

# Resummation of large- $x$ and small- $x$ double logarithms in DIS and semi-inclusive $e^+e^-$ annihilation

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A. Vogt (University of Liverpool)

with G. Soar, A. Lo Presti, C.H. Kom (UoL), A. Almasy (UoL, now DESY),  
K. Yeats (Simon-Frazer U) [ and S. Moch (DESY), J. Vermaseren (NIKHEF) ]

- Splitting and coefficient functions and their endpoint behaviour
- Generalized threshold resummation of  $1/\text{Mellin-}N$  contributions
- Small- $x$  resummation of  $x^{-1}\ln^\ell x$  (SIA) and  $x^0\ln^\ell x$  (DIS) terms

# Conventions and references

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Double-log enhancement: two additional logs  $L$  per additional order in  $\alpha_s$

$$Q|_{\alpha_s^n} \sim L^{-\ell_0} (\#L^{2n} + \#L^{2n-1} + \#L^{2n-2} + \dots) + \dots$$

LL            NLL            NNLL

LL, NLL, ...: leading logarithms, next-to-leading logarithms, ...

Counting of a resummation, cf. small- $x$ , not of a (stronger) exponentiation,  
cf. soft gluons: NNLL resummation  $\Leftrightarrow$  (re-expanded) NLL exponentiation

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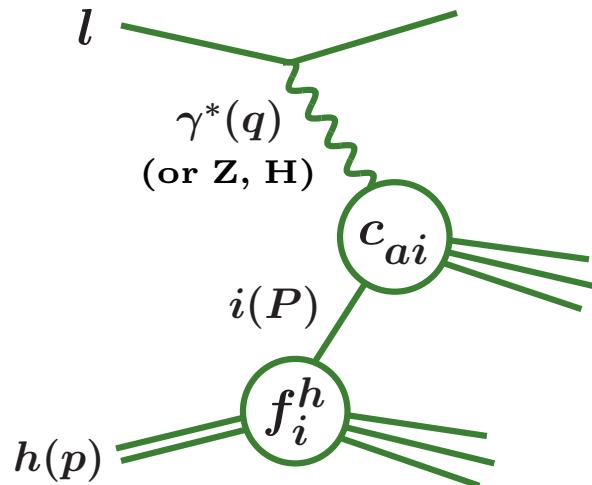
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- Non-singlet NNLL (NLL for DY) resummation from physical kernels  
MV, arXiv:0902.2342 (JHEP), 0909.2124 (JHEP)
- Singlet NNLL for fourth-order splitting functions and  $F_L$  in DIS  
SMVV, 0912.0369 (NPB), 1008.0952 (Loops & Legs 2010)
- Generalized threshold resummation in inclusive DIS and SIA  
A.V., 1005.1606 (PLB); ASV, 1012.3352 (JHEP); ALPV, 1202.5224 (Radcor 2011), ...
- Small- $x$  resummation of splitting & coefficient functions in SIA and DIS  
A.V., arXiv:1108.2993 (JHEP); KKY, arXiv:1207.5631 (JHEP); ...

# Hard lepton-hadron processes in pQCD

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Inclusive deep-inelastic scattering (DIS), semi-incl.  $l^+l^-$  annihilation (SIA)



Left → right: DIS,  $q$  spacelike,  $Q^2 = -q^2$

$P = \xi p$ ,  $f_i^h$  = parton distributions

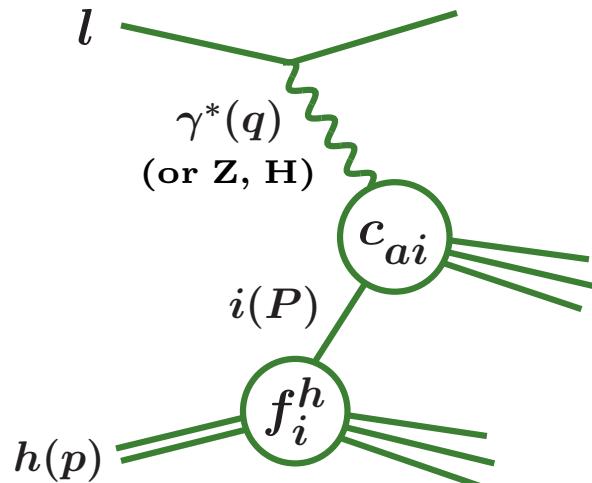
Top → bottom:  $l^+l^-$ ,  $q$  timelike,  $Q^2 = q^2$

$p = \xi P$ , fragmentation distributions

Drell-Yan (DY)  $l^+l^-$  production: bottom → top, 2<sup>nd</sup> hadron from right ( $\{\dots\}$ )

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Structure functions, fragmentation functions etc  $F_a$ : coefficient functions

$$F_a(x, Q^2) = \left[ \textcolor{red}{C}_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2) \{ \otimes f_j^{h'}(\mu^2) \} \right](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables:  $x = Q^2/(2p \cdot q)$  in DIS etc.  $\mu$ : renorm./mass-fact. scale

# Splitting and coefficient functions in pQCD

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Parton/fragmentation distributions  $f_i$ : (renorm. group) evolution equations

$$\frac{d}{d \ln \mu^2} f_i(\xi, \mu^2) = \left[ P_{ik/k_i}^{S/T}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](\xi), \quad \otimes : \text{Mellin convolution}$$

Initial conditions: incalculable, fit-analyses of reference processes

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Expansion in  $\alpha_s$ : splitting functions  $P$ , coefficient fct's  $c_a$  of observables

$$P = \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \dots$$
$$C_a = \underbrace{\alpha_s^{n_a} \left[ c_a^{(0)} + \alpha_s c_a^{(1)} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \dots \right]}_{}$$

NLO: first real prediction of size of cross sections

NNLO,  $P^{(2)}$ ,  $c_a^{(2)}$ : first serious error estimate of pQCD predictions

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$N^3LO$ : for high precision ( $\alpha_s$  from DIS), slow convergence (Higgs in  $pp/p\bar{p}$ )

$F_2/F_3$ : MVV (2005/8),  $P_{ns, N=2}^{(3)}$ : Baikov, Chetyrkin (06), ...;  $\sigma_{H,\text{soft}}$ : MV (05), ...

Endpoint logarithms for  $x \rightarrow 0, 1$ : resummation can be useful or necessary

# $\overline{\text{MS}}$ splitting functions at large $x$ / large $N$

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Mellin trf.  $f(N) = \int_0^1 dx (x^{N-1} \{-1\}) f(x)_{\{+\}}$ : M-convolutions  $\rightarrow$  products

$$\frac{\ln^n(1-x)}{(1-x)_+} \stackrel{\text{M}}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \dots, \quad \ln^n(1-x) \stackrel{\text{M}}{=} \frac{(-1)^n}{N} \ln^n N + \dots$$

Diagonal splitting functions: no higher-order enhancement at  $N^0, N^{-1}$

$$P_{\text{qq/gg}}^{(\ell-1)}(N) = A_{\text{q/g}}^{(\ell)} \ln N + B_{\text{q/g}}^{(\ell)} + C_{\text{q/g}}^{(\ell)} \frac{1}{N} \ln N + \dots, \quad A_{\text{g}} = C_A/C_F A_{\text{q}}$$

$\dots$ ; Korchemsky (89); MVV(04); Dokshitzer, Marchesini, Salam (05)

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Off-diagonal: double-log behaviour, colour structure with  $C_{AF} = C_A - C_F$

$$\begin{aligned} C_F^{-1} P_{\text{gq}}^{(\ell)} / n_f^{-1} P_{\text{qg}}^{(\ell)} &= \frac{1}{N} \ln^{2\ell} N \ # C_{AF}^l \\ &+ \frac{1}{N} \ln^{2\ell-1} N (\ # C_{AF} + \ # C_F + \ # n_f) C_{AF}^{l-1} + \dots \end{aligned}$$

Double logs  $\ln^n N$ ,  $\ell+1 \leq n \leq 2\ell$  vanish for  $C_F = C_A$  ( $\rightarrow$  SUSY case)

# $\overline{\text{MS}}$ coefficient functions at large $x$ / large $N$

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'Diagonal' [ $\mathcal{O}(1)$ ] coeff. fct's for  $F_{2,3,\phi}$  in DIS,  $F_{T,A,\phi}$  in SIA,  $F_{\text{DY}} = \frac{1}{\sigma_0} \frac{d\sigma_{q\bar{q}}}{dQ^2}$

$$C_{2,q/\phi,g/\dots}^{(\ell)} = \# \ln^{2\ell} N + \dots + N^{-1} (\# \ln^{2\ell-1} N + \dots) + \dots$$

$N^0$  parts: threshold exponentiation      Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log ( $N^3\text{LL}$ ) accuracy - mod.  $A^{(4)}$   
⇒ highest seven (DIS, SIA), six (DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05)  
(+ SCET papers, from 06), SIA: Blümlein, Ravindran (06); MV, arXiv:0908.2746 (PLB)

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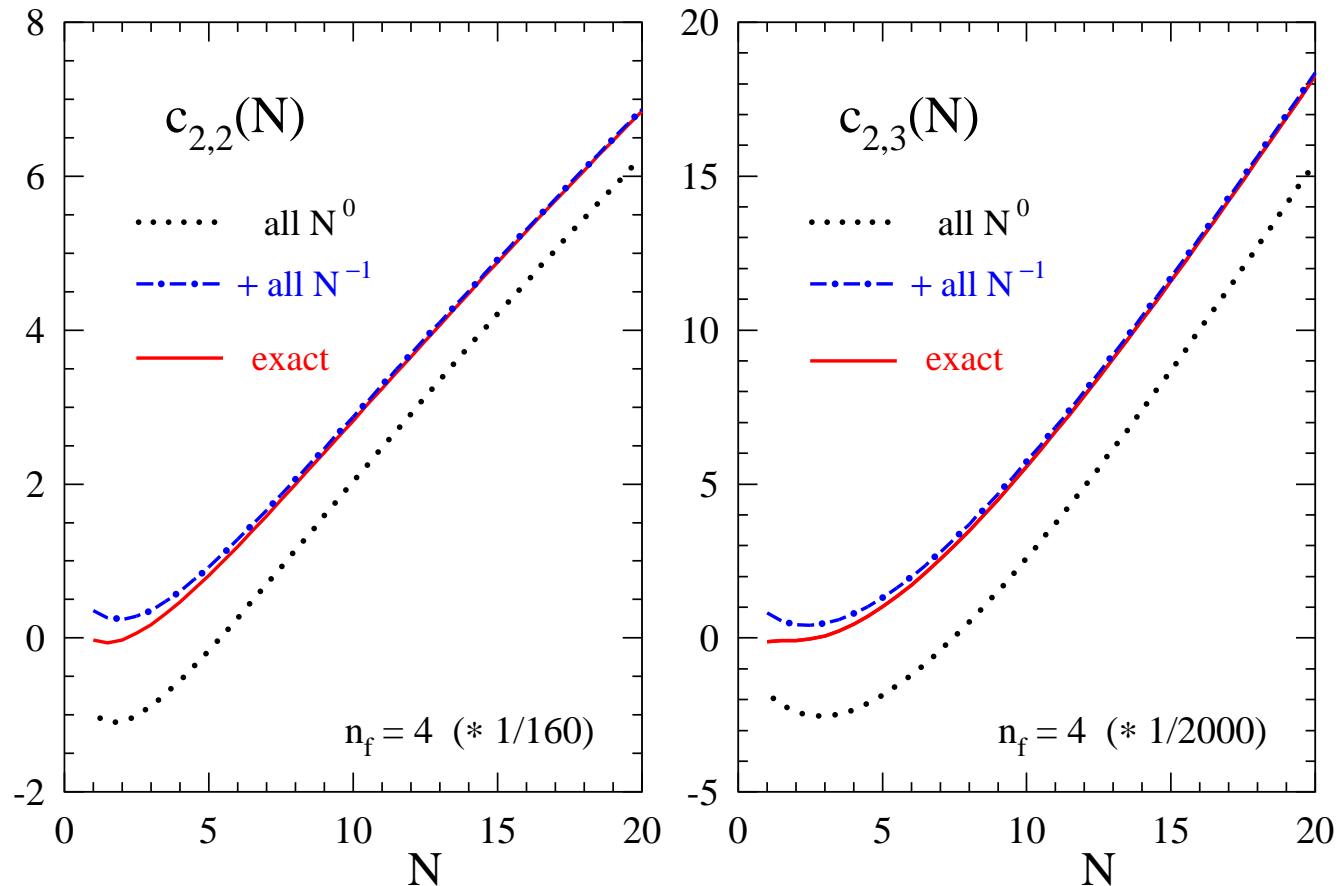
‘Off-diagonal’ [ $\mathcal{O}(\alpha_s)$ ] quantities: leading  $N^{-1}$  double logarithms

$$C_{\phi,q/2,g/\dots}^{(\ell)} = N^{-1} (\# \ln^{2\ell-1} N + \dots) + \dots$$

Longitudinal DIS/SIA structure functions [convention:  $\ell = \text{order in } \alpha_s - 1$ ]

$$C_{L,q}^{(\ell)} = N^{-1} (\# \ln^{2\ell} N + \dots) + \dots, \quad C_{L,g}^{(\ell)} = N^{-2} (\# \ln^{2\ell} N + \dots) + \dots$$

# Second- and third-order $N$ -space $C_{2,\text{ns}}$ in DIS



$N^{-1}$  terms relevant over full range shown,  $\mathcal{O}(N^{-2})$  sizeable only at  $N < 5$

Sum of  $N^{-1} \ln^k N$  looks almost constant: half of maximum only at  $N \simeq 150$

DIS  $\rightarrow$  SIA  $\rightarrow$  DY : increase of the  $N^0$  terms,  $N^{-1}$  corrections less important

# $\overline{\text{MS}}$ splitting functions at small $x/N \rightarrow 1$ or 0

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Logs in  $x$ -space  $\Leftrightarrow$  poles in  $N$ -space,  $x^a \ln^n x \stackrel{\text{M}}{=} \frac{(-1)^n n!}{(N+a)^{n+1}}$

Space-like case, non-singlet: no  $x^{-1}$  terms, leading  $x^0$  double logarithms :

LL: Kirschner, Lipatov (83); Blümlein, A.V. (95)

Singlet quantities: dominant  $x^{-1}$  terms single-log enhanced

$$P_{ij}^{(\ell)S} = x^{-1} (\# \ln^{\ell - \delta_{iq}} x + \dots) + (\# \ln^{2\ell} x + \dots) + \dots$$

$x^{-1}$  part: BFKL (77/78); Jaroszewicz (82); Catani, Fiorani, Marchesini (89);  
Catani, Hautmann (94); ..., Fadin, Lipatov; Camici, Ciafaloni (98)

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Timelike case: huge  $x^{-1}$  double logarithms

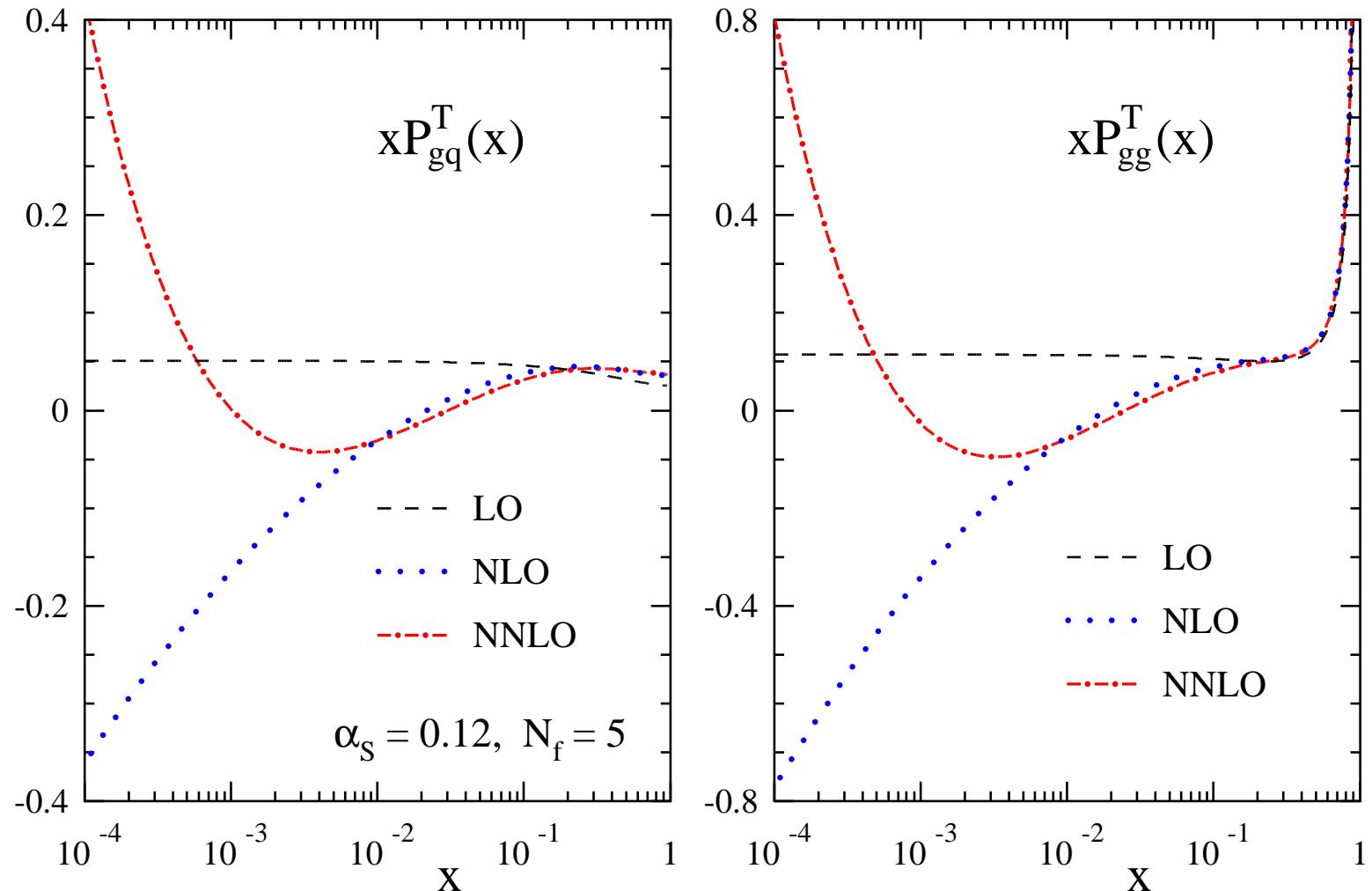
$$P_{ij}^{(\ell)T} = x^{-1} (\# \ln^{2\ell-\delta_{iq}} x + \dots) + (\# \ln^{2\ell} x + \dots) + \dots$$

LL: Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82). NLL: Mueller (83)  
– but latter not in  $\overline{\text{MS}}$ , see Albino, Bolzoni, Kniehl, Kotikov (11)

Behaviour of gauge-boson exchange coefficient functions analogous

# NNLO approximations for $P_{gi}^T(x, \alpha_s)$

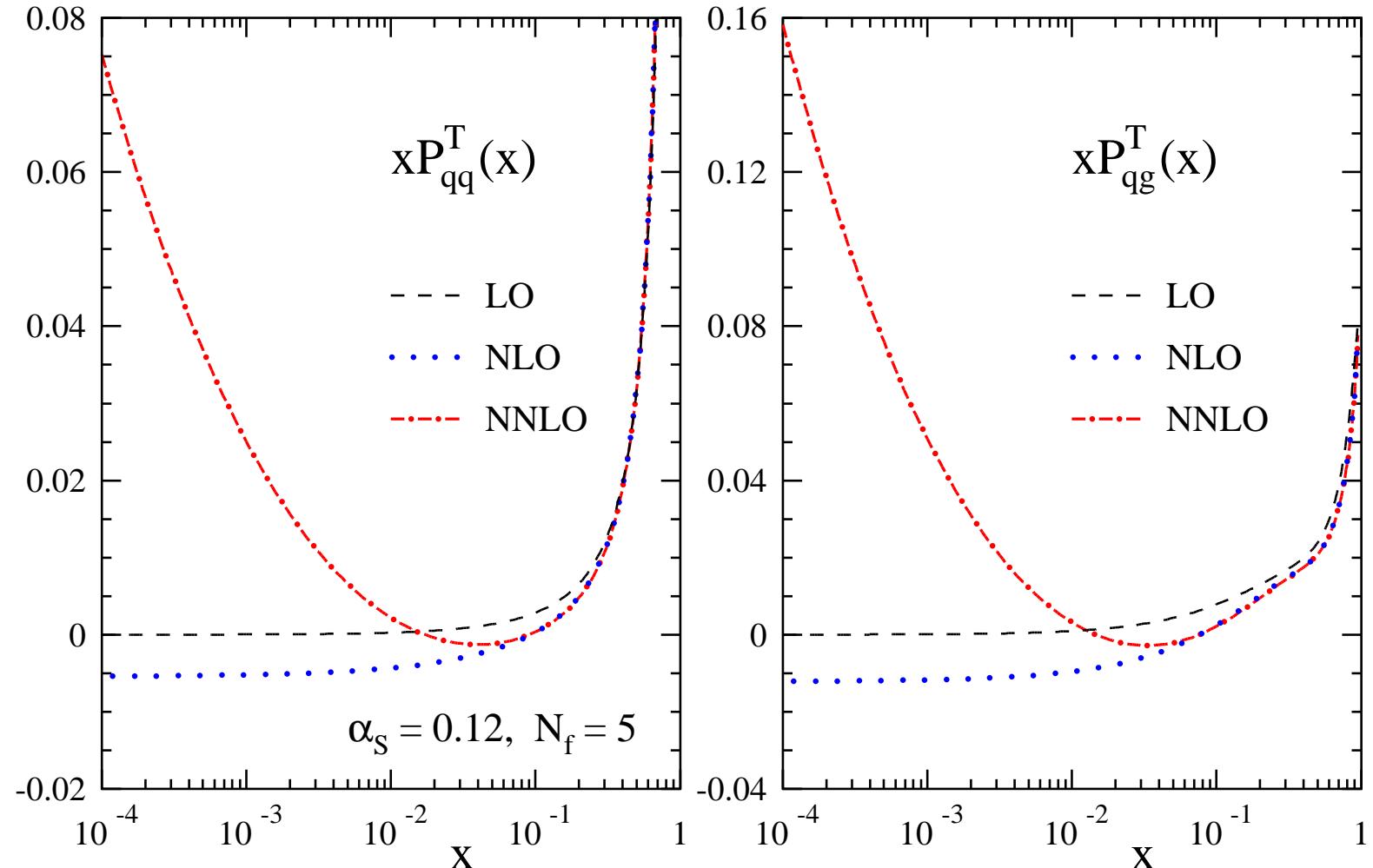
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NLO/NNLO: terms up to  $x^{-1} \ln^2 x / x^{-1} \ln^4 x$ . Unstable at  $x \lesssim 0.005$

# NNLO approximations for $P_{qi}^T(x, \alpha_s)$

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NLO: no  $x^{-1} \ln x$  terms. NNLO: up to  $x^{-1} \ln^3 x$ . Unstable at  $x \lesssim 0.02$

# Large- $N$ cross sections before factorization

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Unfactorized partonic structure functions in  $D = 4 - 2\epsilon$  dimensions

$$T_{a,j} = \tilde{C}_{a,i} Z_{ij}, \quad -\gamma \equiv P = \frac{dZ}{d \ln Q^2} Z^{-1}, \quad \frac{da_s}{d \ln Q^2} = -\epsilon a_s + \beta_{D=4}$$

$N^0$  and  $N^{-1}$  transition functions,  $D$ -dimensional coefficient functions

$$\begin{aligned} Z \Big|_{a_s^n} &= \frac{1}{\epsilon^n} \frac{\gamma_0^{n-1}}{n!} \left[ \gamma_0 - \frac{\beta_0}{2} n(n-1) \right] + \sum_{\ell=1}^{n-1} \frac{1}{\epsilon^{n-\ell}} \sum_{k=1}^{n-\ell-1} \gamma_0^{n-\ell-k-1} \gamma_\ell \gamma_0^k \frac{(\ell+k)!}{n! \ell!} \\ &\quad - \frac{\beta_0}{2} \sum_{\ell=1}^{n-2} \frac{1}{\epsilon^{n-\ell}} \sum_{k=1}^{n-\ell-2} \gamma_0^{n-\ell-k-2} \gamma_\ell \gamma_0^k \frac{(\ell+k)!}{n! \ell!} (n(n-1) - \ell(\ell+k+1)) \\ &\quad + \text{NNLL contributions (explicit expressions)} + \dots \end{aligned}$$

$$\tilde{C}_{a,i} = 1_{(\text{diagonal cases})} + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} a_s^n \epsilon^\ell c_{a,i}^{(n,\ell)}, \quad \ell \text{ additional logs at order } \epsilon^\ell$$

$\alpha_s^n \epsilon^{-n+\ell}$  off-diagonal entries: contributions up to  $N^{-1} \ln^{n+\ell-1} N$

Full  $N^m$ LO calc. of  $T_{a,j}$ : highest  $m+1$  powers of  $\epsilon^{-1}$  to all orders in  $\alpha_s$

Extension to all  $\epsilon$  for highest  $n+1$  logarithms:  $N^n$ LL all-order resummation

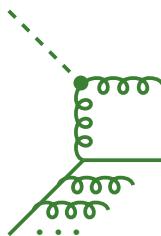
# All-order off-diagonal leading-log amplitudes

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**Example: Leading-log (LL)  $1/N$  terms of  $T_{\phi,q}^{(n)}$  and  $T_{2,g}^{(n)}$ , with  $L \equiv \ln N$**

$$\frac{1}{C_F} T_{\phi,q}^{(n)} = \frac{1}{n_f} T_{2,g}^{(n)} = \frac{L^{n-1}}{N \varepsilon^n} \sum_{k=0}^{\infty} (\varepsilon L)^k \mathcal{L}_{n,k} \left( C_F^{n-1} + C_F^{n-2} C_A + \dots + C_A^{n-1} \right)$$

⇒ all-order relation for one colour structure of either amplitude sufficient



$$T_{\phi,q}^{(n)} \Big|_{C_F \text{ only}} \stackrel{\text{LL}}{=} \frac{1}{n} T_{\phi,q}^{(1)} \underbrace{T_{2,q}^{(n-1)}}_{\frac{1}{(n-1)!} (T_{2,q}^{(1)})^{n-1}} \stackrel{\text{LL}}{=} \frac{1}{n!} T_{\phi,q}^{(1)} (T_{2,q}^{(1)})^{n-1}$$

$$\Rightarrow T_{\phi,q} \Big|_{C_F \text{ only}} \stackrel{\text{LL}}{=} T_{\phi,q}^{(1)} \frac{\exp(a_s T_{2,q}^{(1)}) - 1}{T_{2,q}^{(1)}}$$

Exact  $D$ -dimensional leading-log expressions for the one-loop amplitudes

$$T_{\phi,q}^{(1)} \stackrel{\text{LL}}{=} -2C_F \frac{1}{\varepsilon} (1-x)^{-\varepsilon} \stackrel{\text{M}}{=} -\frac{2C_F}{N} \frac{1}{\varepsilon} \exp(\varepsilon \ln N)$$

$$T_{2,q}^{(1)} \stackrel{\text{LL}}{=} -4C_F \frac{1}{\varepsilon} (1-x)^{-1-\varepsilon} + \text{virtual} \stackrel{\text{M}}{=} 4C_F \frac{1}{\varepsilon^2} (\exp(\varepsilon \ln N) - 1)$$

# Next-to-leading logarithmic iteration for $T_{H,\mathbf{q}}^{(n)}$

---

Ansatz for  $T_{H,\mathbf{q}}^{(n)}$  in terms of first-order quantity and diagonal amplitudes

$$T_{H,\mathbf{q}}^{(n)} \stackrel{\text{NL}}{=} \frac{1}{n} T_{H,\mathbf{q}}^{(1)} \left\{ \sum_{i=0}^{n-1} T_{H,\mathbf{g}}^{(i)} T_{2,\mathbf{q}}^{(n-i-1)} f(n, i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T_{H,\mathbf{g}}^{(i)} T_{2,\mathbf{q}}^{(n-i-2)} g(n, i) \right\}$$

All-order agreement with known highest four powers of  $\varepsilon^{-1}$  for

$$f(n, i) = \binom{n-1}{i}^{-1} \left[ 1 + \varepsilon \left( \frac{\beta_0}{8C_A} (i+1)(n-i) \theta_{i1} - \frac{3}{2} (1 - n \delta_{i0}) \right) \right]$$

$$g(n, i) = \binom{n}{i+1}^{-1}$$

LL: A.V., L&L 2010

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$$g(n, i) = \binom{n}{i+1}^{-1} \quad \text{LL: A.V., L&L 2010}$$

Soft-gluon exponentiation: also  $T_{H,g}^{(n)}$  and  $T_{2,q}^{(n)}$  known at all powers of  $\varepsilon$   
⇒ next-to-leading logarithmic expression for  $T_{H,q}$  completely predicted

Mass factorization ⇒  $P_{gq}^{\text{NLL}}, c_{H,q}^{\text{NLL}}$  to all orders.  $P_{qg}^{\text{NLL}}, c_{2,g}^{\text{NLL}}$  analogous

Extension of this approach to higher-log accuracy (at least) cumbersome

# *D*-dim. structure of unfactorized observables

---

**Maximal phase space for deep-inelastic scattering/semi-incl. annihilation**

NLO :  $2 \rightarrow 2 / 1 \rightarrow 1 + 2$   $(1-x)^{-\varepsilon} x^{\dots} \int_0^1$  **one other variable**

$N^2$ LO :  $2 \rightarrow 3 / 1 \rightarrow 1 + 3$   $(1-x)^{-2\varepsilon} x^{\dots} \int_0^1$  **four other variables**

$N^3$ LO :  $2 \rightarrow 4 / 1 \rightarrow 1 + 4$   $(1-x)^{-3\varepsilon} x^{\dots} \int_0^1$  **seven other variables**

...

**$N^2$ LO:** Matsuura, van Neerven (88), Rijken, vN (95),  **$N^{n \geq 3}$ LO,** indirectly: MV[V] (05)

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$$\text{N}^3\text{LO} : 2 \rightarrow 4 / 1 \rightarrow 1 + 4 \quad (1-x)^{-3\varepsilon} x^{\dots} \int_0^1 \text{seven other variables}$$

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**N<sup>2</sup>LO:** Matsuura, van Neerven (88), Rijken, vN (95), **N<sup>n≥3</sup>LO**, indirectly: MV[V] (05)

**Purely real contributions to unfactorized structure functions**

$$T_{a,j}^{(n)\text{R}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\xi=0} (1-x)^{-1+\xi-n\varepsilon} \left\{ R_{a,j,\xi}^{(n)\text{LL}} + \varepsilon R_{a,j,\xi}^{(n)\text{NLL}} + \dots \right\}$$

**Mixed contributions (2 → r+1 with n-r loops in DIS)**

$$T_{a,j}^{(n)\text{M}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=r}^n \sum_{\xi=0} (1-x)^{-1+\xi-\ell\varepsilon} \left\{ M_{a,j,\ell,\xi}^{(n)\text{LL}} + \varepsilon M_{a,j,\ell,\xi}^{(n)\text{NLL}} + \dots \right\}$$

**Purely virtual part (diagonal cases,  $\xi = 0$  present):  $\gamma^* \text{qq}$ ,  $H_{gg}$  form factors**

$$T_{a,j}^{(n)\text{V}} = \delta(1-x) \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\text{LL}} + \varepsilon V_{a,j}^{(n)\text{NLL}} + \dots \right\}$$

# Resulting resummation of large- $x$ double logs

---

KLN cancellation between purely real, mixed and purely virtual contributions

$$T_{a,j}^{(n)} = T_{a,j}^{(n)\text{R}} + T_{a,j}^{(n)\text{M}} \left( + T_{a,j}^{(n)\text{V}} \right) = \frac{1}{\varepsilon^n} \left\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \dots \right\}$$

$\Rightarrow$  Up to  $n-1$  relations between the coeff's of  $(1-x)^{-\ell\varepsilon}$ ,  $\ell = 1, \dots, n$

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Present situation: (a)  $N^3$ LO for non-singlet  $F_{a \neq L}$  in DIS – recall DMS (05)  
(b)  $N^2$ LO for SIA, non-singlet  $F_L$  in DIS, and singlet DIS

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Soft-gluon exponentiation of  $(1-x)^{-1}/N^0$  diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \dots, (1-x)^{-1-(n-1)\varepsilon}$  at order  $n$ : products of lower-order quantities

$\Rightarrow N^n$ LO  $[+A^{(n+1)}] \rightarrow N^n$ LL exponentiation;  $2n[+1]$  highest logs predicted

# NS results, off-diagonal splitting fct's and $C_{L,g}$

---

NS: physical-kernel results confirmed and extended by fourth log for  $c_{a,\text{ns}}^{(n \geq 4)}$

also: Grunberg (2010)

## Off-diagonal splitting functions

$$NP_{\text{qg}}^{\text{NL}}(N, \alpha_s) = 2a_s n_f \mathcal{B}_0(\tilde{a}_s) + a_s^2 \ln \tilde{N} n_f \left\{ (6C_F - \beta_0) \left( \frac{2}{\tilde{a}_s} \mathcal{B}_{-1}(\tilde{a}_s) + \mathcal{B}_1(\tilde{a}_s) \right) + \frac{\beta_0}{\tilde{a}_s} \mathcal{B}_{-2}(\tilde{a}_s) \right\}$$

$$NP_{\text{gq}}^{\text{NL}}(N, \alpha_s) = 2a_s C_F \mathcal{B}_0(-\tilde{a}_s) + a_s^2 \ln \tilde{N} C_F \left\{ (12C_F - 6\beta_0) \frac{1}{\tilde{a}_s} \mathcal{B}_{-1}(-\tilde{a}_s) - \frac{\beta_0}{\tilde{a}_s} \mathcal{B}_{-2}(-\tilde{a}_s) + (14C_F - 8C_A - \beta_0) \mathcal{B}_1(-\tilde{a}_s) \right\}$$

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Gluon contribution to  $F_L$  – ‘non-singlet’  $C_F = 0$  part done before MV (09)

$$N^2 C_{L,g}^{\text{NL}}(N, \alpha_s) = 8a_s n_f \exp(2C_A a_s \ln^2 \tilde{N}) + 4a_s C_F N C_{2,g}^{\text{LL}}(N, \alpha_s) + 16a_s^2 \ln \tilde{N} n_f \left\{ 4C_A - C_F + \frac{1}{3} a_s \ln^2 \tilde{N} C_A \beta_0 \right\} \exp(2C_A a_s \ln^2 \tilde{N})$$

New: also NNLL terms now in closed form ( $\mathcal{B}_{-4} \dots \mathcal{B}_2$ ) A. Almasy, A.V.

# Resummed gluon coefficient function for $F_2$

---

$$NC_{2,g}(N, \alpha_s) =$$

$$\begin{aligned}
& \frac{1}{2 \ln \tilde{N}} \frac{n_f}{C_A - C_F} \left[ \exp(2a_s C_F \ln^2 \tilde{N}) \mathcal{B}_0(a_s^3) - \exp(2a_s C_A \ln^2 \tilde{N}) \right] \\
& - \frac{1}{8 \ln^2 \tilde{N}} \frac{n_f (3C_F - \beta_0)}{(C_A - C_F)^2} \left[ \exp(2a_s C_F \ln^2 \tilde{N}) \mathcal{B}_0(a_s^3) - \exp(2a_s C_A \ln^2 \tilde{N}) \right] \\
& - \frac{a_s}{4} \frac{n_f}{C_A - C_F} \exp(2a_s C_A \ln^2 \tilde{N}) (8C_A + 4C_F - \beta_0) \\
& - \frac{a_s}{4} \frac{n_f}{C_A - C_F} \exp(2a_s C_F \ln^2 \tilde{N}) \left[ -6C_F \mathcal{B}_0(a_s^3) - (6C_F - \beta_0) \mathcal{B}_1(a_s^3) \right. \\
& \quad \left. - (12C_F - 4\beta_0) \frac{1}{a_s^3} \mathcal{B}_{-1}(a_s^3) - \frac{\beta_0}{a_s^3} \mathcal{B}_{-2}(a_s^3) \right] \\
& - \frac{a_s^2}{3} \beta_0 \ln^2 \tilde{N} \frac{n_f}{C_A - C_F} \left[ C_A \exp(2a_s C_A \ln^2 \tilde{N}) - C_F \exp(2a_s C_F \ln^2 \tilde{N}) \mathcal{B}_0(a_s^3) \right] \\
& + \text{known NNLL contributions (now in closed form)} + \dots
\end{aligned}$$

$C_{H,q}$  analogous. Analytic form identified via the physical kernel for  $(F_2, F_H)$

Resummed timelike splitting and coefficient functions: same structure

# $\mathcal{B}$ -functions: $\mathcal{B}_0$ and general definition

---

Relation between even- $n$  Bernoulli numbers and the Riemann  $\zeta$ -function

$$\mathcal{B}_0(x) \equiv \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n = 1 - \frac{x}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \zeta_{2n} \left(\frac{x}{2\pi}\right)^{2n}$$

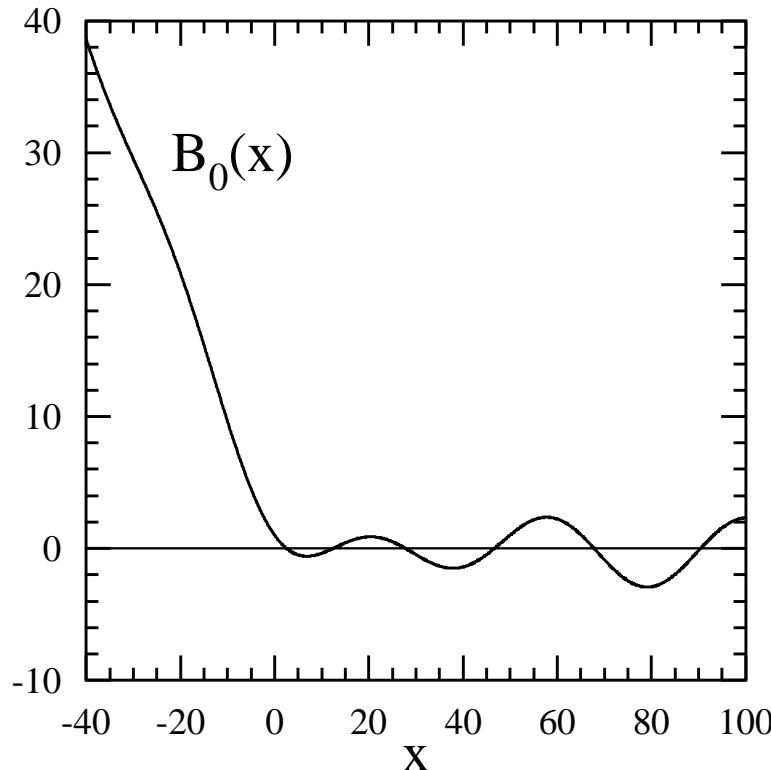
$\mathcal{B}_0(2\pi i)$  numerically known (Wolfram MathWorld, Sloane's A093721), no closed form

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## Further $\mathcal{B}$ -functions

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!} x^n$$

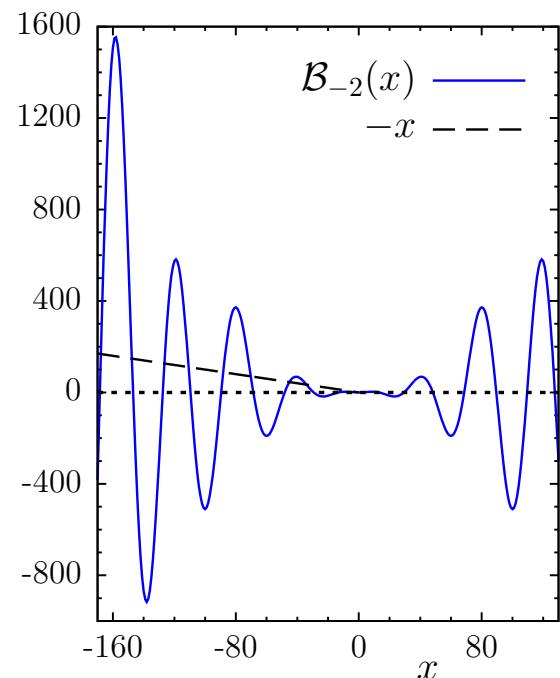
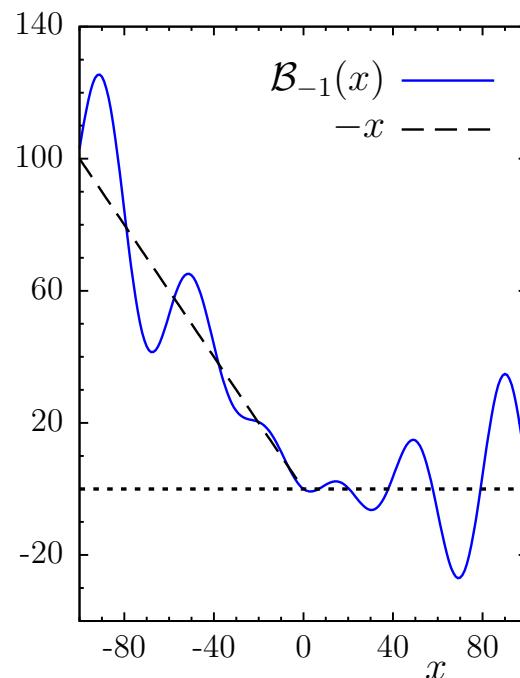
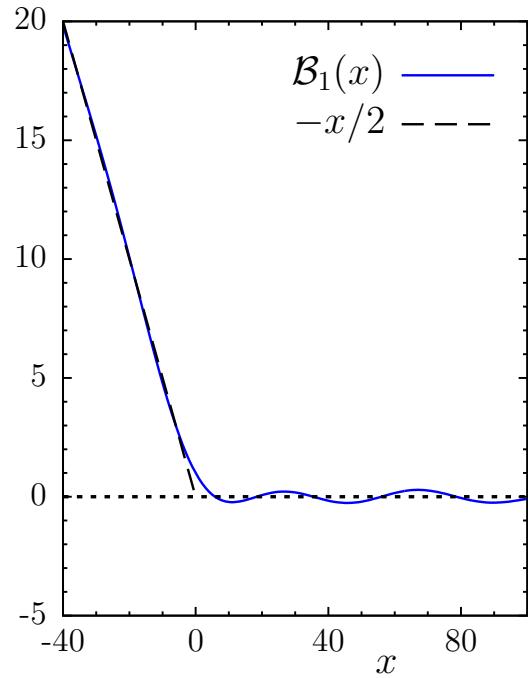
$$\mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!} x^n$$

## Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k} (x^k \mathcal{B}_k) = \mathcal{B}_0, \quad x^k \frac{d^k}{dx^k} \mathcal{B}_0 = \mathcal{B}_{-k}$$

A.V., L&L 2010, arXiv:1005.1606 (PLB)

# $\mathcal{B}$ -functions with index unequal zero



$x > 0$ : all functions  $\mathcal{B}_k(x)$  oscillate about  $y = 0$

$x < 0$ : oscillations about  $y = -\frac{x}{(k+1)!}$  for  $k \geq 0$  and  $y = -x$  for  $k < 0$

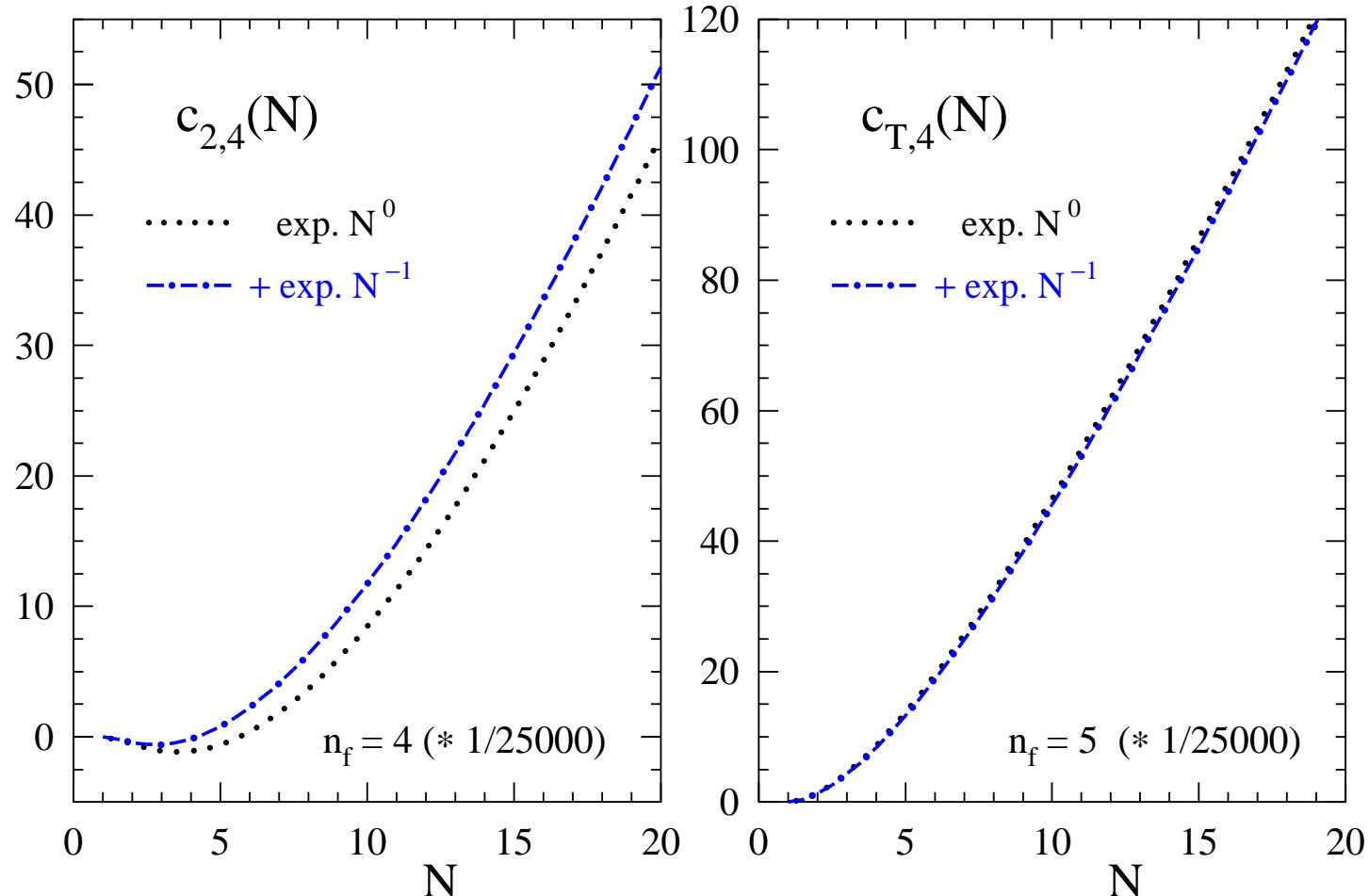
Amplitudes increase very rapidly with decreasing  $k$

Oscillation of  $\mathcal{B}_0$  continues (much more irregularly) to very large  $x$

D. Broadhurst, private communication

# Fourth-order $C_2$ (DIS) and $C_T$ (SIA) at large $N$

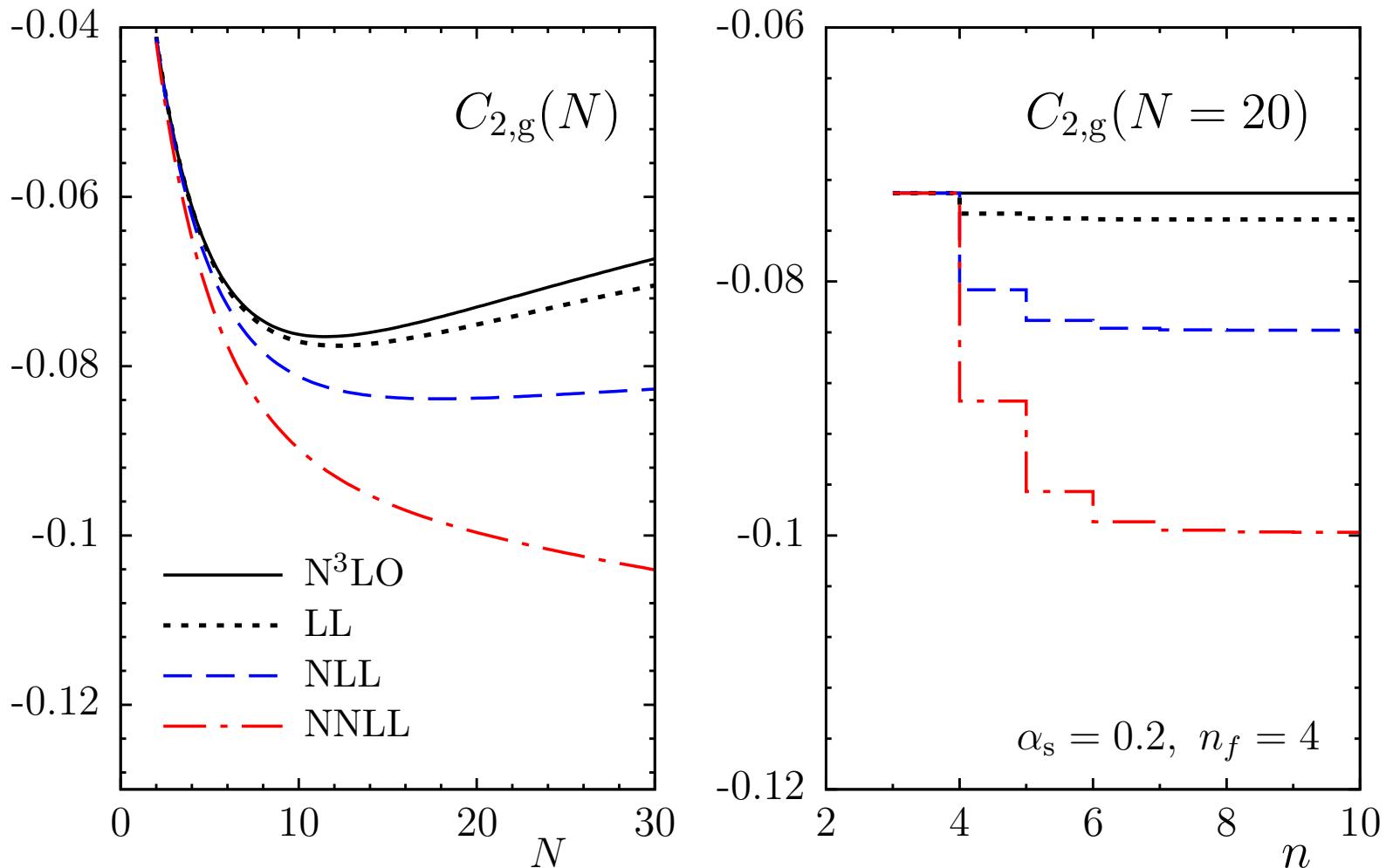
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**Exp.  $N^0$ : 7 of 8 logs, exp.  $N^{-1}$ : 4 of 7 logs  $\Rightarrow$  large- $x$  higher-twist analyses**

$N^{-1}$  contributions again relevant for  $F_2$ , but small for  $F_T$  at least at  $N > 5$

# Numerical illustration of $C_{2,g}$



NNLL terms dominate  $\Rightarrow$  impact of high orders presumably underestimated  
About 35% correction at  $N = 20$ , 4<sup>th</sup>-order coefficient  $\approx$  Padé estimate

# Small- $x$ resummation via unfactorized SIA

---

Phase-space integrations:  $x^{a\varepsilon}$  terms analogous to  $(1-x)^{b\varepsilon}$  large- $x$  factors

2<sup>nd</sup> order: Matsuura, van Neerven (88), Rijken, vN (95)

Decomposition of the  $D$ -dim. partonic fragmentation functions for  $a = T, H$

$$\widehat{F}_{a,g}^{(n)} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} \left\{ A_{a,g}^{(\ell,n)} + \varepsilon B_{a,g}^{(\ell,n)} + \varepsilon^2 C_{a,g}^{(\ell,n)} + \dots \right\}$$

Leading log: terms of the form  $x^{-1} \ln^{n+\delta-1} x$  at all orders  $\varepsilon^{-n+\delta}$  with  $\delta = 0, 1, 2, \dots$ , and  $\widehat{F}_{a,g}^{(n)}$  is decomposed into  $n$  contributions of the form

$$\varepsilon^{-2n+1} x^{-1-k\varepsilon} = \varepsilon^{-2n+1} x^{-1} \left[ 1 - k\varepsilon \ln x + \frac{1}{2}(k\varepsilon)^2 \ln^2 x + \dots \right],$$
$$k = 2, 4, \dots, 2n$$

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$n-1$  KLN-type cancellations –  $\widehat{F}_{a,g}^{(n)}$  starts at order  $1/\varepsilon^n$  – plus 3 constraints from the NNLO results  $\Rightarrow n+2$  linear equations for  $n$  coefficients  $A_{a,g}^{(\ell,n)}$

Thus: N<sup>n</sup>LO known  $\Rightarrow$  highest  $n+1$  (N<sup>n</sup>LL) double logs fixed at all orders

'All-order' mass factorization: NNLL timelike splitting & coefficient functions

# Splitting & coefficient functions, August 2011

---

$$\frac{C_A}{C_F} P_{\text{gq}}^T(N, \alpha_s) \stackrel{\text{LL}}{=} P_{\text{gg}}^T(N, \alpha_s) \stackrel{\text{LL}}{=} \frac{1}{4}(N-1) \left\{ (1 - 4\xi)^{1/2} - 1 \right\}, \quad \xi = -\frac{8C_A a_s}{(N-1)^2}$$

Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82)

NLL contributions to the  $\overline{\text{MS}}$  splitting functions: only partially in closed form

$$\begin{aligned} \left[ P_{\text{gg}}^T \right]_{C_F=0}^{\text{NLL}} &= \left\{ (1 - 4\xi)^{-1/2} + 1 \right\} a_s \left( \frac{11}{6} C_A + \frac{1}{3} n_f \right) \\ \left[ \frac{C_A}{C_F} P_{\text{gq}}^T \right]_{C_F=0}^{\text{NLL}} &= \left[ P_{\text{gg}}^T \right]_{C_F=0}^{\text{NLL}} + \left\{ (1 - 4\xi)^{1/2} - 1 \right\} \frac{1}{24} (N-1)^2 (1 + n_f/C_A) \end{aligned}$$

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LL coefficient functions for  $F_T$  &  $F_H$  [also: Albino, Bolzoni, Kniehl, Kotikov (11)]

$$C_{T,g}^{\text{LL}} = \frac{C_F}{C_A} \left( C_{H,g}^{T,\text{LL}} - 1 \right) = \frac{C_F}{C_A} \left\{ (1 - 4\xi)^{-1/4} - 1 \right\} \quad \text{in } \overline{\text{MS}}$$

‘Everything else’, including all of  $P_{\text{qq}}^T$ ,  $P_{\text{qg}}^T$ , the quark coefficient fct’s,  $C_{L,i}$ :  
 Tables of coefficients to order  $\alpha_s^{16}$  – numerically sufficient for  $x \gtrsim 10^{-4}$  – e.g.

$$P_{\text{gg},\text{NLL}}^{(n)T}(N) = -\frac{(-8)^n C_A^{n-1}}{3(N-1)^{2n}} \left[ (11C_A^2 + 2C_A n_f) B_{\text{gg},1}^{(n)} - 2C_F n_f B_{\text{gg},2}^{(n)} \right]$$

# Normalized LL, NLL splitting-fct. coefficients

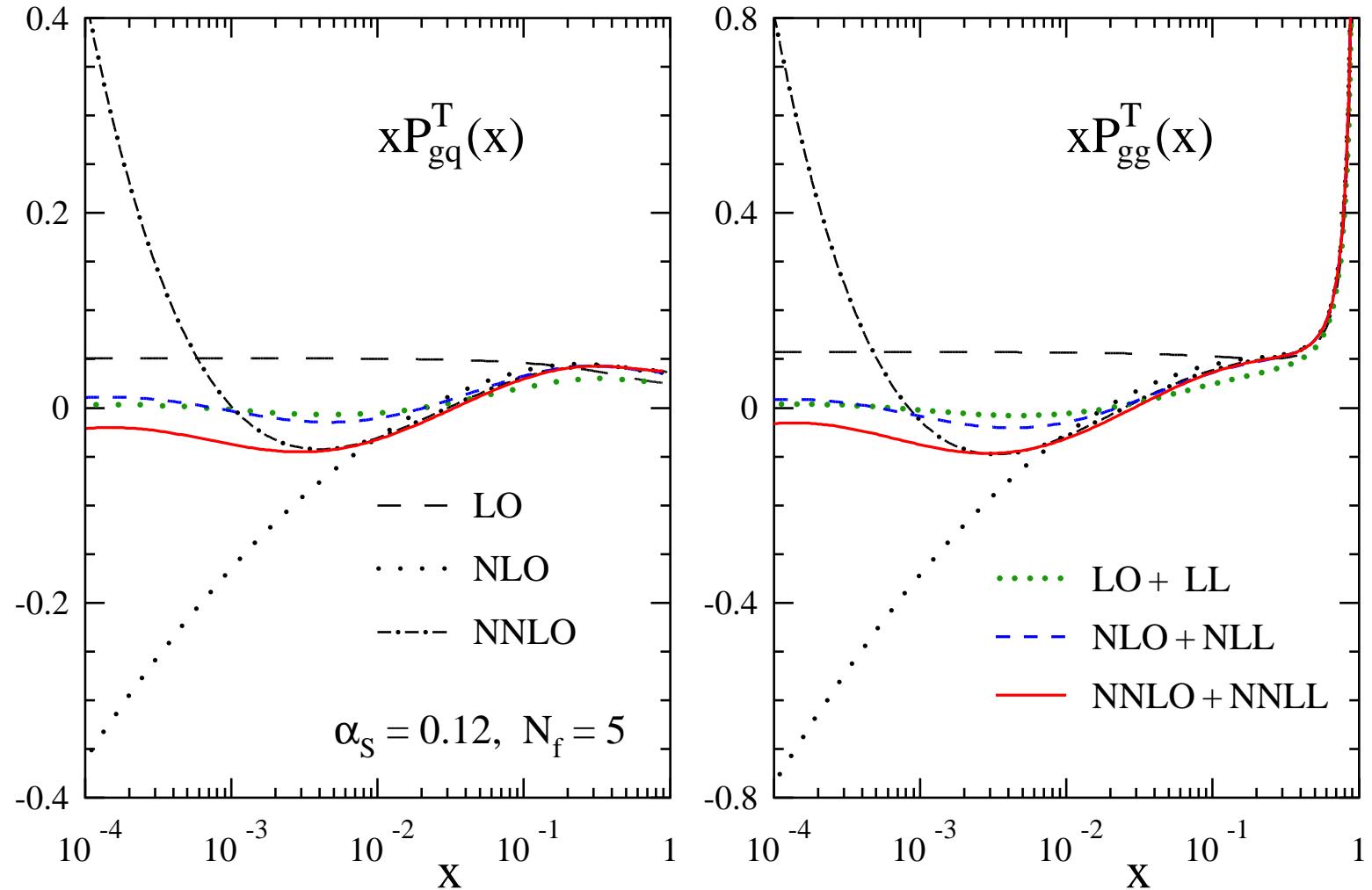
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$n$	$A_{gi}^{(n)}$	$B_{gg,1}^{(n)}$	$B_{gg,2}^{(n)}$	$B_{gq,1}^{(n)}$	$B_{gq,2}^{(n)}$	$B_{gq,3}^{(n)}$	$A_{qi}^{(n)}$
0	1	1	—	9	—	—	—
1	1	1	2	9	—	—	—
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

All integer series known,  $B_{gg,2}^{(n)} - B_{gq,3}^{(n)} = 2A_{gi}^{(n)}$ ,  $A_{qi}^{(n)} + B_{gg,2}^{(n)} = 2B_{gg,1}^{(n)}$

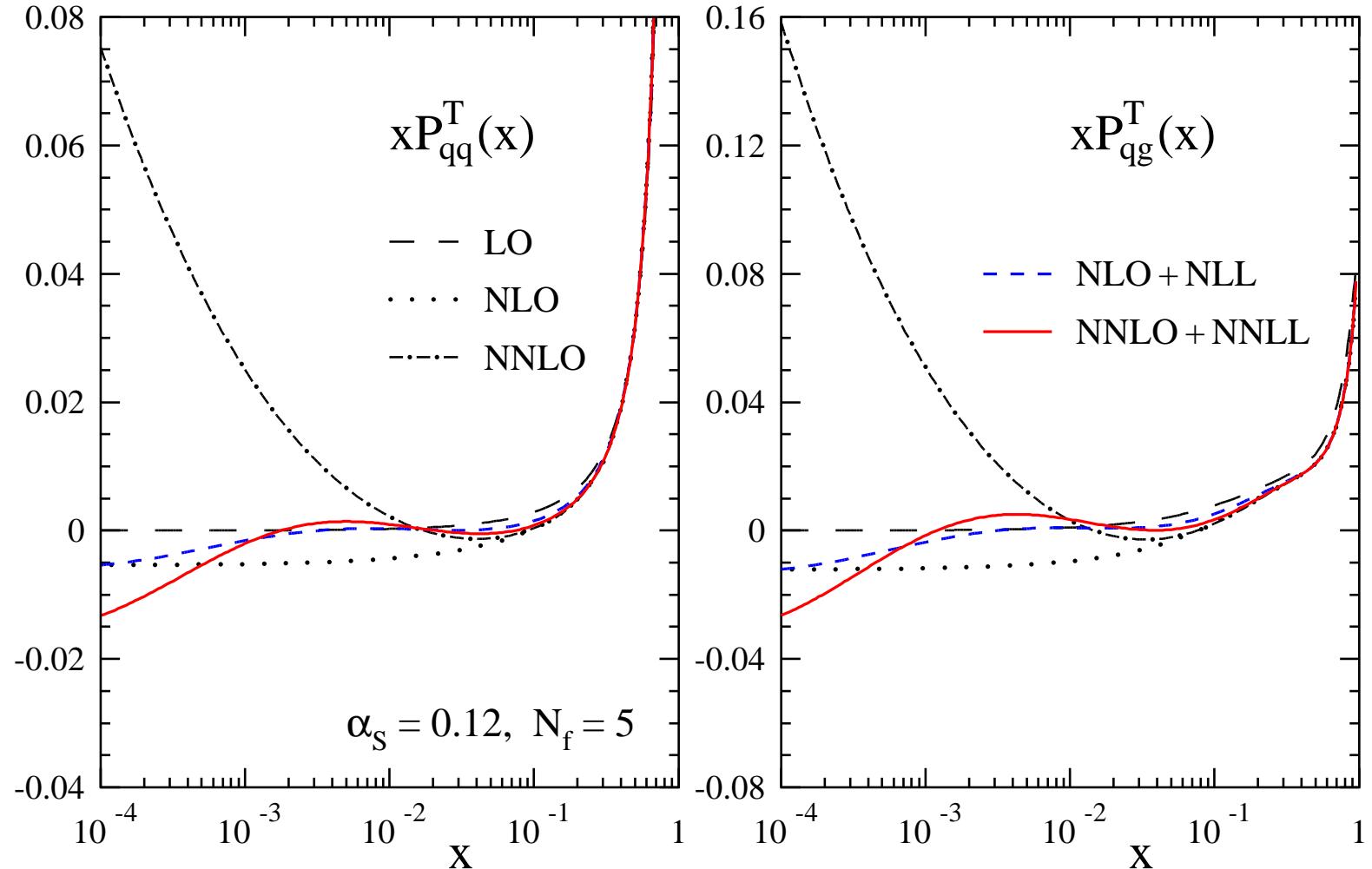
Solution of one non-integer series: analytic structure of all NLL contributions

# Small- $x$ gluon-parton splitting functions



Approximation sequence LO+LL, NLO+NLL, NNLO+NNLL rather stable to very small  $x$

# Small- $x$ quark-parton splitting functions



Also consistent with  $xP_{j_i}^T \approx 0$  at  $x < 10^{-2}$  (N<sup>3</sup>LL corr's known and positive)

# 2012 progress: solution of the $A_{\text{qi}}^{(n)}$ series

---

Denominators  $\leftrightarrow$  triangular numbers (A025555 in OEIS) + ‘playing around’

$$\begin{aligned} A_{\text{qi}}^{(2)} &= 1 = \frac{1}{1}, & A_{\text{qi}}^{(5)} &= \frac{1207}{30} = \frac{14}{10} + \frac{19}{6} + \frac{23}{3} + \frac{28}{1}, \\ A_{\text{qi}}^{(3)} &= \frac{11}{3} = \frac{2}{3} + \frac{3}{1}, & A_{\text{qi}}^{(6)} &= \frac{2015}{15} = \frac{42}{15} = \frac{56}{10} + \frac{66}{6} + \frac{76}{3} + \frac{90}{1}, \\ A_{\text{qi}}^{(4)} &= \frac{73}{6} = \frac{5}{6} + \frac{7}{3} + \frac{9}{1}, & \dots \end{aligned}$$

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$$\Rightarrow A_{\text{qi}}^{(n)} = \frac{2(2n-2)!}{(n-1)!(n+1)!} \left( \frac{1}{n-1} + \frac{1}{n} + \frac{6}{n+1} - 2 \right) + \frac{2(2n)!}{n!(n+1)!} \sum_{k=n}^{2n-3} \frac{1}{k}$$

K. Yeats

Checked to 136<sup>th</sup> entries by ‘brute-force’ determination of  $A_{\text{qi}}^{(n \leq 17)}$

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Generating function

$$A_{\text{qi}}(\xi) = \left( \sqrt{1 - 4\xi} - 1 \right) \left( 1 + \ln \left( \frac{1}{2} [\sqrt{1 - 4\xi} + 1] \right) \right) + 2\xi$$

C.H. Kom

The logarithm is the key for solving all sequences in the 2011 article

# Diagonal splitting functions at NNLL accuracy

---

**Notation:**  $S = (1 - 4\xi)^{1/2}$ ,  $\mathcal{L} = \ln(\frac{1}{2}(1 + S))$  with  $\xi = -8C_A a_s / \bar{N}^2$ ,  $\bar{N} \equiv N - 1$

$$\begin{aligned} P_{\text{qq}}^T(N) &= \frac{4}{3} \frac{C_F n_f}{C_A} a_s \left\{ \frac{1}{2\xi} (S - 1)(\mathcal{L} + 1) + 1 \right\} \\ &+ \frac{1}{18} \frac{C_F n_f}{C_A^3} a_s \bar{N} \left\{ (-11 C_A^2 + 6 C_A n_f - 20 C_F n_f) \frac{1}{2\xi} (S - 1 + 2\xi) + 10 C_A^2 \frac{1}{\xi} (S - 1) \mathcal{L} \right. \\ &- (51 C_A^2 - 6 C_A n_f + 12 C_F n_f) \frac{1}{2} (S - 1) + (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) S^{-1} \mathcal{L} \\ &\left. + (5 C_A^2 - 2 C_A n_f + 6 C_F n_f) \frac{1}{\xi} (S - 1) \mathcal{L}^2 + (51 C_A^2 - 14 C_A n_f + 36 C_F n_f) \mathcal{L} \right\} \end{aligned}$$

$$\begin{aligned} P_{\text{gg}}^T(N) &= \frac{1}{4} \bar{N} (S - 1) - \frac{1}{6 C_A} a_s (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) (S^{-1} - 1) - P_{\text{qq}}^T(N) \\ &+ \frac{1}{576 C_A^3} a_s \bar{N} \left\{ \left( [1193 - 576 \zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 - 56 C_A^2 C_F n_f - 48 C_F^2 n_f^2 \right. \right. \\ &+ 16 C_A C_F n_f^2 \Big) (S - 1) + \left( [830 - 576 \zeta_2] C_A^4 + 96 C_A^3 n_f - 8 C_A^2 n_f^2 - 208 C_A^2 C_F n_f \right. \\ &\left. \left. + 64 C_A C_F n_f^2 - 96 C_F^2 n_f^2 \right) (S^{-1} - 1) + (11 C_A^2 + 2 C_A n_f - 4 C_F n_f)^2 (S^{-3} - 1) \right\} \end{aligned}$$

**First line: LL (for  $P_{\text{gg}}^T$ ) and NLL contributions. Rest: NNLL corrections**

**Off-diagonal splitting functions: similar. For  $P_{\text{qi}}^T$  also  $N^3$ LL terms known**

# NLO + resummed first moments

---

**Fixed-order  $N = 1$  poles removed by the resummation (NLO requires NNLL)**

$$P_{\text{qg}}^T(N=1) = \frac{8}{3} n_f a_s - \frac{1}{3 C_A^2} \left( 17 C_A^2 n_f - 2 C_A n_f^2 + 4 C_F n_f^2 \right) (2 C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2)$$

$$P_{\text{qq}}^T(N=1) = \frac{C_F}{C_A} \left( P_{\text{qg}}^T(N=1) - \frac{4}{3} n_f a_s \right) + \mathcal{O}(a_s^2)$$

$$\begin{aligned} P_{\text{gg}}^T(N=1) &= (2 C_A a_s)^{1/2} - \frac{1}{6 C_A} (11 C_A^2 + 2 C_A n_f + 12 C_F n_f) a_s \\ &\quad + \frac{1}{144 C_A^3} \left( [1193 - 576 \zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 + 760 C_A^2 C_F n_f \right. \\ &\quad \left. - 80 C_A C_F n_f^2 + 144 C_F^2 n_f^2 \right) (2 C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2) \end{aligned}$$

$$P_{\text{gq}}^T(N=1) = \frac{C_F}{C_A} \left( P_{\text{gg}}^T(N=1) + \frac{4}{3} \frac{C_F n_f}{C_A} a_s \right) + \mathcal{O}(a_s^2).$$

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 &\quad \left. - 80 C_A C_F n_f^2 + 144 C_F^2 n_f^2 \right) (2 C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2) \\
 P_{\text{gq}}^T(N=1) &= \frac{C_F}{C_A} \left( P_{\text{gg}}^T(N=1) + \frac{4}{3} \frac{C_F n_f}{C_A} a_s \right) + \mathcal{O}(a_s^2).
 \end{aligned}$$

**Numerically for QCD with  $n_f = 5$ , including  $\mathbf{N^3LL}$  for  $P_{\text{qi}}^T$  ( $\alpha_s^2$  contributions)**

$$\begin{aligned}
 P_{\text{qq}}^T(N=1) &\cong 0.2358 \alpha_s - 0.6773 \alpha_s^{3/2} + 0.5880 \alpha_s^2 \\
 P_{\text{qg}}^T(N=1) &\cong 1.0610 \alpha_s - 1.5240 \alpha_s^{3/2} + 1.8089 \alpha_s^2 \\
 P_{\text{gq}}^T(N=1) &\cong 0.3071 \alpha_s^{1/2} - 0.3059 \alpha_s + 0.2884 \alpha_s^{3/2} \\
 P_{\text{gg}}^T(N=1) &\cong 0.6910 \alpha_s^{1/2} - 0.9240 \alpha_s + 0.6490 \alpha_s^{3/2}
 \end{aligned}$$

# Partial $x$ -space expressions

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**Non-log parts (integer series): Bessel functions in  $z = (32 C_A a_s)^{1/2} \ln \frac{1}{x}$**

$$\begin{aligned} x P_{\text{gg}}^T + x P_{\text{qq}}^T \Big|_{\text{NNLL}} &= \left\{ 4 C_A a_s + \frac{8}{3} (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) a_s^2 \ln \frac{1}{x} \right\} \frac{2}{z} J_1(z) \\ &+ \left\{ \frac{4}{9} (26 C_F n_f - 23 C_A n_f) a_s^2 + \frac{8}{9 C_A} (11 C_A^2 + 2 C_A n_f - 4 C_F n_f)^2 a_s^3 \ln^2 \frac{1}{x} \right\} \frac{2}{z} J_1(z) \\ &+ \frac{32}{9 C_A} \left( [134 - 72 \zeta_2] C_A^4 + 23 C_A^3 n_f - 48 C_A^2 C_F n_f + 4 C_A C_F n_f^2 - 8 C_F^2 n_f^2 \right) a_s^3 \ln^2 \frac{1}{x} \frac{4}{z^2} J_2(z) \end{aligned}$$

$$x P_{\text{gq}}^T(N) - \frac{C_F}{C_A} x P_{\text{gg}}^T \Big|_{\text{NLL}} = - \frac{32}{3} \frac{C_F}{C_A} (C_A^2 + C_A n_f - 2 C_F n_f) a_s^2 \ln \frac{1}{x} \frac{4}{z^2} J_2(z)$$

**Single-logarithmic enhancement of the oscillations at extremely small  $x$**

**$x \rightarrow 0$  dominant  $a_s (a_s \ln \frac{1}{x})^\ell \frac{2}{z} J_1(z)$  terms  $\propto (11 C_A^2 - 2 C_A n_f + 4 C_F n_f)^\ell$**

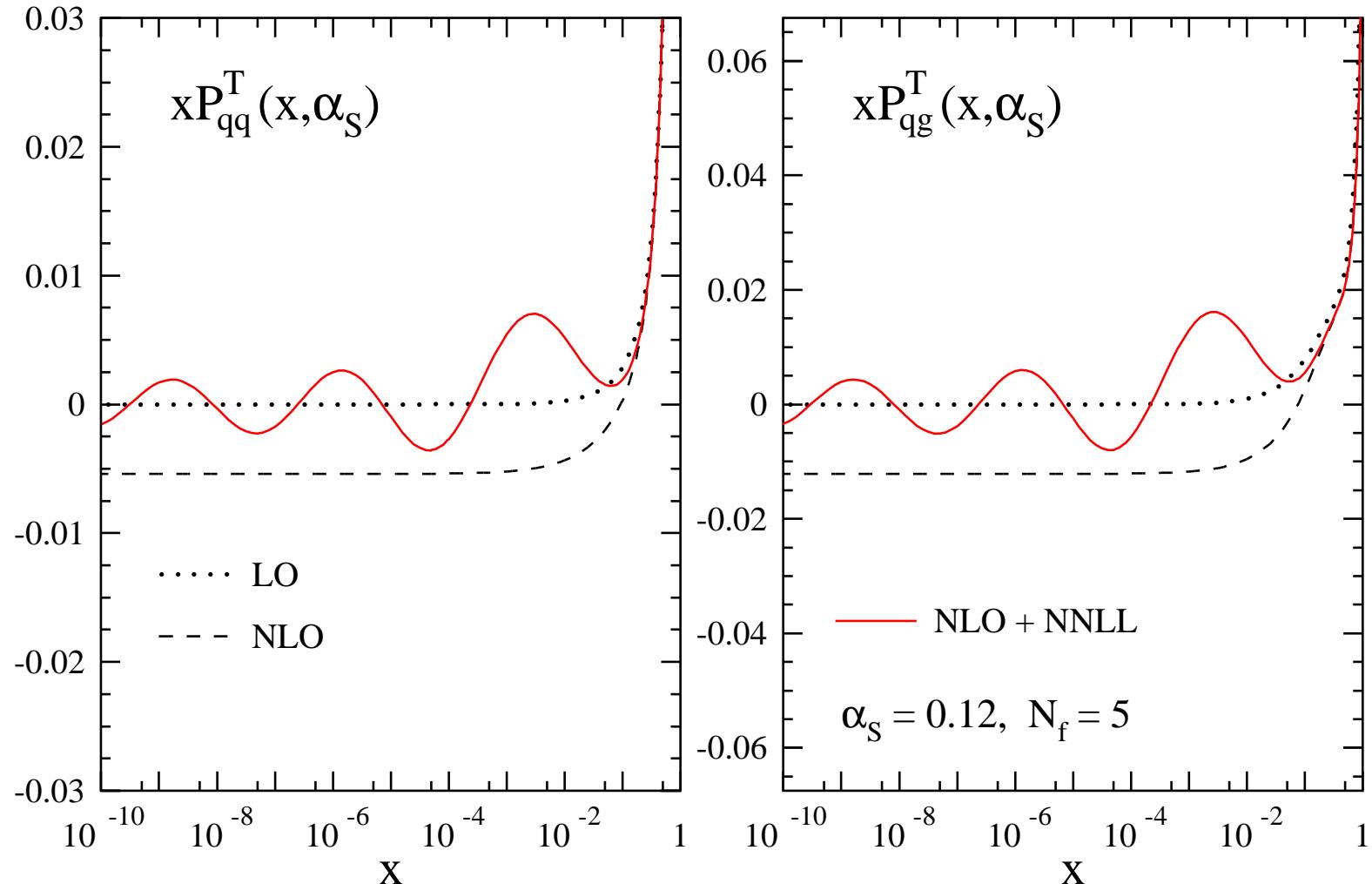
**(non- $C_F$  known to  $\ell = 4$ , see below). Possibility of a ‘second resummation’?**

**For the basic logarithm  $\mathcal{L}$ , unlike  $A_{\text{qi}}(\xi)$ , we found a simple Mellin inverse**

$$\int_0^1 dx x^{N-2} \frac{1}{\ln x} (J_0(2\sqrt{a} \ln x) - 1) = \ln \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4a}{(N-1)^2}} \right)$$

# Resummed quark-parton splitting functions

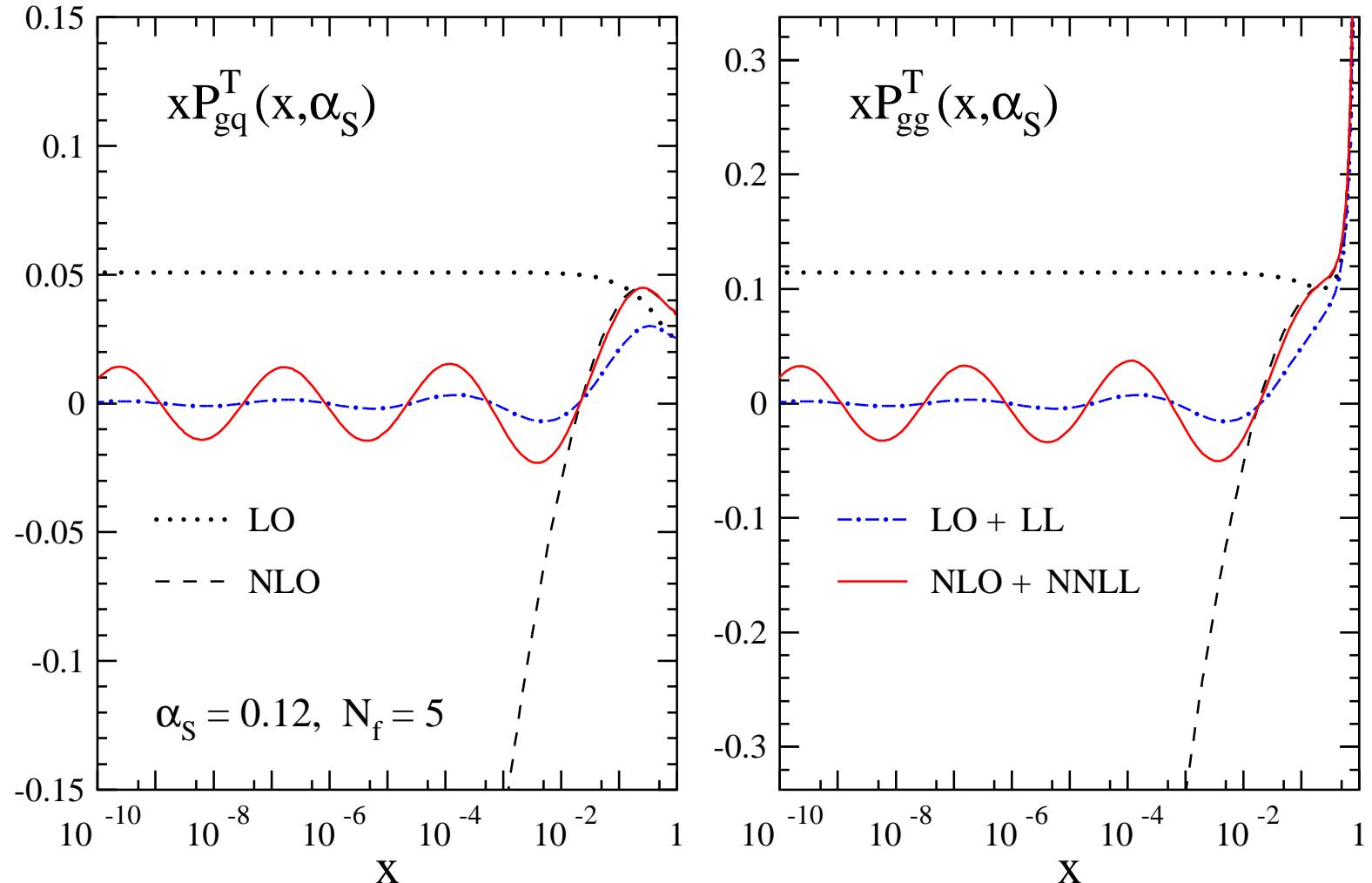
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**Small- $x$   $P_{ig}^T$  vs.  $P_{iq}^T$ :** approximate ‘Casimir scaling’ by factor  $C_A/C_F = 9/4$

# Resummed gluon-parton splitting functions

---



LL contributions numerically small; expect large corrections beyond NNLL

# Towards higher logarithmic accuracy

---

$N=1$  finite NNLO + small- $x$  resummed evolution requires  $N^4\text{LL}$  accuracy

Relation between SIA ( $\sigma = 1, P^T$ ) and DIS ( $\sigma = -1, P^S$ ) parton evolution

$$\frac{\partial}{\partial \ln Q^2} f_\sigma(x, Q^2) = [P_u(\alpha_s(Q^2)) \otimes f_\sigma(z^\sigma Q^2)](x)$$

with  $P_u$  independent of  $\sigma$

Dokshitzer, Marchesini, Salam (2005)

Non-singlet relation ( $n_f = 0$  for  $P_{gg}$ ), but found to hold for all non- $C_F$  terms

$\alpha_s^3$ : Moch, A.V. (2007)

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In  $N$ -space :

$\alpha_s^3$ : Moch, A.V. (2007)

$$\partial_{\ln Q^2} f_\sigma(N, Q^2) = P_\sigma(N) f_\sigma(N, Q^2) = P_u(N + \sigma \partial_{\ln Q^2}) f_\sigma(N, Q^2)$$

$$\Rightarrow P_\sigma(N) = P_u(N) + \sum_{n=1}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^{n-1}}{\partial N^{n-1}} \left( \frac{\partial P_u}{\partial N} [P_u(N)]^n \right)$$

Difference  $\delta P_{gg} = P_{gg}^T - P_{gg}^S$  given by lower-order quantities at any order

$P_{gg}^S$  single-log enhanced (BFKL)  $\rightarrow$  resummation of non- $C_F$  double logs in  $P_{gg}^T$

To NNLL as above, plus  $N^3$ LL and  $N^4$ LL (mod. one BFKL coeff.) corrections

# Large- $x$ summary and outlook

---

- Non-singlet physical kernels for nine observables in DIS, SIA and DY:  
Single-log behaviour  $\Rightarrow$  leading three (DY: two) logs of higher-order  $C_a$
- Singlet kernels for  $(F_2, F_\phi)$  and  $(F_2, F_L)$  in DIS also single-logarithmic  
 $\Rightarrow$  Prediction of three logs in N<sup>3</sup>LO  $\alpha_s^4$  splitting and  $F_L$  coefficient fct's

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- Iterative structure of (next-to) leading-log  $N^{-1}$  amplitudes for  $C_{2,g/\phi,q}$   
 $\Rightarrow$  All-order (N)LL off-diagonal splitting functions and coefficient fct's
- $D$ -dimensional structure of unfactorized DIS/SIA structure functions  
Verification, extension of above results to  $N^3\text{LL}$  or  $N^2\text{LL}$  for  $N^{-1}$  terms

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- Complementary: Grunberg; Laenen, Gardi, Magnea, Stavenga, White
- Applications, now: assess relevance of NS  $1/N$  terms, large- $x$  DIS fits
- Near/mid future: combine with other results, esp. fixed- $N$  calculations  
(close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)
- Beyond-LL extension to Drell-Yan, Higgs prod'n needs more insights

# Small- $x$ summary and outlook

---

- $D$ -dimensional structure of unfactorized SIA/DIS structure functions  
⇒ NNLL small- $x$  resummation of timelike splitting & coefficient fct's  
Required for using NNLO results in SIA below  $x \approx 10^{-2} \dots 10^{-3}$
- Analogous results for (singlet case: subdominant)  $x^0 \ln^\ell x$  terms in DIS  
Formally similar, numerically very different: diff. sign in roots,  $(1 - \dots)^r$

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Formally similar, numerically very different: diff. sign in roots,  $(1 - \dots)^r$
- Unlike large- $x$  case: no direct generalization to all (higher)  $a$  in  $x^a \ln^\ell x$   
But works for higher even  $a$  in SIA – DIS case not checked yet
- Does not work for the odd- $N$  quantities  $F_3$  and  $g_1$  in DIS,  $F_A$  in SIA  
E.g., leading logs with group factor  $d_{abc} d^{abc}$  at third order in  $F_3$  and  $F_A$   
cf. Dokshitzer, Marchesini (2007)

All large- $x$  and many, but not all, small- $x$  double logarithms in SIA and DIS appear to be 'inherited' from lower-order results.