{-S_B} \det M$$

- if $e^{-S_B} \det M > 0$, interpret as probability weight
- evaluate using importance sampling

LATTICE QCD

IMPORTANCE SAMPLING

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- evaluate using importance sampling

QCD at finite baryon chemical potential:

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

fermion determinant is complex!

- importance sampling not possible

sign problem

- basic tool of all lattice QCD algorithms breaks down

WHY IS THE SIGN PROBLEM DIFFICULT?

PHASE QUENCHED THEORY

write $\det M = |\det M| e^{i\varphi}$

- phase quenched theory with weight $e^{-S_B} |\det M| > 0$

$$\langle O \rangle_{\rm full} = \frac{\int DU\, e^{-S_B} \det M\, O}{\int DU\, e^{-S_B} \det M}$$

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$$= \frac{\langle e^{i\varphi} O \rangle_{\text{pq}}}{\langle e^{i\varphi} \rangle_{\text{pq}}}$$

Why is the sign problem difficult?

PHASE QUENCHED THEORY

write $\det M = |\det M| e^{i\varphi}$ $\Omega =$ lattice volume

- phase quenched theory with weight $e^{-S_B}|\det M| > 0$

$$\langle O\rangle_{\rm full} = \frac{\int DU\, e^{-S_B} \det M\, O}{\int DU\, e^{-S_B} \det M} = \frac{\int DU\, e^{-S_B} |\det M|\, e^{i\varphi} O}{\int DU\, e^{-S_B} |\det M|\, e^{i\varphi}}$$

$$= \frac{\langle e^{i\varphi} O\rangle_{\rm pq}}{\langle e^{i\varphi}\rangle_{\rm pq}} \quad \to \quad \frac{0}{0} \quad \to \quad ??$$

- average phase factor

$$\langle e^{i\varphi}\rangle_{\rm pq} = \frac{\int DU\, e^{-S_B} |\det M|\, e^{i\varphi}}{\int DU\, e^{-S_B} |\det M|} = \frac{Z_{\rm full}}{Z_{\rm pq}} = e^{-\Omega\Delta f} \to 0$$

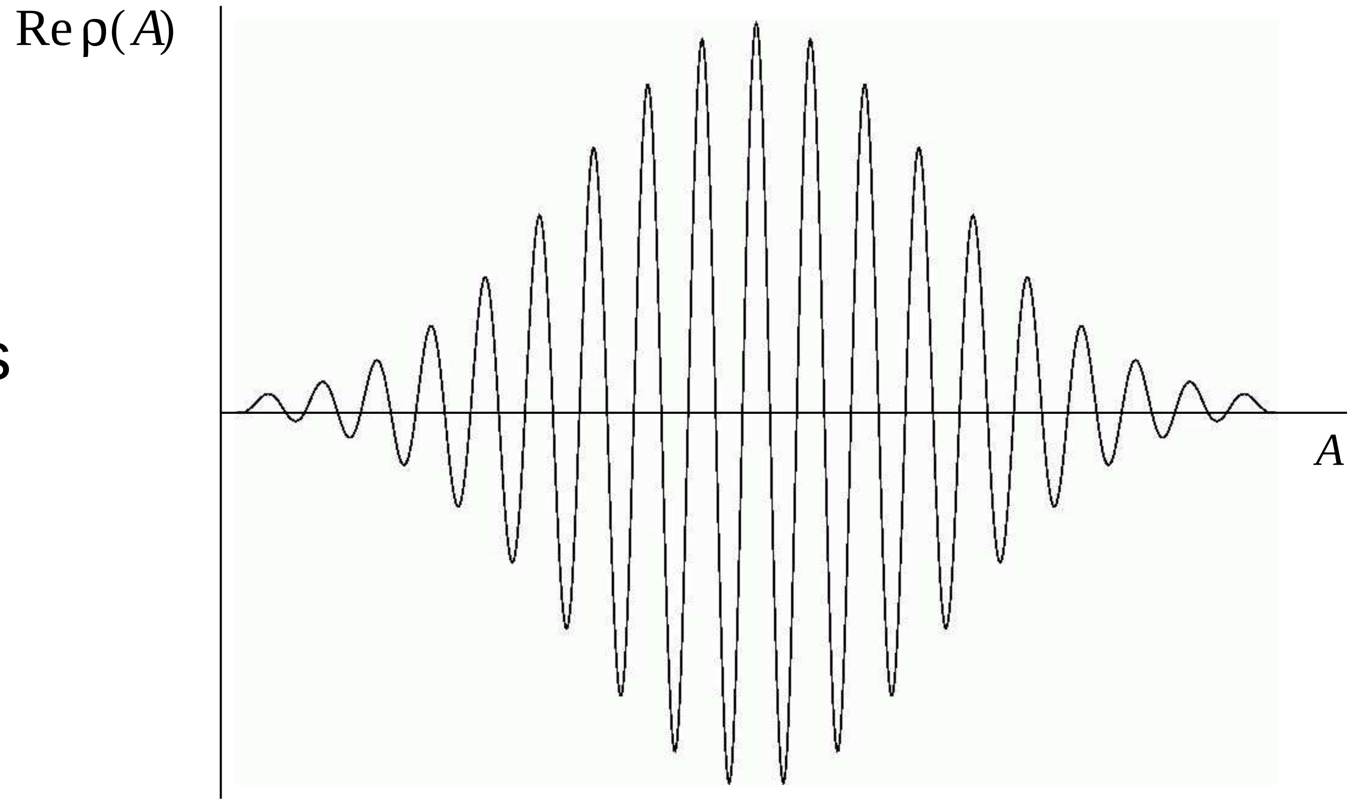
overlap problem, exponentially hard in thermodynamic limit

OVERLAP PROBLEM

COMPLEX WEIGHT $\rho(A;\mu)$

- configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
- cancelation between configurations with 'positive' and 'negative' weight

dominant configurations in the path integral?



OVERLAP PROBLEM

COMPLEX WEIGHT $\rho(A;\mu)$

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- cancelation between configurations with ‘positive’ and ‘negative’ weight

radically different approach:

- complexify all degrees of freedom: $A \to A_{\mathrm{R}} + iA_{\mathrm{I}}$
- enlarged complexified field space
- new directions to explore

complex Langevin dynamics

Parisi, Klauder 83

MAIN IDEA

ONE DEGREE OF FREEDOM

- consider complex Gaussian integral

$$Z(a,b) = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 - ibx} \qquad \left(= \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2}b^2/a} \right)$$

  complex action $S^*(b) = S(-b^*)$ [assume $a > 0$ and real]

- phase quenched theory

$$Z_{\rm pq} = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = Z(a,0) = \sqrt{\frac{2\pi}{a}}$$

- sign problem: average phase factor

$$\langle e^{-ibx} \rangle_{\rm pq} = \frac{Z(a,b)}{Z(a,0)} = e^{-\frac{1}{2}b^2/a}$$

MAIN IDEA

ONE DEGREE OF FREEDOM

- average phase factor: one degree of freedom only

$$\langle e^{-ibx}\rangle_{\rm pq} = \frac{Z(a,b)}{Z(a,0)} = e^{-\frac{1}{2}b^2/a}$$

  sign problem only bad when b gets large

- for N degrees of freedom x_j, $j = 1, \ldots, N$

$$\langle e^{-ib\sum_j x_j}\rangle_{\rm pq} = e^{-\frac{1}{2}Nb^2/a}$$

  limits $b \to 0$, $N \to \infty$ do not commute

  severe sign problem for all $b \neq 0$ in $N \to \infty$ limit!

MAIN IDEA

ONE DEGREE OF FREEDOM

$$Z(a,b) = \int dx\, e^{-\frac{1}{2}ax^2 - ibx} \qquad \langle x^2 \rangle = -2\frac{\partial \ln Z}{\partial a} = \frac{a-b^2}{a^2}$$

goal: compute numerically without importance sampling

first take $b=0$:

- use analogy with Brownian motion

Parisi & Wu 81

particle in a fluid: friction (a) and kicks (η)

- Langevin equation

$$\frac{d}{dt}x(t) = -ax(t) + \eta(t) \qquad \langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$$

MAIN IDEA

ONE DEGREE OF FREEDOM

- Langevin equation $\dot{x}(t) = -ax(t) + \eta(t)$
- analytical solution

$$x(t) = e^{-at}x(0) + \int_0^t ds\, \eta(s) e^{-a(t-s)}$$

- correlator [with $x(0) = 0$, no i.c. dependence]

$$\langle x^2(t)\rangle = \int_0^t ds \int_0^t ds'\, \langle \eta(s)\eta(s')\rangle e^{-a(2t-s-s')}$$

- noise averaged correlator, use $\langle \eta(s)\eta(s')\rangle = 2\delta(s-s')$

$$\lim_{t\to\infty} \langle x^2(t)\rangle = \frac{1}{a}$$

- no importance sampling, solution of stochastic process

MAIN IDEA

ONE DEGREE OF FREEDOM

$$Z(a,b)=\int dx\, e^{-S(x)} \qquad\qquad S(x)=\frac{1}{2}ax^2+ibx$$

$b\neq 0$:

- analytically: complete the square

  shift in the complex plane $x\to x+i\frac{b}{a}$

- achieve the same with Langevin equation

  "complexify" $x\to z=x+iy$

$$\begin{aligned}\dot{x} &= -\mathrm{Re}\,\partial_z S(z)+\eta = -ax+\eta\\ \dot{y} &= -\mathrm{Im}\,\partial_z S(z) \qquad\;\; = -ay-b\end{aligned}$$

with $S(z)=S(x+iy)$

MAIN IDEA

ONE DEGREE OF FREEDOM

- solution: $$x(t) = x(0)e^{-at} + \int_0^t ds\, e^{-a(t-s)}\eta(s)$$
$$y(t) = [y(0) + b/a]e^{-at} - b/a$$

- correlators:

$$\begin{aligned} \langle x^2(t)\rangle &= x^2(0)e^{-2at} + \left(1 - e^{-2at}\right)/a \to 1/a \\ \langle x(t)y(t)\rangle &= x(0)e^{-at}\left([y(0) + b/a]e^{-at} - b/a\right) \to 0 \\ \langle y^2(t)\rangle &= \left([y(0) + b/a]e^{-at} - b/a\right)^2 \to b^2/a^2 \end{aligned}$$

- combination:

$$\lim_{t\to\infty} \langle [x(t) + iy(t)]^2\rangle = \langle x^2 - y^2 + 2ixy\rangle = \frac{1}{a} - \frac{b^2}{a^2} = \frac{a - b^2}{a^2}$$

correct!

MAIN IDEA

ONE DEGREE OF FREEDOM

complex Langevin process should have an associated distribution $P(x, y; t)$ in complex plane

- real and positive distribution (if it exists)

$$\langle O(x + iy)(t)\rangle \quad = \quad \int dxdy\, P(x, y; t)O(x + iy)$$

LHS: Langevin equation for $x(t)$ and $y(t)$

RHS: Fokker-Planck equation for $P(x, y; t)$

Fokker-Planck equation:

$$\dot{P}(x, y; t) = \left[\partial_x \left(\partial_x + \mathrm{Re}\, \partial_z S\right) + \partial_y \mathrm{Im}\, \partial_z S\right] P(x, y; t)$$

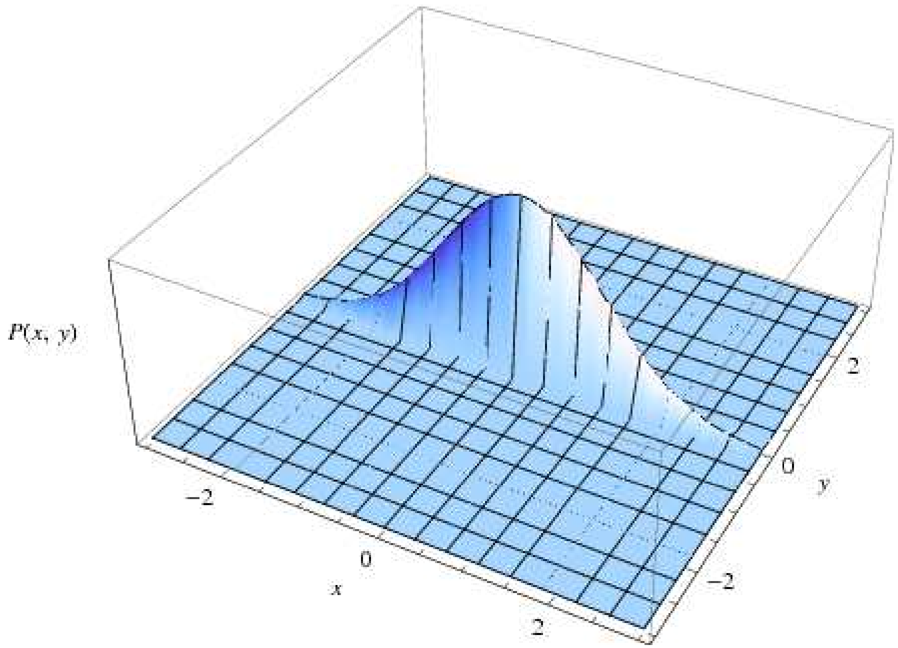
- solvable in Gaussian models (like here)
- in general case: no generic solutions known!

MAIN IDEA

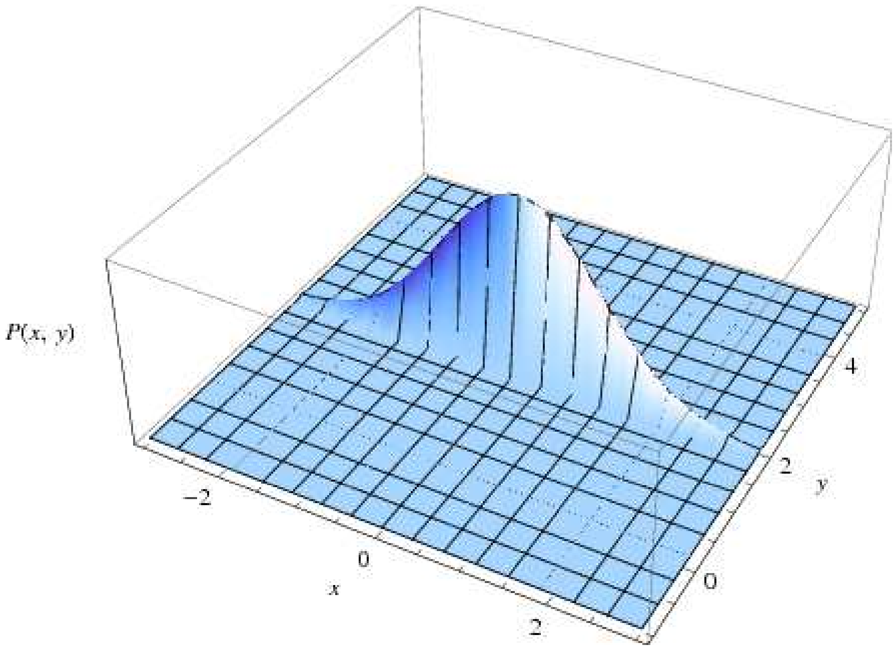
ONE DEGREE OF FREEDOM

distribution $P(x,y)$ in the complex plane

$b = 0$



$b = -2$



shift in the complex plane: $y \to -b/a$

Langevin process "finds" this distribution

MAIN IDEA

ONE DEGREE OF FREEDOM

final Gaussian example:

- $S = \frac{1}{2}(a + ib)x^2$ $\qquad\qquad$ $\langle x^2 \rangle = \frac{1}{a+ib}$
- coupled Langevin equations

$$\dot{x} = -ax + by + \eta \qquad\qquad \dot{y} = -ay - bx$$

- solve and find correlators when $t \to \infty$

$$\langle x^2 \rangle = \frac{1}{2a}\frac{2a^2 + b^2}{a^2 + b^2} \qquad \langle y^2 \rangle = \frac{1}{2a}\frac{b^2}{a^2 + b^2} \qquad \langle xy \rangle = -\frac{1}{2}\frac{b}{a^2 + b^2}$$

- correlator $\langle z^2 \rangle = \langle x^2 - y^2 + 2ixy \rangle = \dfrac{a - ib}{a^2 + b^2} = \dfrac{1}{a + ib}$

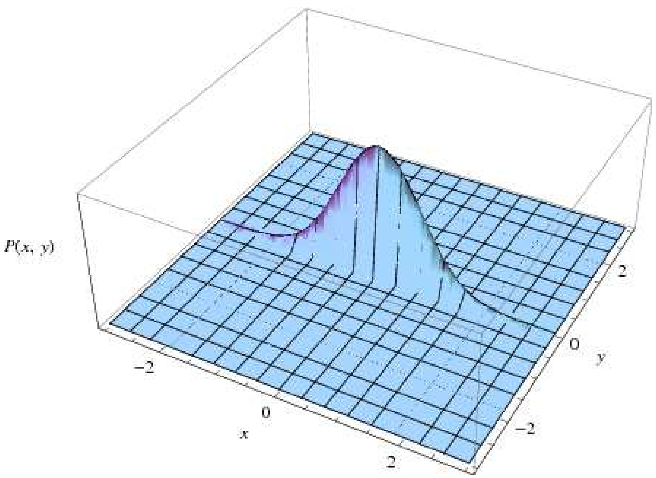
correct!

MAIN IDEA

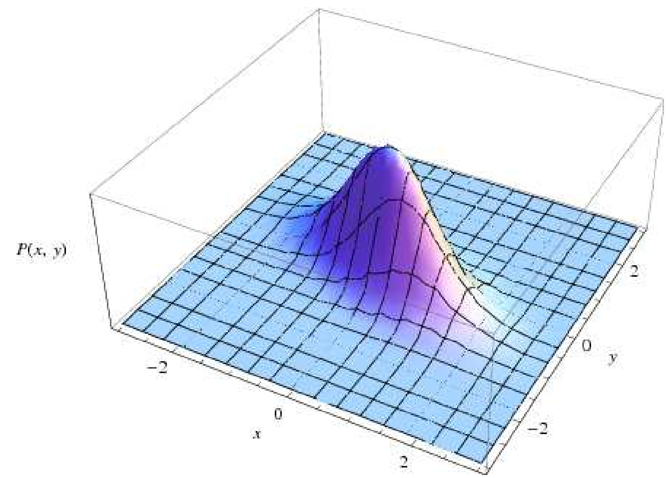
ONE DEGREE OF FREEDOM

distribution $P(x, y)$ in the complex plane

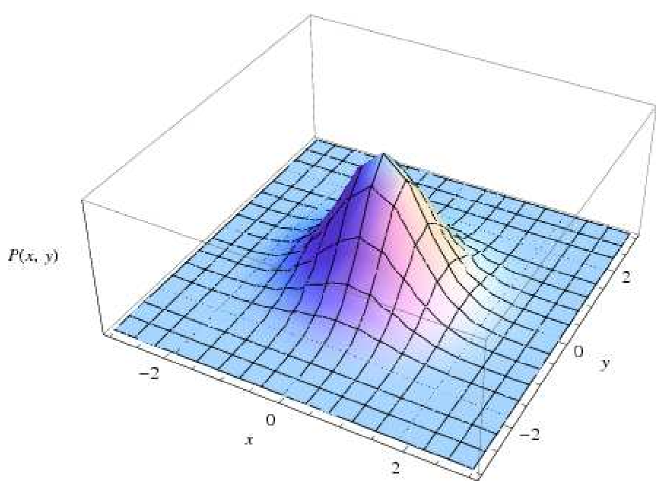
$b = 0.01$



$b = 1$



$b = 10$



Langevin process “finds” this distribution

original weight e^{-S} is complex

this distribution is real and positive

DISCRETIZATION

MOST CASES NOT ANALYTICALLY SOLVABLE

numerical solution of Langevin equation:

- discretize stochastic equation (Itō calculus)

$$x_{n+1} = x_n + \epsilon K_n^{\mathrm{R}} + \sqrt{\epsilon}\eta_n$$
$$y_{n+1} = y_n + \epsilon K_n^{\mathrm{I}}$$

- drift terms

$$K_n^{\mathrm{R}} = -\mathrm{Re}\,\frac{\partial S}{\partial z} \qquad\qquad K_n^{\mathrm{I}} = -\mathrm{Im}\,\frac{\partial S}{\partial z}$$

- noise

$$\langle \eta_n \eta_{n'} \rangle = \delta_{nn'}$$

- use adaptive stepsize if necessary

STOCHASTIC QUANTIZATON

LANGEVIN DYNAMICS

adapt to field theory Parisi & Wu 81, Parisi, Klauder 83

- path integral $Z = \int D\phi\, e^{-S}$
- Langevin dynamics in “fifth” time direction

$$\frac{\partial\phi(x,\theta)}{\partial\theta} = -\frac{\delta S[\phi]}{\delta\phi(x,\theta)} + \eta(x,\theta)$$

- Gaussian noise

$$\langle\eta(x,\theta)\rangle = 0 \qquad \langle\eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$$

- compute expectation values $\langle\phi(x,\theta)\phi(x',\theta)\rangle$, etc
- study converge as $\theta \to \infty$

PHASE TRANSITIONS AND THE SILVER BLAZE

can complex Langevin dynamics handle:

- a severe sign problem?
- the thermodynamic limit?
- phase transitions?
- the Silver Blaze problem? Cohen 03
- . . .

study in a model with a phase diagram with similar features as QCD at low temperature

$\Rightarrow$ relativistic Bose gas at nonzero μ

0810.2089, 0902.4686

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

- scalar O(2) model with global symmetry

- continuum action

$$S = \int d^4x \Big[\, |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu \left(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi\right) + \lambda |\phi|^4 \Big]$$

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu^*)$ as in QCD

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

- scalar O(2) model with global symmetry

- lattice action

$$S = \sum_x \left[\left(2d + m^2\right) \phi_x^* \phi_x + \lambda \left(\phi_x^* \phi_x\right)^2 - \sum_{\nu=1}^{4} \left(\phi_x^* e^{-\mu\delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^* e^{\mu\delta_{\nu,4}} \phi_x \right) \right]$$

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu^*)$ as in QCD

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

tree level potential in the continuum

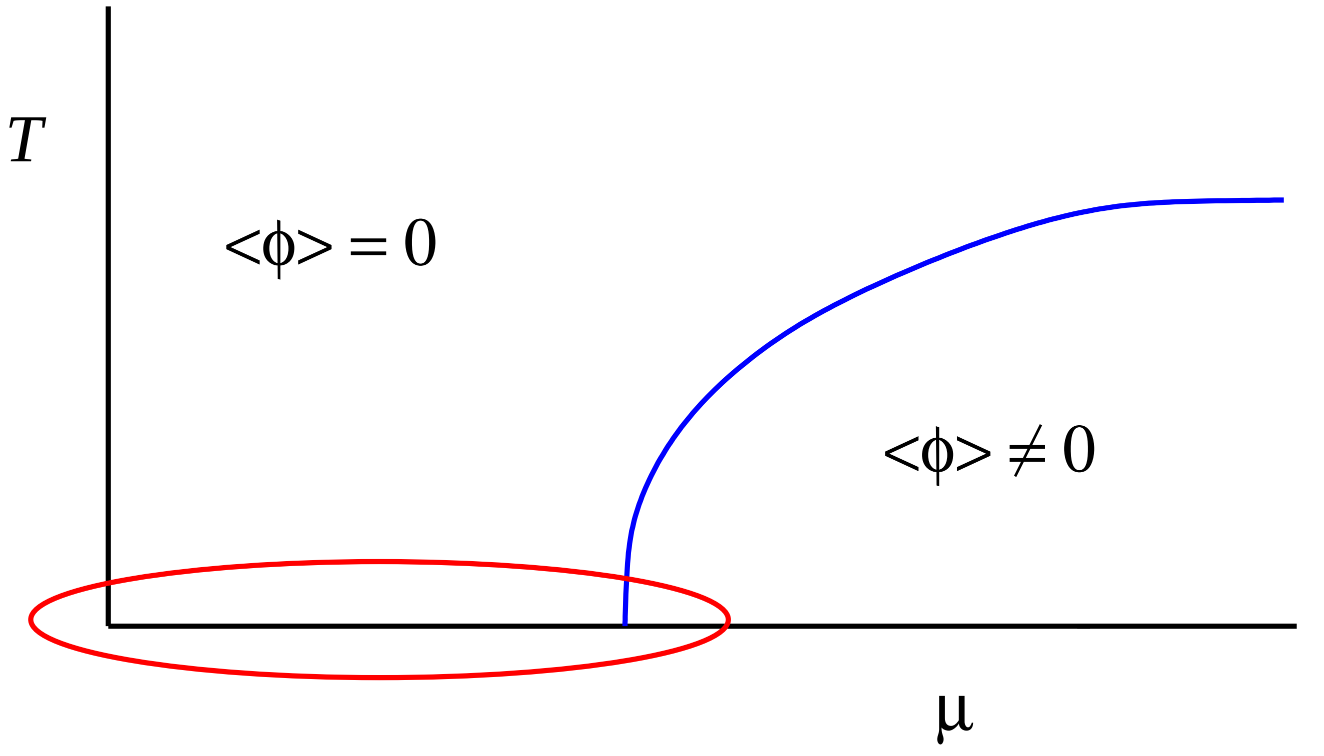
$$V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

condensation when $\mu^2 > m^2$, SSB

when $T = 0$
and $\mu < \mu_c$:

μ independence

Silver Blaze
problem



RELATIVISTIC BOSE GAS AT NONZERO μ

COMPLEX LANGEVIN

- write $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \;\Rightarrow\; \phi_a \;\; (a = 1, 2)$
- complexification $\phi_a \;\to\; \phi_a^{\mathrm{R}} + i\phi_a^{\mathrm{I}}$
- complex Langevin equations

$$\frac{\partial \phi_a^{\mathrm{R}}}{\partial \theta} = -\mathrm{Re}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a \to \phi_a^{\mathrm{R}} + i\phi_a^{\mathrm{I}}} + \eta_a$$

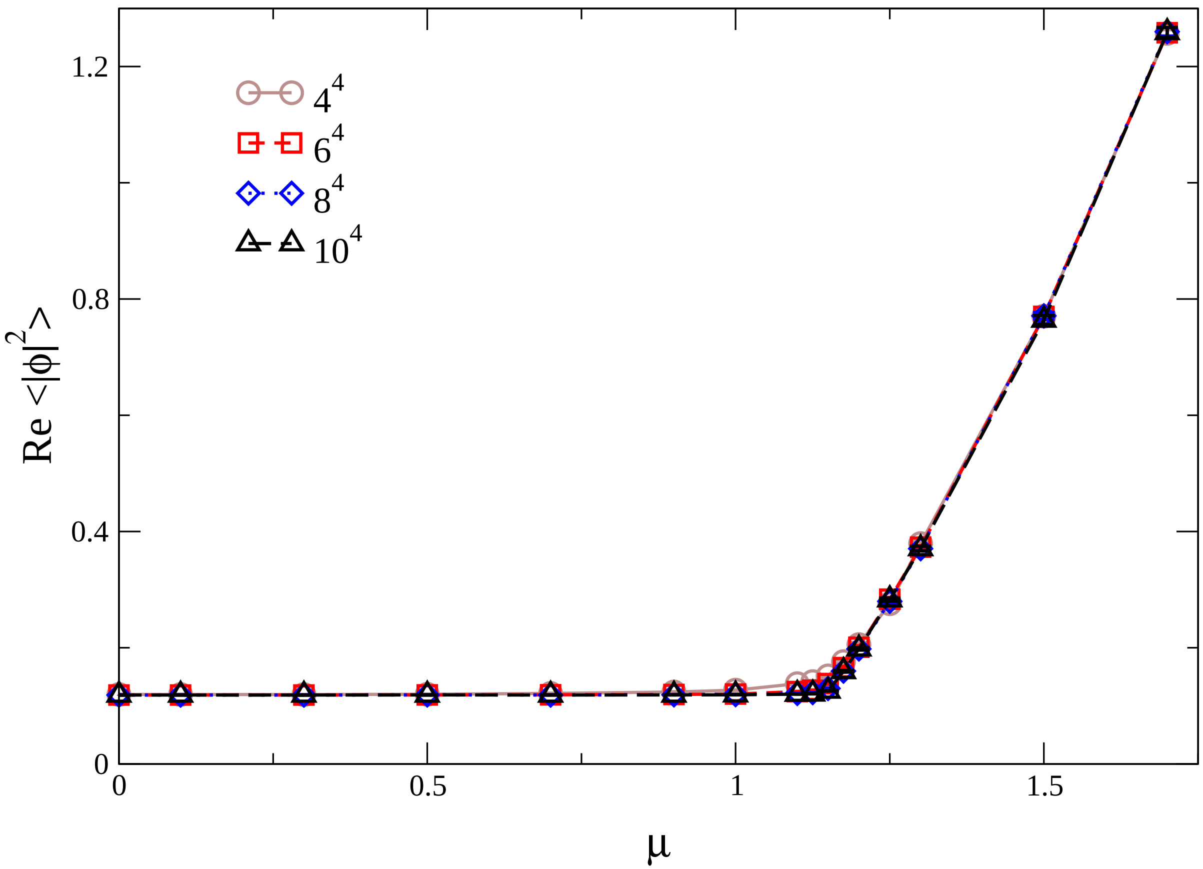
$$\frac{\partial \phi_a^{\mathrm{I}}}{\partial \theta} = -\mathrm{Im}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a \to \phi_a^{\mathrm{R}} + i\phi^{\mathrm{I}}}$$

- straightforward to solve numerically, $m = \lambda = 1$
- lattices of size N^4, with $N = 4, 6, 8, 10$

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

field modulus squared $$|\phi|^2 \to \frac{1}{2}\left({\phi_a^{\rm R}}^2 - {\phi_a^{\rm I}}^2\right) + i\phi_a^{\rm R}\phi_a^{\rm I}$$



Silver Blaze!

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

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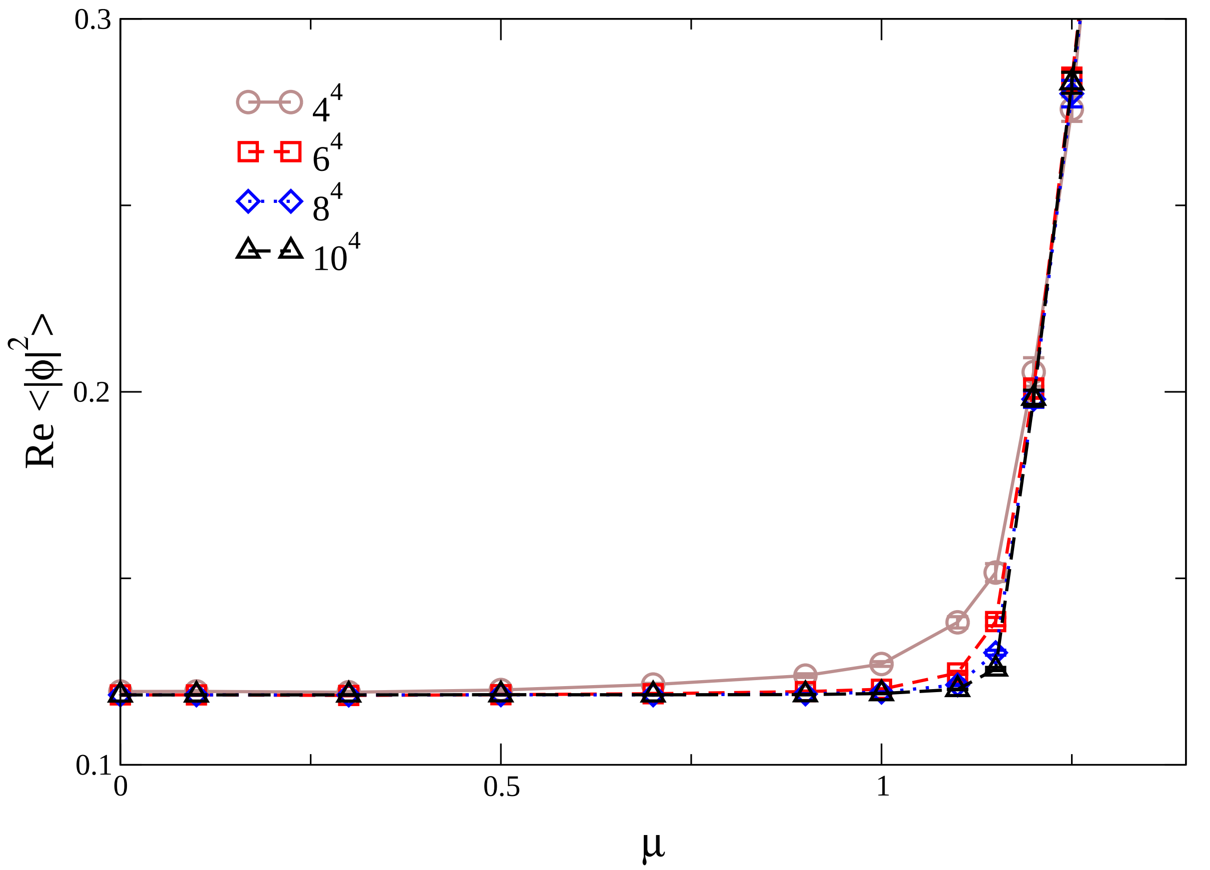
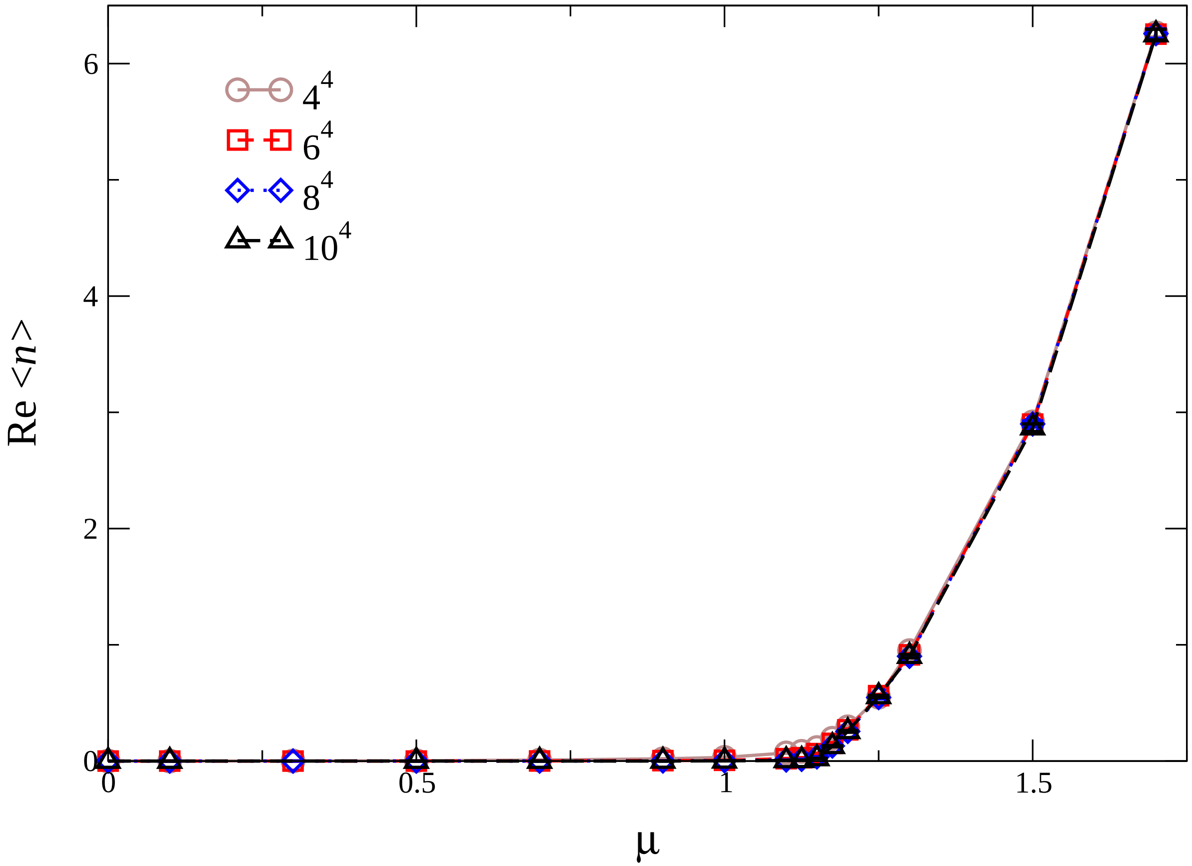


second order phase transition in thermodynamic limit

Relativistic Bose gas

complex Langevin

density $\langle n \rangle = (1/\Omega)\partial \ln Z/\partial \mu$



Silver Blaze

RELATIVISTIC BOSE GAS

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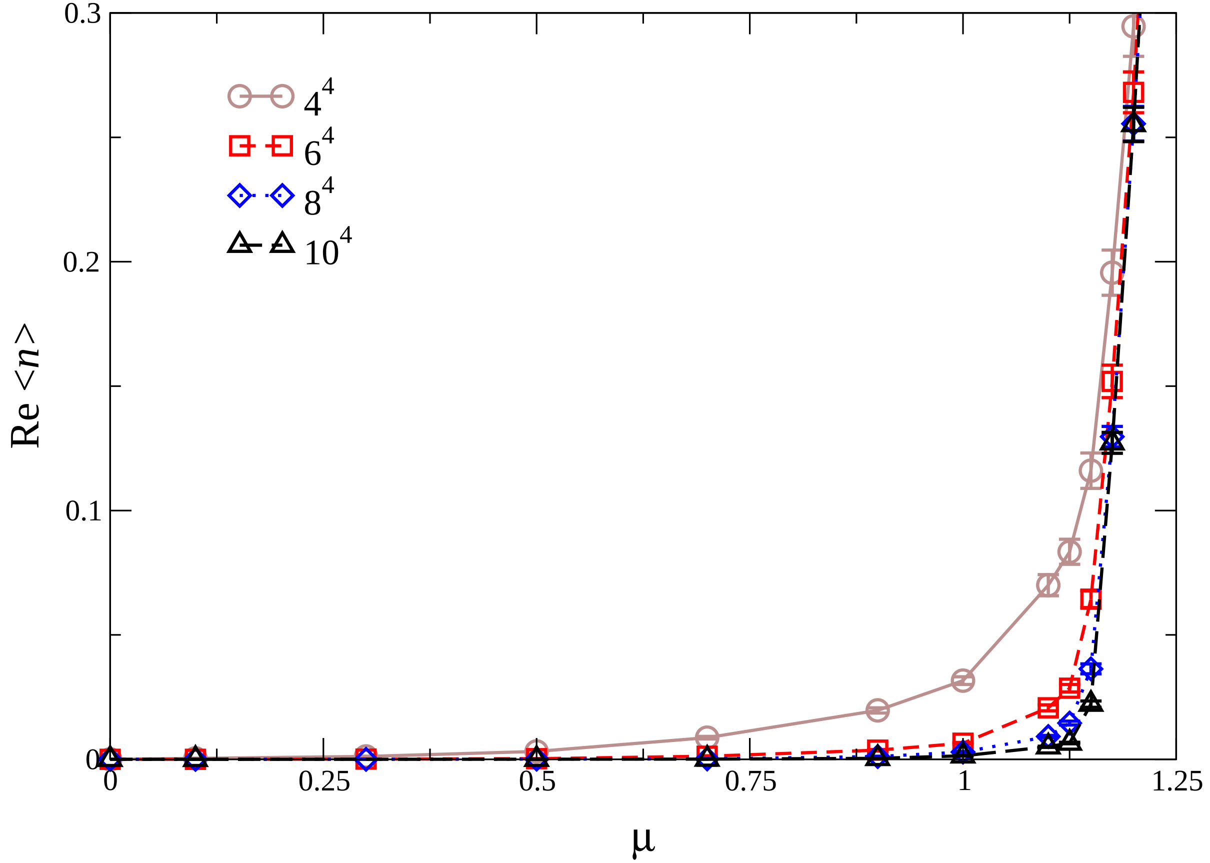


second order phase transition in thermodynamic limit

Silver Blaze and the sign problem

relativistic Bose gas

Silver Blaze and sign problems are intimately related

- phase quenched theory $Z_{\rm pq} = \int D\phi |e^{-S}|$

physics of phase quenched theory:

- chemical potential appears only in mass parameter (in continuum notation)

$$V = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

- dynamics of symmetry breaking, no Silver Blaze

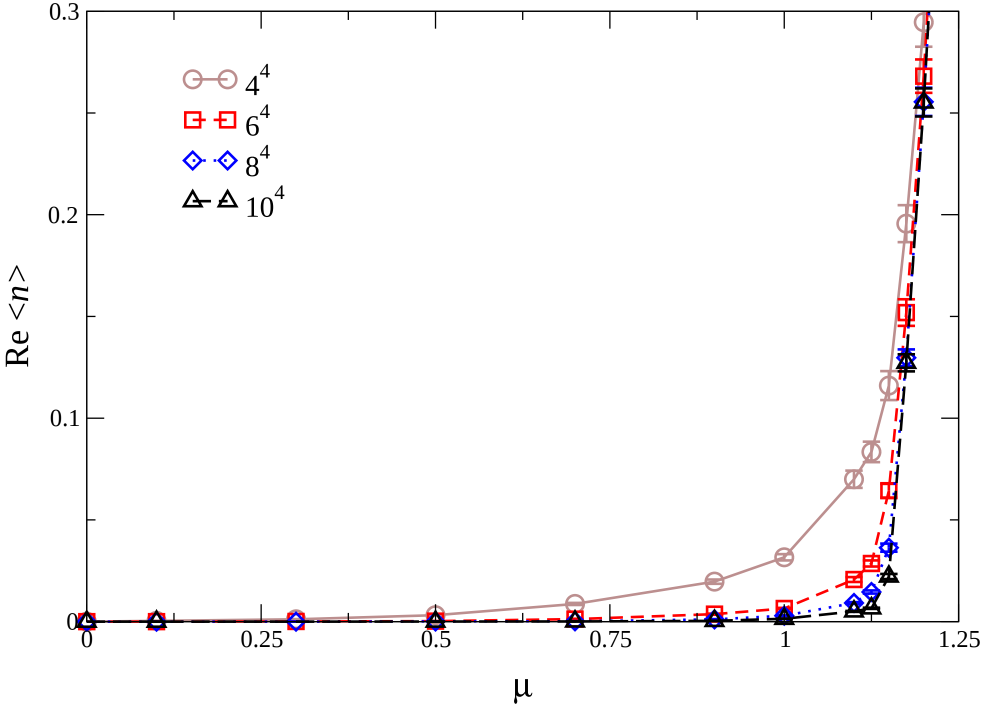
in QCD: phase quenched = finite isospin

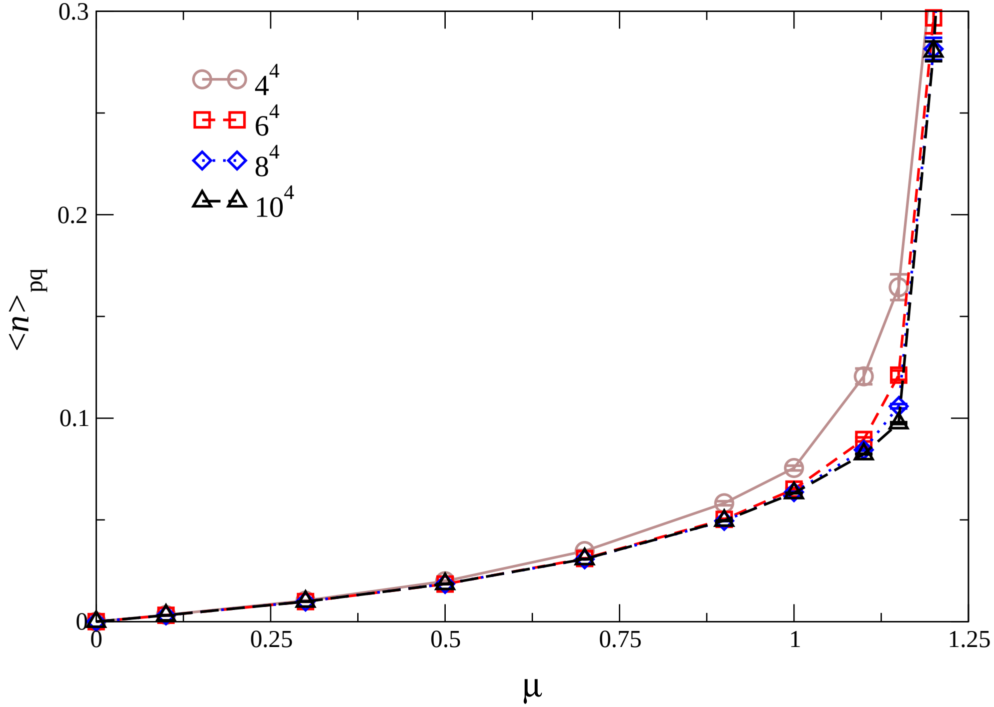
onset at $\mu = m_\pi/2$ instead of $m_B/3$

SILVER BLAZE AND THE SIGN PROBLEM

COMPLEX VS PHASE QUENCHED

density





complex — phase quenched

phase $e^{i\varphi} = e^{-S}/|e^{-S}|$ does precisely what is expected

How severe is the sign problem?

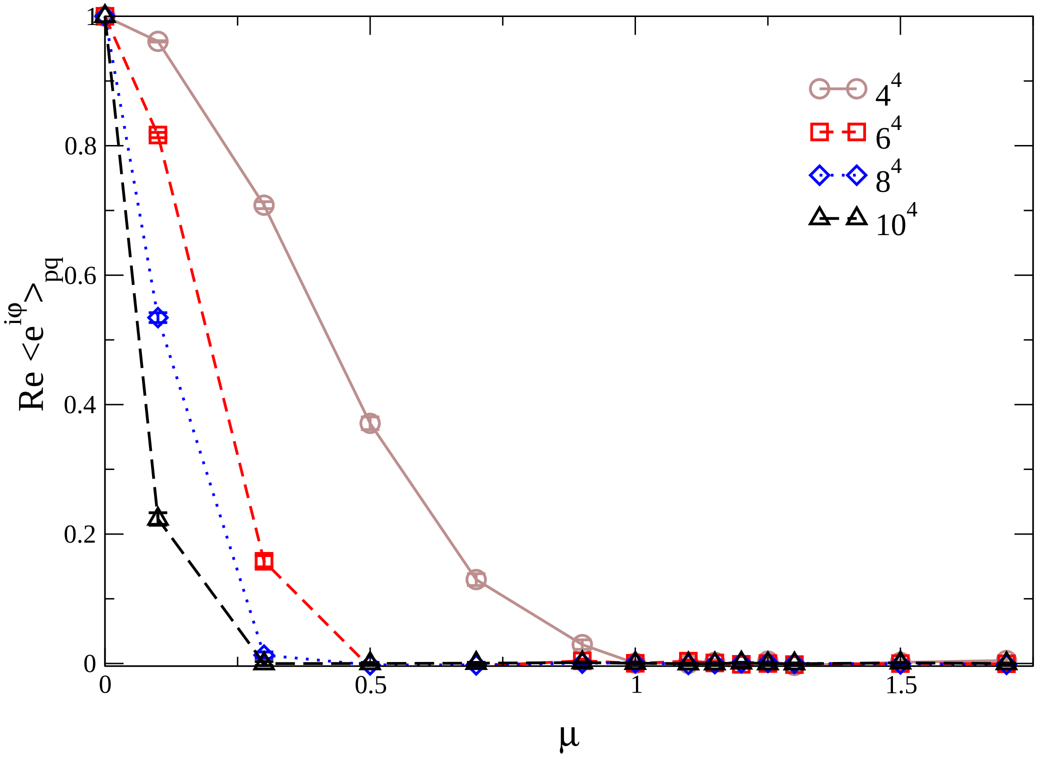
AVERAGE PHASE FACTOR

- complex action $e^{-S} = |e^{-S}|e^{i\varphi}$
- average phase factor in phase quenched theory

$$\langle e^{i\varphi}\rangle_{\rm pq} = \frac{Z_{\rm full}}{Z_{\rm pq}}$$

$$= e^{-\Omega\Delta f} \to 0$$

as $\Omega \to \infty$



- exponentially hard in thermodynamic limit

LATTICE GAUGE THEORY

- partition function

$$Z = \int DU \, e^{-S_B} \det M$$

- M is the fermion matrix
- fermion determinant is complex

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

Complex Langevin dynamics

In lattice gauge theory

Langevin update for link variables $U_{x,\nu}$:

$$U_{x,\nu}(\theta+\epsilon) = R_{x,\nu}(\theta)\, U_{x,\nu}(\theta) \qquad R_{x,\nu} = \exp\left[i\lambda_a\left(\epsilon K_{x\nu a} + \sqrt{\epsilon}\eta_{x\nu a}\right)\right]$$

Gell-mann matrices λ_a ($a = 1, \ldots 8$)

- drift term

$$K_{x\nu a} = -D_{x\nu a} S_{\text{eff}}[U] \qquad S_{\text{eff}} = S_B + S_F \qquad S_F = -\ln\det M$$

- noise

$$\langle \eta_{x\nu a}\rangle = 0 \qquad \langle \eta_{x\nu a}\eta_{x'\nu' a}\rangle = 2\delta_{xx'}\delta_{\nu\nu'}\delta_{aa'}$$

real action: $\Rightarrow K^\dagger = K \Leftrightarrow R^\dagger R = 1 \Leftrightarrow U \in$ SU(3)

complex action: $\Rightarrow K^\dagger \neq K \Leftrightarrow R^\dagger R \neq 1 \Leftrightarrow U \in$ SL$(3, \mathbb{C})$

HEAVY DENSE QCD

TOWARDS QCD

- bosonic action: standard SU(3) Wilson action

$$S_B = -\beta \sum_P \left(\frac{1}{6} \left[\mathrm{Tr}\, U_P + \mathrm{Tr}\, U_P^{-1} \right] - 1 \right)$$

- determinant $\det M$ for Wilson fermions

  fermion matrix:

$$M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left(e^{\mu} \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_{-4} \right)$$

HEAVY DENSE QCD

TOWARDS QCD

- hopping expansion:

$$\det M \approx \det \left[1 - \kappa \left(e^{\mu}\Gamma_{+4}U_{x,4}T_4 + e^{-\mu}\Gamma_{-4}U^{-1}_{x,4}T_{-4}\right)\right]$$
$$= \prod_{\mathbf{x}} \det \left(1 + h e^{\mu/T}\mathcal{P}_{\mathbf{x}}\right)^2 \det \left(1 + h e^{-\mu/T}\mathcal{P}^{-1}_{\mathbf{x}}\right)^2$$

  with $h = (2\kappa)^{N_\tau}$ and (conjugate) Polyakov loops $\mathcal{P}^{(-1)}_{\mathbf{x}}$

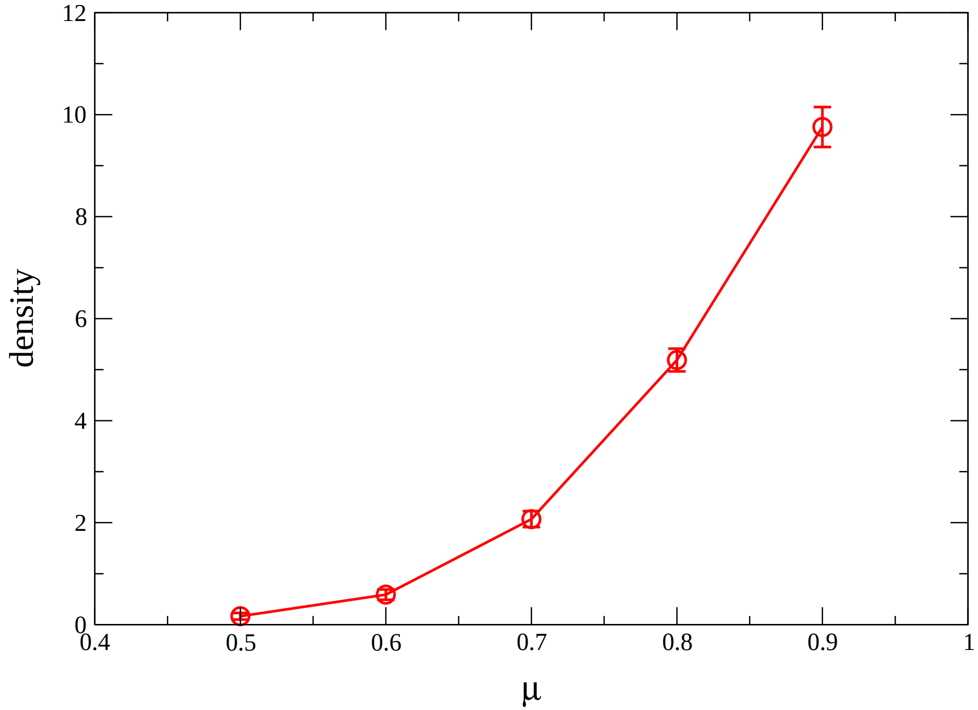
- static quarks propagate in temporal direction only: Polyakov loops

- full gauge dynamics included

with Stamatescu 0807.1597

DENSITY

HEAVY DENSE QCD



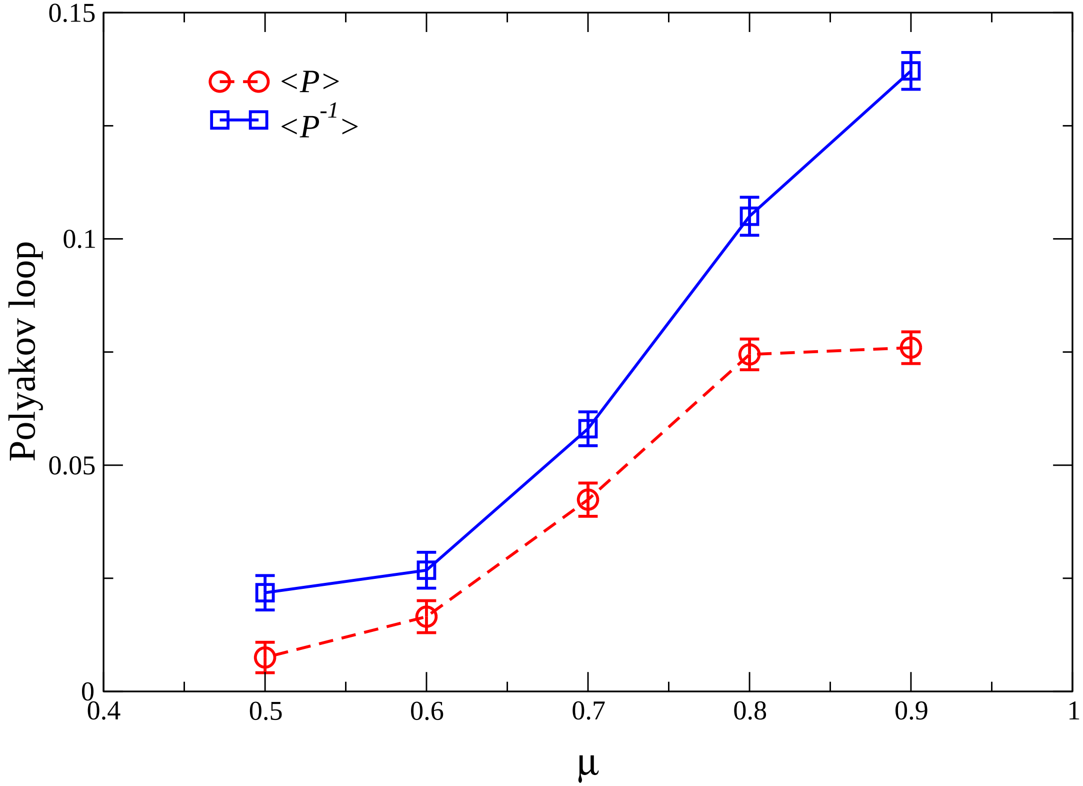
first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density phase $\Rightarrow$ high-density phase

(CONJUGATE) POLYAKOV LOOPS

HEAVY DENSE QCD

results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$



low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase

$\mathbf{SU(3)} \to \mathbf{SL(3,\mathbb{C})}$

HEAVY DENSE QCD

- complex Langevin dynamics: no longer in SU(3)
- instead $U \in$ SL(3,$\mathbb{C}$)
- in terms of gauge potentials $U = e^{i\lambda_a A_a/2}$
  A_a is now complex
- how far from SU(3)?

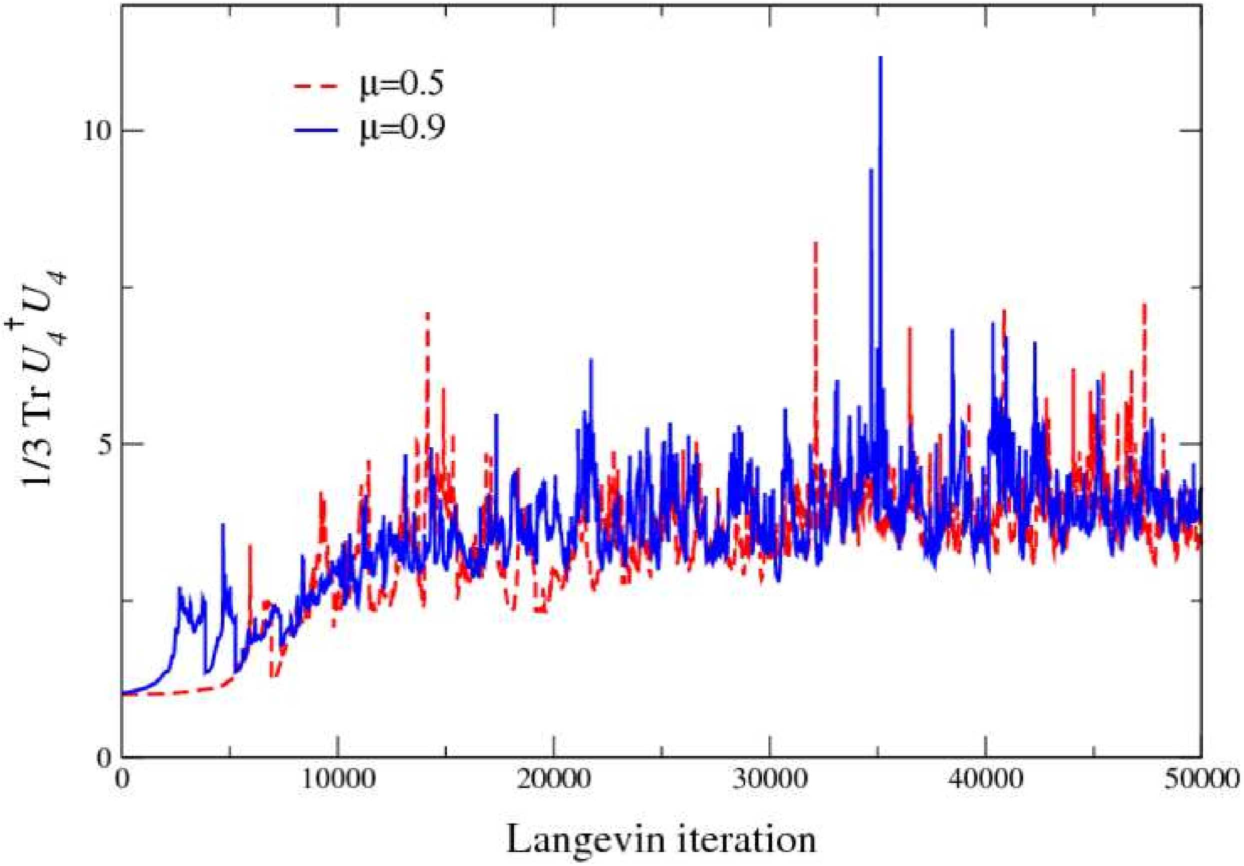
consider

$$\frac{1}{N}\text{Tr}\, U^\dagger U \begin{cases} = 1 \ \text{ if } \ U \in \text{SU}(N) \\ \\ \geq 1 \ \text{ if } \ U \in \text{SL}(N,\mathbb{C}) \end{cases}$$

$\mathbf{SU(3)} \to \mathbf{SL(3,\mathbb{C})}$

HEAVY DENSE QCD

$$\frac{1}{3}\mathrm{Tr}\, U^\dagger U \geq 1 \qquad\qquad = 1 \text{ if } U \in \text{SU(3)}$$



OPEN QUESTIONS

- complex Langevin works very well for Bose gas
- first results in heavy dense QCD promising

but . . . problems from the 80s:

- instabilities and runaways
- convergence to wrong result
- lack of theoretical understanding

Klauder & Petersen 85

Ambjørn et al 85,86

...

surprisingly, all present in three-dimensional XY model

XY MODEL

WITH FRANK JAMES

three-dimensional XY model at nonzero μ

$$S = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0}\right)$$

- μ couples to the conserved Noether charge
- symmetry $S^*(\mu) = S(-\mu^*)$

unexpectedly difficult to simulate with complex Langevin!

numerics shares many features with heavy dense QCD

also studied by Banerjee & Chandrasekharan using worldline formulation
hep-lat/1001.3648

Instabilities and runaways

with Frank James, Erhard Seiler and Ion-Olimpiu Stamatescu

- unstable classical trajectories: $\phi_{\rm I} \to \infty$
- force not always restoring

careful integration mandatory

Ambjørn et al 85,86

adaptive stepsize

XY model at nonzero μ and heavy dense QCD

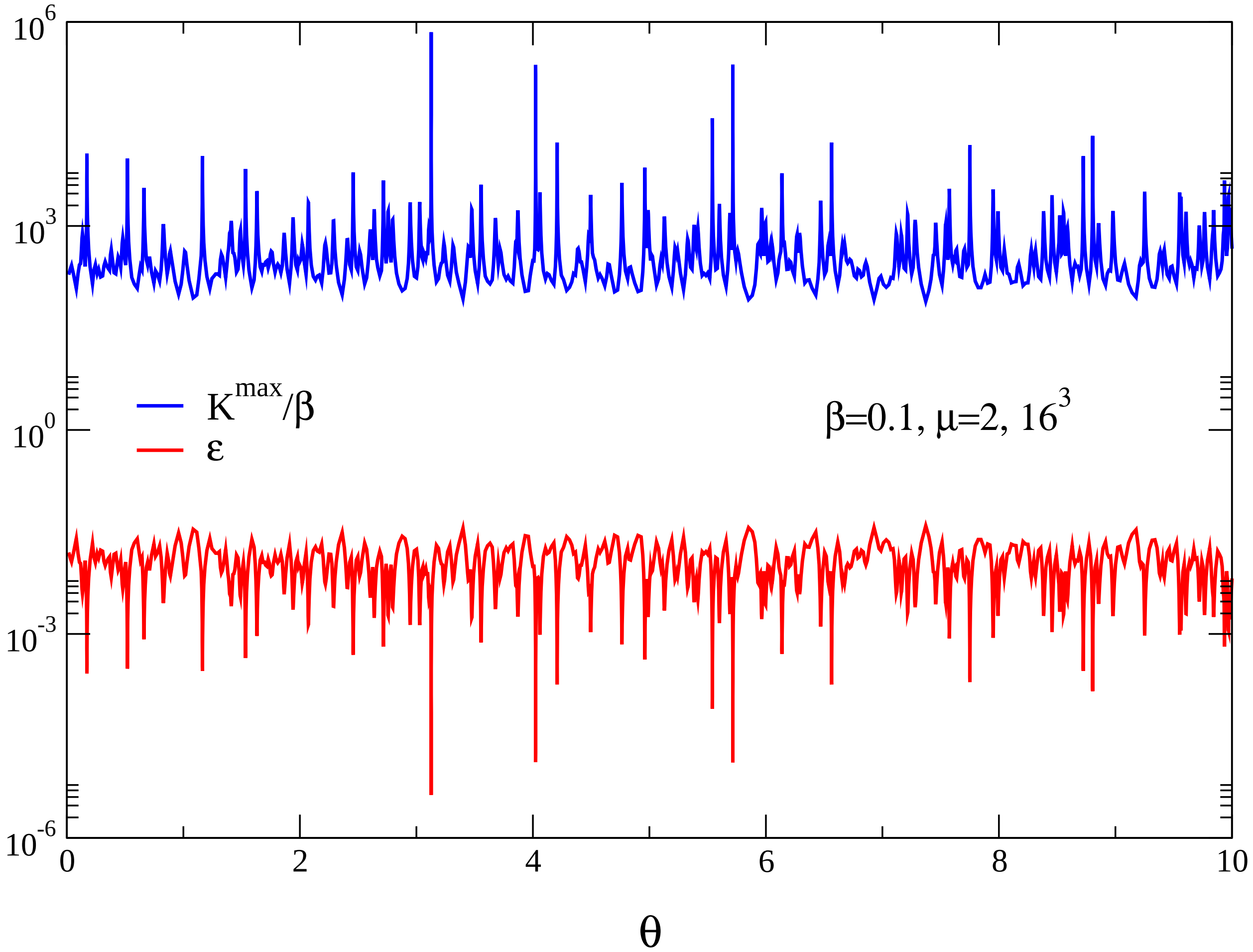
0912.0617

- no runaways encountered at all
- wide range of parameter values

INSTABILITIES

XY MODEL

$K^{\max}$ and adaptive time step during the evolution

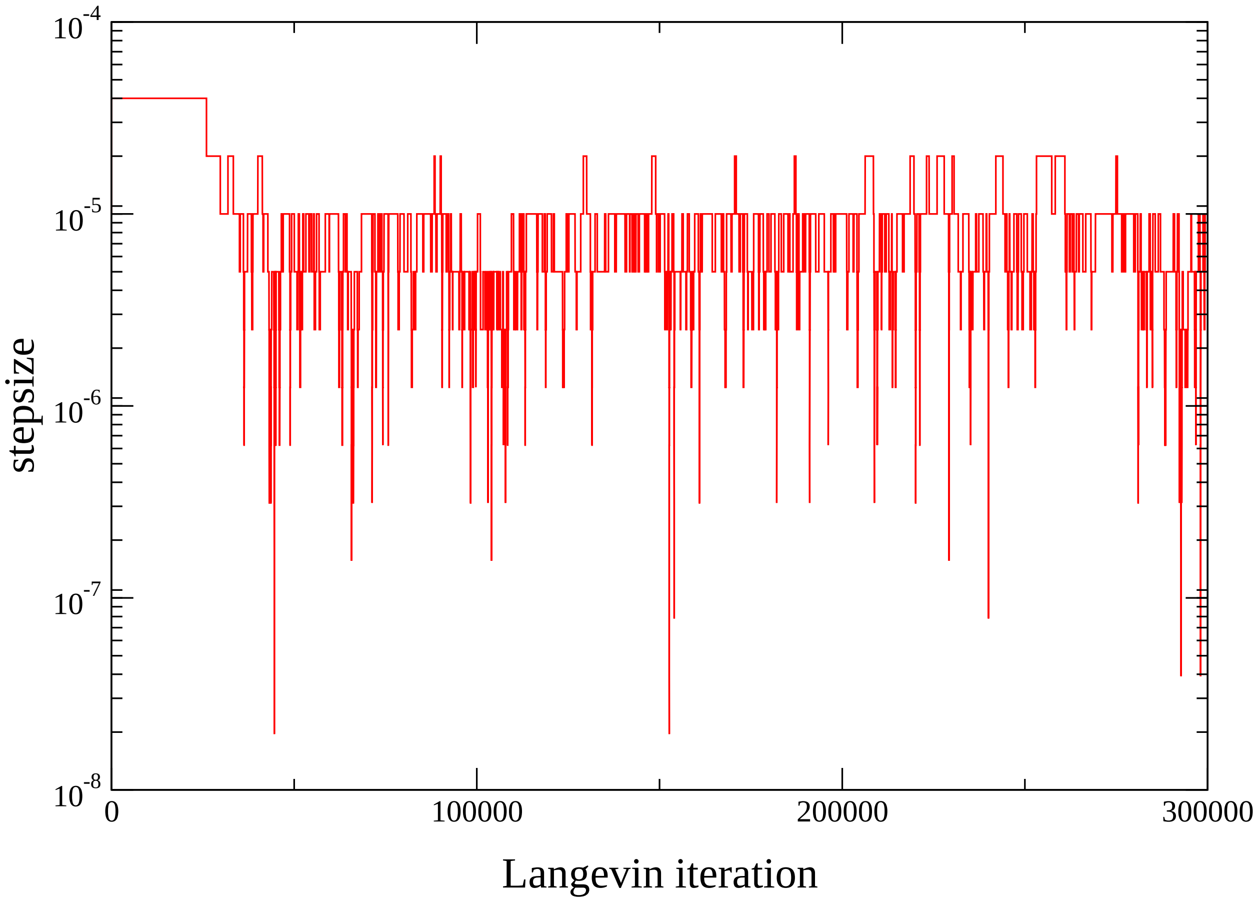


$K^{\max}$ is the maximal drift term at each time step

INSTABILITIES

HEAVY DENSE QCD

$K^{\max}$ and adaptive time step during the evolution



occasionally *very* small stepsize required
can go to longer Langevin times without problems

CONVERGENCE

XY MODEL, WITH FRANK JAMES, 1005.3468

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_{\rm I}$

real μ, complex action:

$$S = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0}\right)$$

imaginary $\mu = i\mu_{\rm I}$, real action:

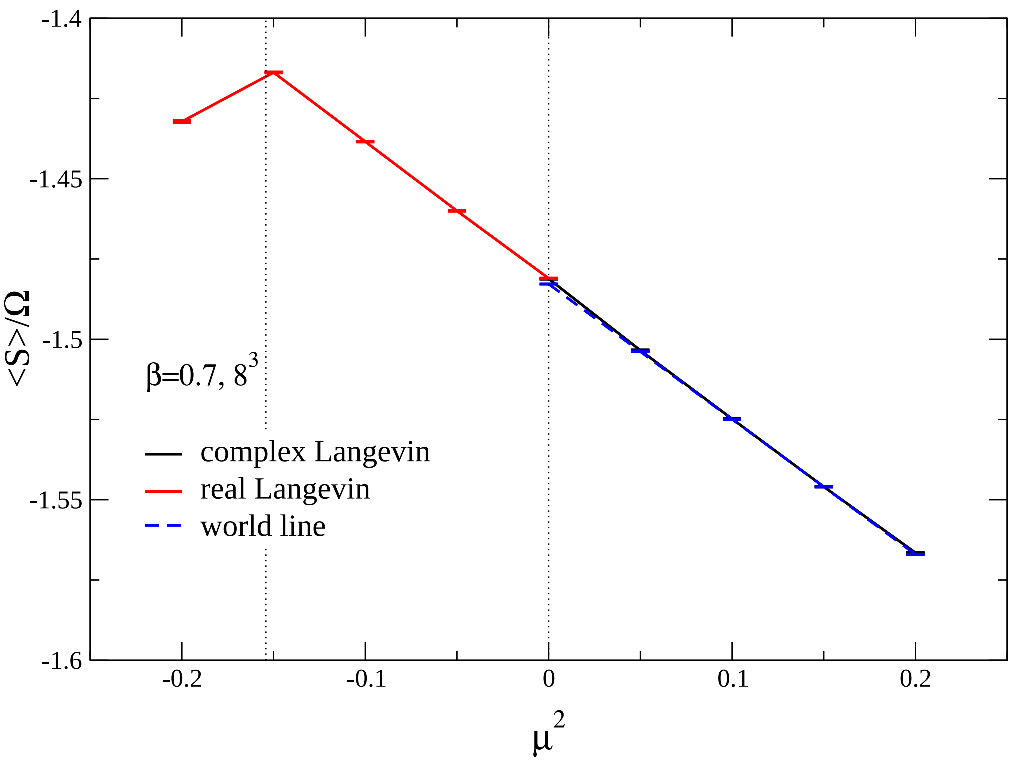
$$S_{\rm I} = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} + \mu_{\rm I}\delta_{\nu,0}\right)$$

- real and imag μ results analytic in μ^2

CONVERGENCE

XY MODEL

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_{\rm I}$



action density versus μ^2

$\beta = 0.7$

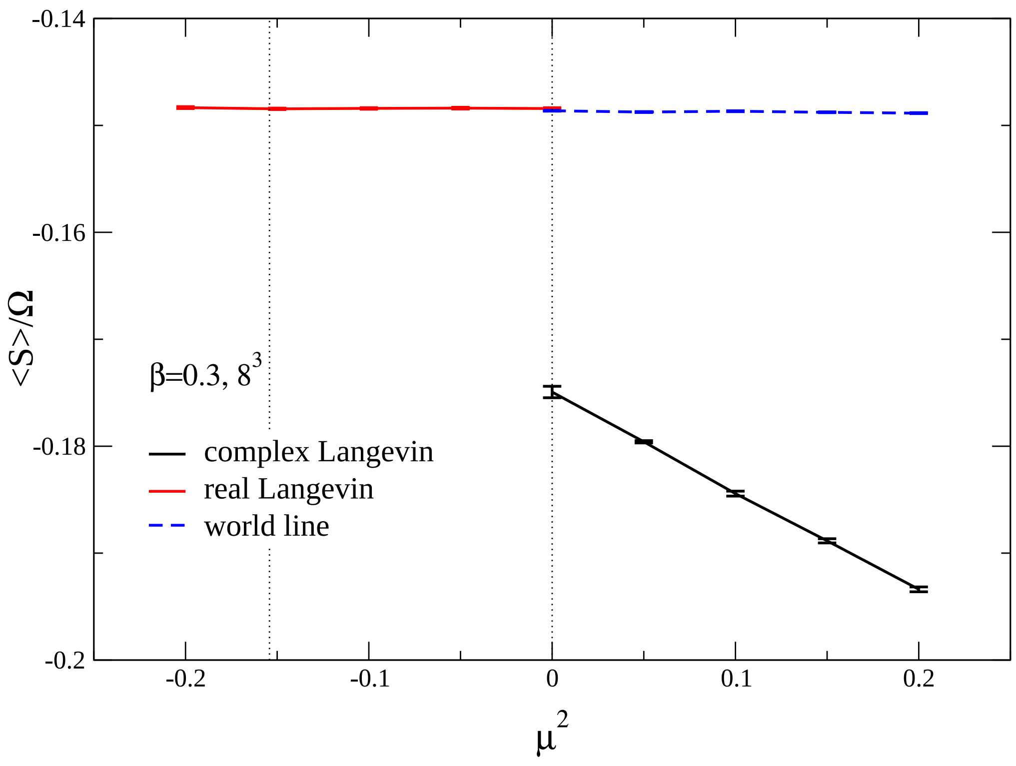
ordered phase

- "Roberge-Weiss" transition at $\mu_{\rm I} = \pi/N_\tau$

CONVERGENCE

XY MODEL

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_{\mathrm{I}}$



action density versus μ^2

$\beta = 0.3$

disordered phase

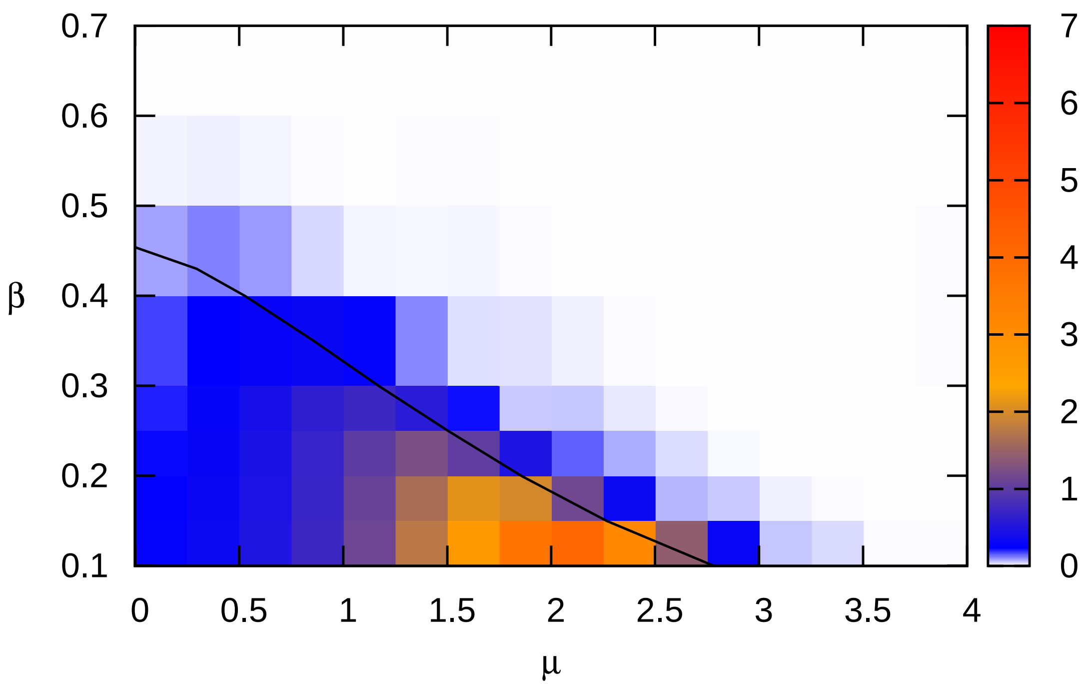
- Silver Blaze feature at small β and μ

CONVERGENCE

XY MODEL

- comparison with known result (world line formulation)

phase diagram:



relative deviation:

$$\Delta S = \frac{\langle S \rangle_{\text{cl}} - \langle S \rangle_{\text{wl}}}{\langle S \rangle_{\text{wl}}}$$

high β: ordered

low β: disordered

- phase boundary from Banerjee & Chandrasekharan
- highly correlated with ordered/disordered phase

CONVERGENCE

XY MODEL

- apparent correct results in the ordered phase
- incorrect result in the disordered/transition region

diagnostics:

- distribution $P[\phi_{\rm R}, \phi_{\rm I}]$ qualitatively different
- classical force distribution qualitatively different
- complexified dynamics $\neq$ real dynamics when $\mu = 0$

but:

- independent of strength of the sign problem

conclusion: failure not due to sign problem

TOWARDS UNDERSTANDING

- formal justification and analytical understanding
- assumptions: explicit and implicit
- relation between complex weight e^{-S} and (real and positive) solution of Fokker-Planck equation P
- properties of the distribution $P[\phi_{\rm R}, \phi_{\rm I}]$
- necessary conditions for correctness (even when exact result is not known)

with FJ, ES and IOS, 0912.3360, 1101.3270

SU(3) SPIN MODEL

WITH FRANK JAMES, IN PREPARATION

3-dimensional SU(3) spin model $S = S_B + S_F$

$$S_B = -\beta \sum_{x,i} \left[\mathrm{Tr}\, U_x \mathrm{Tr}\, U_{x+i}^\dagger + \mathrm{Tr}\, U_x^\dagger \mathrm{Tr}\, U_{x+i} \right]$$

$$S_F = -h \sum_x \left[e^\mu \mathrm{Tr}\, U_x + e^{-\mu} \mathrm{Tr}\, U_x^\dagger \right]$$

- SU(3) matrices U_x, complex action $S_F^*(\mu) = S_F(-\mu^*)$
- solved with complex Langevin Karsch & Wyld 85

  Bilic, Gausterer & Sanielevici 88
- reformulated as a flux model without sign problem

  Gattringer 11

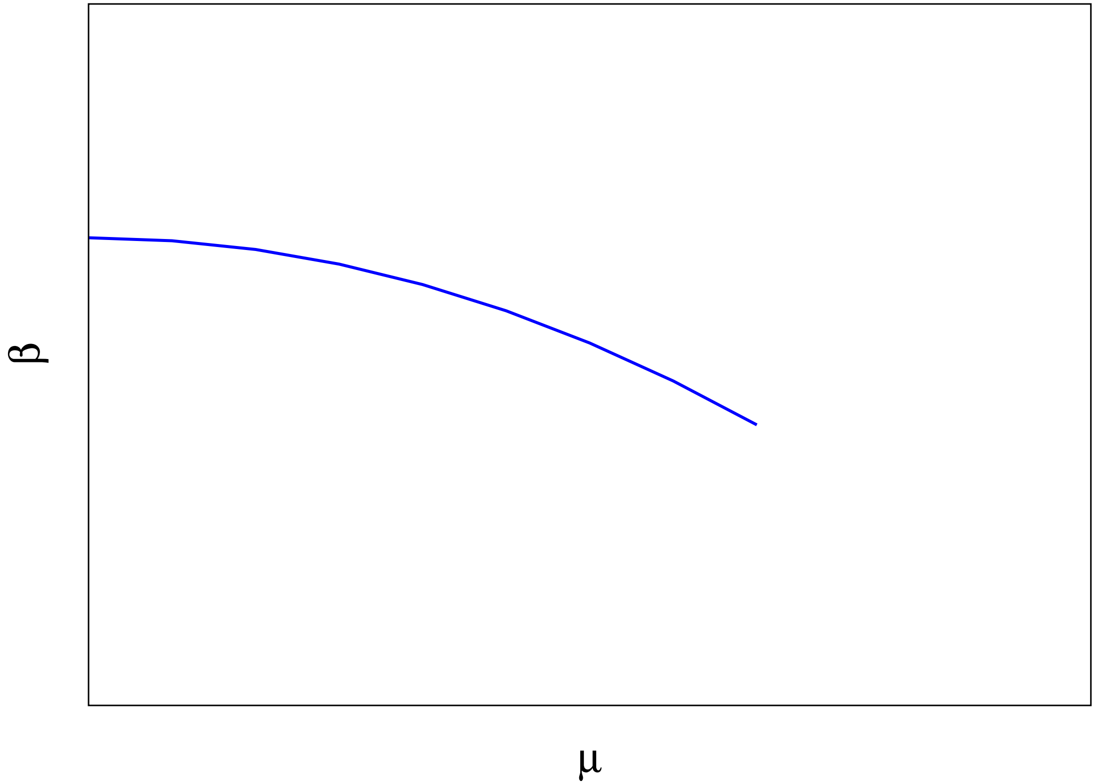
test reliability of complex Langevin using developed tools

SU(3) SPIN MODEL

IN PREPARATION

phase structure (for small h):

first-order transition in $\beta - \mu$ plane



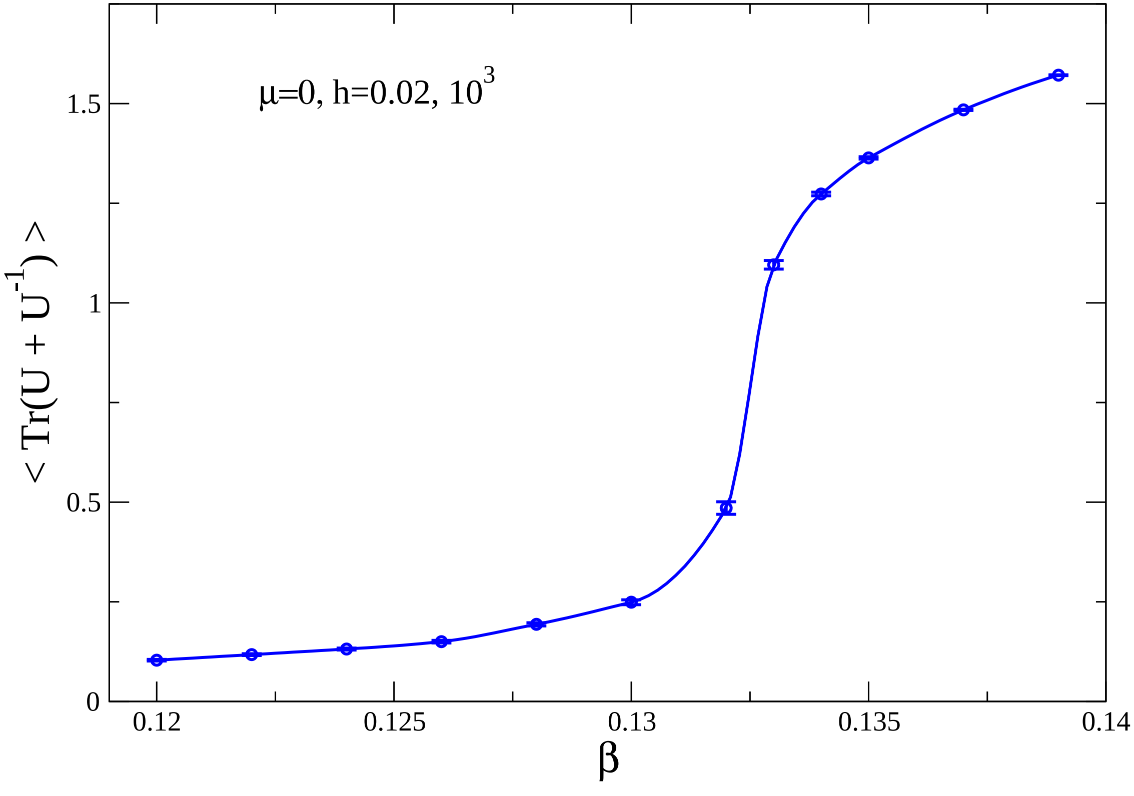
ends at a critical endpoint

SU(3) SPIN MODEL

IN PREPARATION

phase structure (for small h):

first-order transition in $\beta - \mu$ plane

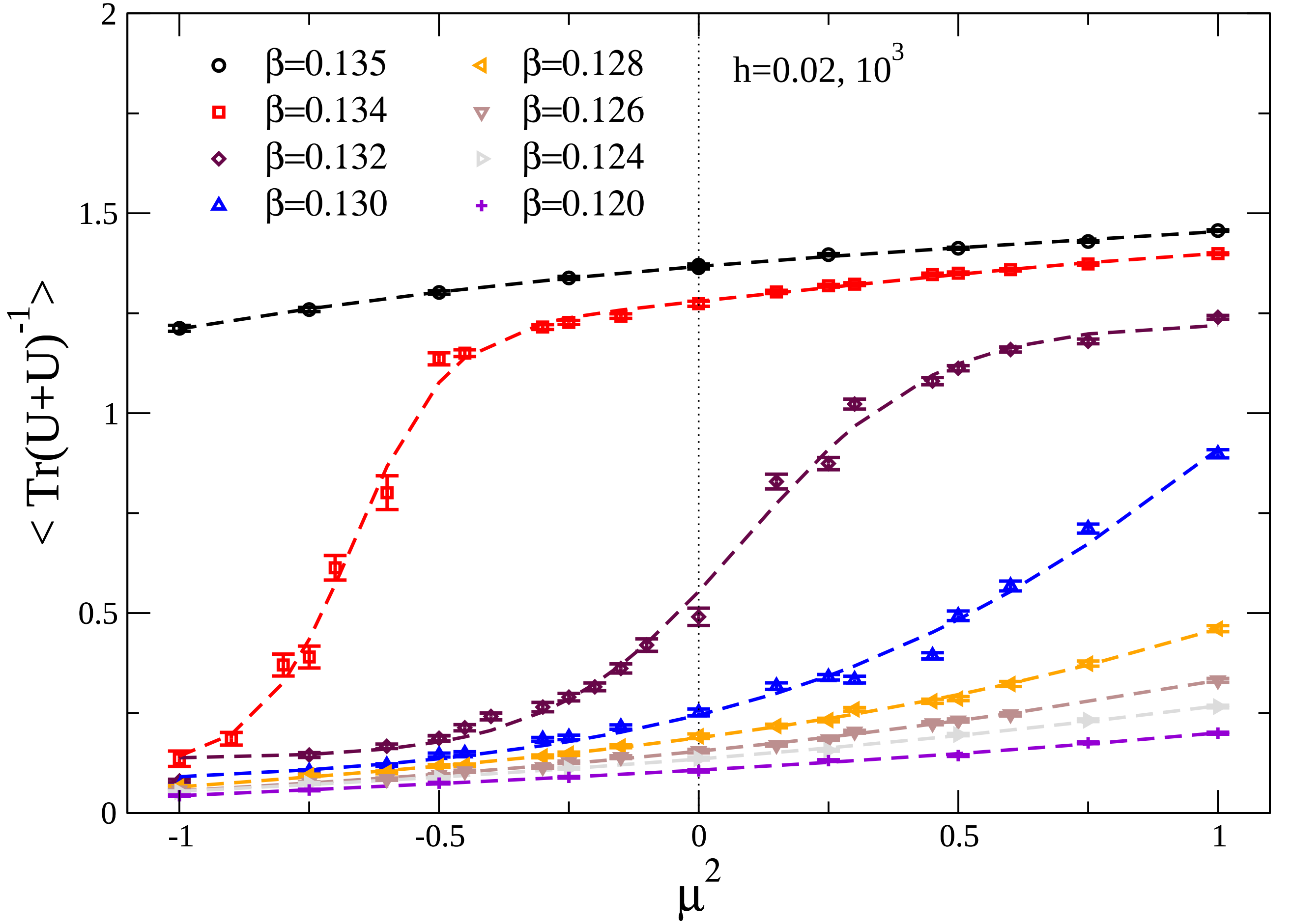


varying β at $\mu = 0$ (real Langevin)

SU(3) SPIN MODEL

REAL AND IMAGINARY POTENTIAL

first-order transition in $\beta - \mu^2$ plane

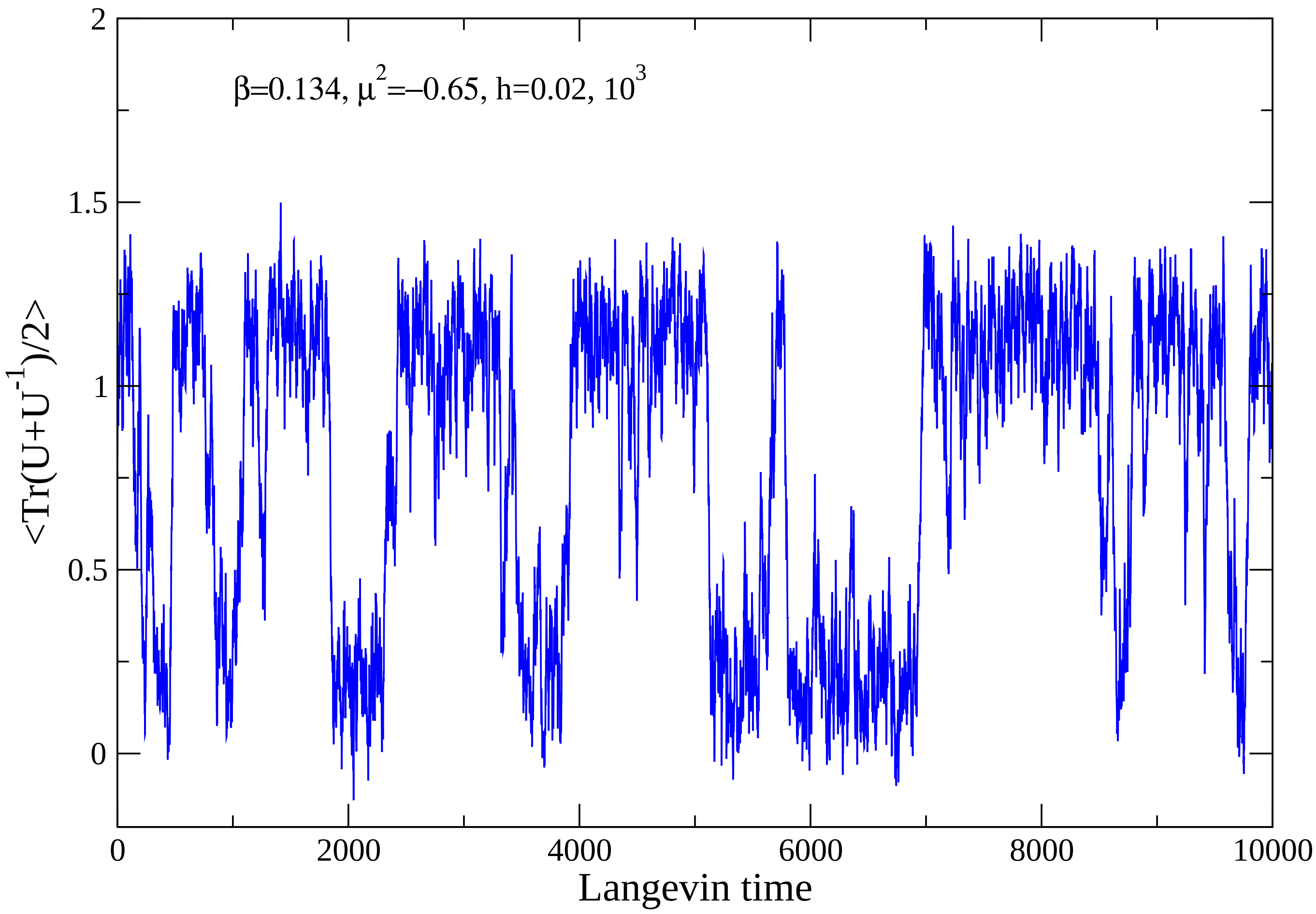


negative μ^2: real Langevin — positive μ^2: complex Langevin

SU(3) SPIN MODEL

IN PREPARATION

first order behaviour



$\sim 10^8$ Langevin steps

SU(3) SPIN MODEL

WITH FRANK JAMES, IN PREPARATION

tests of complex Langevin dynamics:

- analyticity around $\mu^2 = 0$
- properties of the distribution
- more detailed criteria (developed with Seiler and Stamatescu)

for this model all tests are passed

- complex Langevin can be trusted (in both phases)
- compare with flux representation?

SUMMARY

FINITE CHEMICAL POTENTIAL

many stimulating results: examples where complex Langevin can handle

- sign problem
- Silver Blaze problem
- phase transition
- thermodynamic limit

problems from the 80s:

- instabilities and runaways $\rightarrow$ adaptive stepsize
- convergence: correct result not guaranteed

resolution in progress, important:

- failure does not depend on strength of sign problem
- distinct from all other approaches