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Operator product expansion (OPE) of the Energy Momentum Tensor Correlator in QCD

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Outline

- Motivation
- Operator product expansions
- The energy-momentum tensor of QCD
- Calculation and Results

Motivation: Operator products

Dispersion relations for sum rules

$$\Pi(Q^2) = \int \mathrm{d}^4x\, e^{iqx} \langle 0| T\, O(x) O(0) |0\rangle = \int_0^\infty \frac{\mathrm{d}s}{(2\pi i)} \frac{i\rho(s)}{s+Q^2-i\varepsilon}$$

spectral density $\rho(s) = \sum_{\lambda(p=0)} \delta(s - M_\lambda^2) \left|\langle 0| O(0) |\lambda(p=0)\rangle\right|^2$

M_λ: invariant mass of the state λ

$Q^2 = -q^2$

Properties of hadrons:

For $O = \bar{\Psi}_f \Psi_{f'}$ scalar mesons
For $O = G^{a\mu\nu} G^a_{\mu\nu}$ scalar glueballs (0^{++})
For $O = G^a_{\mu\nu} G^a_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}$ pseudoscalar glueballs (0^{-+})
For $O = T^{\mu\nu}$ tensor glueballs (2^{++})

[Shifman, Vainshtein, Zakharov; Forkel]

Motivation: Energy momentum tensor correlator

$$T^{\mu\nu;\rho\sigma}(q) := i\int \mathrm{d}^4x\, e^{iqx}\, T\left(T^{\mu\nu}(x)\, T^{\rho\sigma}(0)\right)$$

- Sum rules in the Quark-Gluon-Plasma
  - shear viscosity $\eta(T) \sim \lim_{q\to 0} \rho_{12,12}(q,T)$
  - bulk viscosity $\zeta(T) \sim \lim_{q\to 0} \rho_{ii,jj}(q,T)$

where $\rho_{\mu\nu,\rho\sigma}$ is a spectral density related to the correlator $T_{\mu\nu,\rho\sigma}(q,T) = \langle\langle T^{\mu\nu;\rho\sigma}(q)\rangle\rangle_T$ through

$$\langle\langle T^{\mu\nu}(x)\, T^{\rho\sigma}(0)\rangle\rangle \sim \int_0^\infty d\omega\, \rho_{\mu\nu,\rho\sigma}(\omega) \frac{\cosh\left[\frac{1}{2}\beta - x_0\right]\omega}{\sinh\frac{\beta\omega}{2}}$$

$\beta = \frac{1}{T}, x = (x_0, \mathbf{x})$

[Laine, Meyer, Schröder, Vepsäläinen, Vuorinen, Zhu]

How can we produce results valid in the low energy (non-perturbative) region of momentum space?

Idea of the Operator Product Expansion (OPE)

[K.Wilson, 1969]

Separate perturbative and non-perturbative physics

through a large Q / small x expansion in local operators

$$\Pi(Q^2) = \int \mathrm{d}^4x\, e^{iqx} \langle 0| T\, O(x) O(0) |0\rangle$$
$$= C_0(Q^2)\langle 0|\mathbb{1}|0\rangle + C_m(Q^2)\langle 0|m^2|0\rangle$$
$$+ C_1(Q^2)\langle 0| -\frac{1}{4}[G^a_{\mu\nu}G^{a\mu\nu}]|0\rangle + C_2(Q^2)\langle 0|[m\bar{\psi}\psi]|0\rangle + \ldots$$

$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

in massless QCD $\langle 0|G^a_{\mu\nu}G^{a\mu\nu}|0\rangle_{pert} = 0$ (massless tadpol)

for the physical vacuum at low energy:

$\frac{\alpha_s}{\pi}\langle 0|G^a_{\mu\nu}G^{a\mu\nu}|0\rangle \simeq 0.012 GeV^4$ [Novikov, Shifman, Vainshtein, Zakharov, 1979]

(Gluon Condensate)

OPE as an operator identity

$$\int \mathrm{d}^4x\, e^{iqx}\, T\, O(x)O(0) = \sum_{d=0,2,4,\ldots} \sum_n C_{d,n}(Q^2) O_{d,n}(0)$$

d: mass dimension of Operators

Example:

$$\int \mathrm{d}^4x\, e^{iqx}\, T\, O(x)O(0) = C_0(Q^2)\mathbb{1} + C_1(Q^2)\left(-\frac{1}{4}[G^a_{\mu\nu}G^{a\mu\nu}]\right) + \ldots$$

Renormalization of O and OP

Operators:

- $[G^a_{\mu\nu} G^{a\mu\nu}] = Z_G \cdot G^{a,Bare}_{\mu\nu} G^{a\mu\nu,Bare}$

  $Z_G = 1 + \alpha_s \frac{\partial}{\partial \alpha_s} \ln Z_{\alpha_s} = \left(1 - \frac{\beta(\alpha_s)}{\varepsilon}\right)^{-1}$ [Spiridonov]
- $[T^{\mu\nu}] = 1 \cdot T^{\mu\nu,Bare}$ [Nielsen] (no anomalous dimension!)

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Operator products:

- $Z_O^2 \cdot \int \mathrm{d}^4x\, e^{iqx}\, T\, O^{Bare}(x) O^{Bare}(0)$

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Operator products:

- $Z_O^2 \cdot \int d^4x\, e^{iqx}\, T\, O^{Bare}(x) O^{Bare}(0) - \sum_i Z_i O_i(0)$

Renormalization of O and OP

Operators:

- $[G^a_{\mu\nu} G^{a\mu\nu}] = Z_G \cdot G^{a,Bare}_{\mu\nu} G^{a\mu\nu,Bare}$

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- $[T^{\mu\nu}] = 1 \cdot T^{\mu\nu,Bare}$ [Nielsen] (no anomalous dimension!)

Operator products:

- $Z_O^2 \cdot \int d^4x\, e^{iqx}\, T\, O^{Bare}(x) O^{Bare}(0) - \sum_i Z_i O_i(0)$

  subtract contact terms
  ($\sim \delta(x) \Longleftrightarrow$ polynomial in q)

  only non-local/scale dependent part of physical importance!

Renormalization of OPE

Multiplicative Renormalization of O

Contact terms (local subtractions)

$$
\begin{aligned}
& Z_O^2 \cdot \int \mathrm{d}^4x\, e^{iqx}\, T\, O^{Bare}(x) O^{Bare}(0) && - \sum_i Z_i O_i(0) \\
&= \sum_n Z_O^2 \cdot C_n^{Bare}(Q^2) O_n^{Bare}(0) && - \sum_i Z_i O_i(0) \\
&= \sum_n \underbrace{Z_O^2 \cdot C_n^{Bare}(Q^2)(Z_{O_n})^{-1}}_{C_n(Q^2)} \underbrace{Z_{O_n} O_n^{Bare}(0)}_{[O_n(0)]} && - \sum_i Z_i O_i(0)
\end{aligned}
$$

Finite Coefficients

Finite Operators

OPE of $T^{\mu\nu;\rho\sigma}$

$$T^{\mu\nu;\rho\sigma}(q) := i\int \mathrm{d}^4x\, e^{iqx}\, T\left(T^{\mu\nu}(x)\, T^{\rho\sigma}(0)\right)$$

$$\boxed{T^{\mu\nu;\rho\sigma}(q) = C_0^{\mu\nu;\rho\sigma}(q)\mathbb{1} + C_1^{\mu\nu;\rho\sigma}(q)[O_1] + \ldots}$$

for $Q^2 = -q^2 \to \infty$

where $[O_1] = -\frac{1}{4}[G^a_{\mu\nu}G^{a\mu\nu}]$, $C_1^{\mu\nu;\rho\sigma}(q) = \frac{1}{Z_G}C_1^{\mu\nu;\rho\sigma,\,Bare}(q)$

<u>Motivation for C_1:</u>

shift in a spectral density from T=0 to finite T

$$\boxed{\Delta\rho(\omega, T) = \rho(\omega, T) - \rho(\omega, 0)}$$

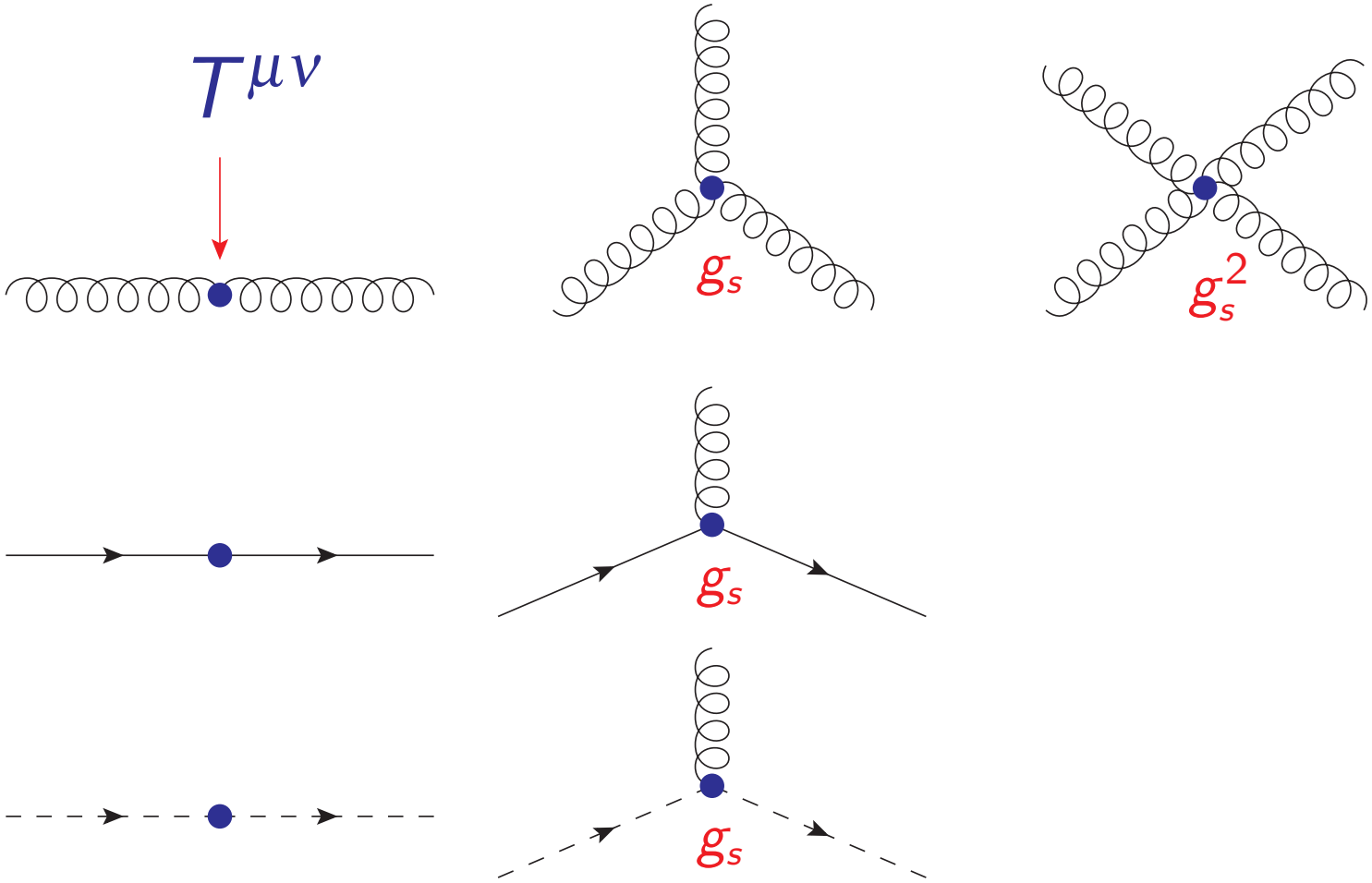
The Energy Momentum Tensor of QCD

$$\mathscr{L} = -\frac{1}{4} Z_3\, G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2\lambda}\left(\partial_\mu A^{a\mu}\right)^2 + \tilde{Z}_3 \partial_\rho \bar{c}^a \partial^\rho c^a + g\tilde{Z}_1 f^{abc} \partial_\rho \bar{c}^a A^{b\rho} c^c$$
$$+ \frac{i}{2} Z_2 \bar{\psi}_i \overleftrightarrow{\not{\partial}} \psi_i + g Z_{1\psi} \bar{\psi}_i \not{A}^a T^a_{ij} \psi_j$$

$$T_{\mu\nu} = -Z_3 G_{\mu\rho} G_\nu^{\ \rho} + \frac{1}{\lambda}(\partial_\mu \partial_\rho A^\rho) A_\nu + \frac{1}{\lambda}(\partial_\nu \partial_\rho A^\rho) A_\mu$$
$$+ \tilde{Z}_3 (\partial_\mu \bar{c} \partial_\nu c + \partial_\nu \bar{c} \partial_\mu c) + g\tilde{Z}_1 \left(\partial_\mu \bar{c}(A_\nu \times c) + \partial_\nu \bar{c}(A_\mu \times c)\right)$$
$$+ \frac{i}{4} Z_2 \bar{\psi} \left(\overleftrightarrow{\partial}_\mu \gamma_\nu + \overleftrightarrow{\partial}_\nu \gamma_\mu\right) \psi + \frac{g}{2} Z_{1\psi} \bar{\psi} \left(A_\mu T \gamma_\nu + A_\nu\ T \gamma_\mu\right) \psi$$
$$- g_{\mu\nu} \Big\{ -\frac{1}{4} Z_3\, G_{\rho\sigma} G^{\rho\sigma} + \frac{1}{\lambda}\left(\partial_\sigma \partial_\rho A^\rho\right) A^\sigma + \frac{1}{2\lambda}\left(\partial_\rho A^\rho\right)^2$$
$$+ \tilde{Z}_3 \partial_\rho \bar{c} \partial^\rho c + g\tilde{Z}_1 \left(\partial_\rho \bar{c}(A_\rho \times c)\right) + \frac{i}{2} Z_2 \bar{\psi} \overleftrightarrow{\not{\partial}} \psi + g Z_{1\psi} \bar{\psi} \not{A} T \psi \Big\}$$

[Nielsen, 1977]

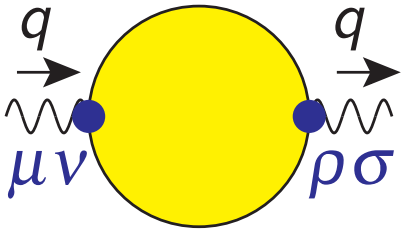
Feynman Rules



Calculation of $C_0^{\mu\nu;\rho\sigma}(q)$

[QGRAF, MINCER, FORM]

$$C_0^{\mu\nu;\rho\sigma}(q) = \langle 0|\, T^{\mu\nu;\rho\sigma}(q)|0\rangle_{pert} =$$



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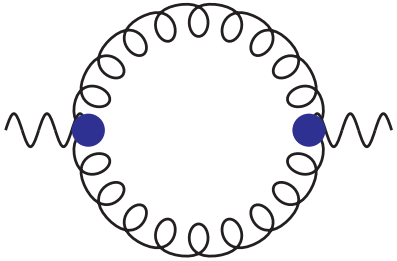

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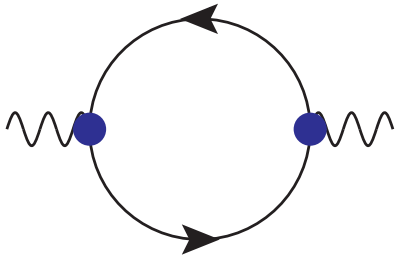

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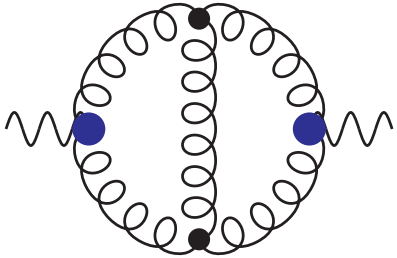

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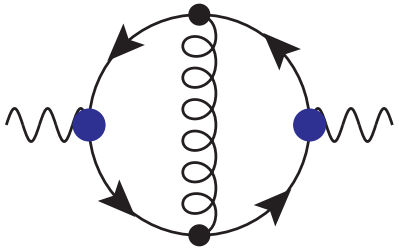

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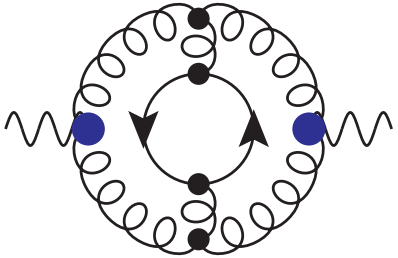


$+\ldots$

OPE of the correlator of two $T^{\mu\nu}$

3 loop result for C_0 (Coefficient in front of $O_0 = \mathbb{1}$) in the $\overline{MS}$-scheme:

$$C_0^{\mu\nu;\rho\sigma}(q) = C_0^{(S)}(Q^2)\, t_S^{\mu\nu\rho\sigma}(q) + C_0^{(T)}(Q^2)\, t_T^{\mu\nu\rho\sigma}(q)$$

Tensor structures: $\eta^{\mu\nu}(q) = q^2 g^{\mu\nu} - q^\mu q^\nu$

$$t_S^{\mu\nu\rho\sigma}(q) = \eta^{\mu\nu}\eta^{\rho\sigma}$$

$$t_T^{\mu\nu\rho\sigma}(q) = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{D-1}\eta^{\mu\nu}\eta^{\rho\sigma} \qquad \text{(traceless)}$$

$$\begin{aligned}
Q^2 \frac{d}{dQ^2} C_0^{(T)} = \frac{1}{16\pi^2} \Bigg[& -\frac{1}{10} n_g - \frac{1}{20} n_f N_c \\
& + a_s \left\{ \frac{1}{18} C_A n_g - \frac{7}{144} n_f T_F n_g \right\} \\
& + a_s^2 \Bigg\{ \frac{67}{12960} C_A^2 n_g + \frac{3}{128} n_f T_F C_F n_g - \frac{10663}{51840} n_f C_A T_F n_g + \frac{473}{6480} n_f^2 T_F^2 n_g \\
& \quad + \frac{11}{216} l_{\mu q} C_A^2 n_g - \frac{109}{1728} l_{\mu q} n_f C_A T_F n_g + \frac{7}{432} l_{\mu q} n_f^2 T_F^2 n_g \\
& \quad + \frac{11}{40} \zeta_3 C_A^2 n_g + \frac{3}{80} \zeta_3 n_f C_A T_F n_g - \frac{1}{20} \zeta_3 n_f^2 T_F^2 n_g \Bigg\} \Bigg] .
\end{aligned}$$

$$Q^2 \frac{d}{dQ^2} C_0^{(S)} = \frac{a_s^2}{16\pi^2} \left\{ -\frac{121}{1296} C_A^2 n_g + \frac{11}{162} n_f C_A T_F n_g - \frac{1}{81} n_f^2 T_F^2 n_g \right\}$$

$\alpha_s = \frac{g_s^2}{4\pi}$ $\quad a_s = \frac{\alpha_s}{\pi}$ $\quad l_{\mu q} = \log\left(\frac{\mu^2}{Q^2}\right)$

Trace anomaly and $C_0^{(S)}$

$$\boxed{T^\mu_\mu = \frac{\beta(a_s)}{2}\,[G^a_{\rho\sigma}G^{a\rho\sigma}] = -2\,\beta(a_s)\,[O_1]}$$

$$\int \mathrm{d}^4x\, e^{iqx}\,\langle 0|\,T[\,T^\mu_\mu(x)\,T^\nu_\nu(0)]|0\rangle$$

$$= \underbrace{4\,\beta^2(\alpha_s)}_{\mathscr{O}(\alpha_s^2)}\left\{\int \mathrm{d}^4x\, e^{iqx}\,\langle 0|\,T\,[O_1(x)\,O_1(0)]|0\rangle\right\} + \text{contact terms}$$

- Leading contribution to $Q^2\frac{d}{dQ^2}\,C_0^{(S)}(Q^2)$ of $\mathscr{O}(\alpha_s^2)$
- No $\log\frac{\mu^2}{Q^2}$ and ζ functions up to $\mathscr{O}(\alpha_s^2)$.
- Result for $Q^2\frac{d}{dQ^2}\,C_0^{(S)}(Q^2)$ extendable up to $\mathscr{O}(\alpha_s^5)$

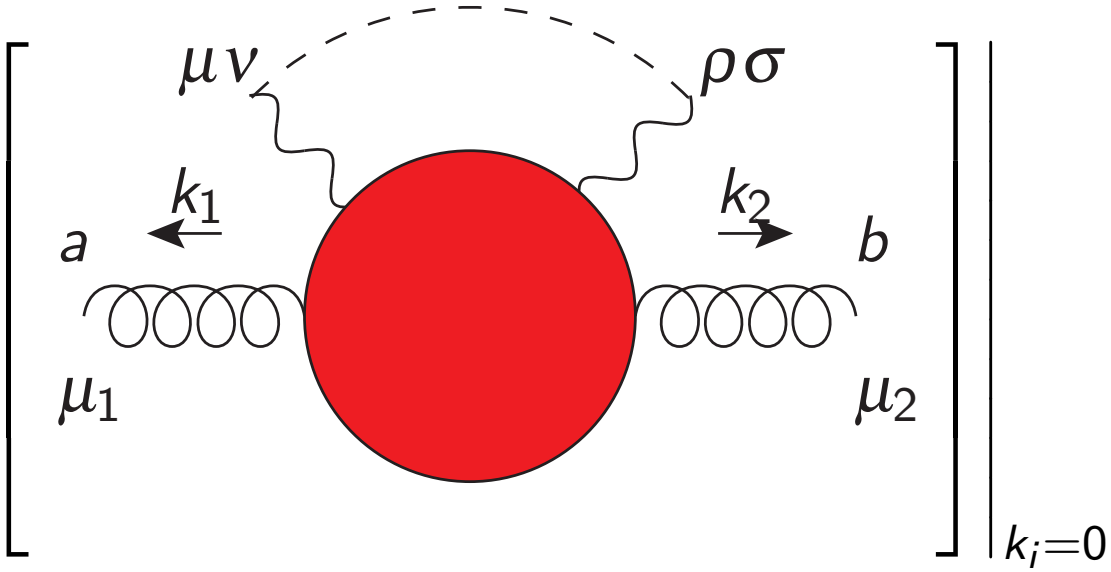
Calculation of C_1

$$\int d^4x\, e^{iqx}\, T\left[T^{\mu\nu}(x)\, T^{\rho\sigma}(0)\right] = C_0^{\mu\nu;\rho\sigma}(Q^2)\mathbb{1} + C_1^{\mu\nu;\rho\sigma}(Q^2)[O_1] + \dots$$

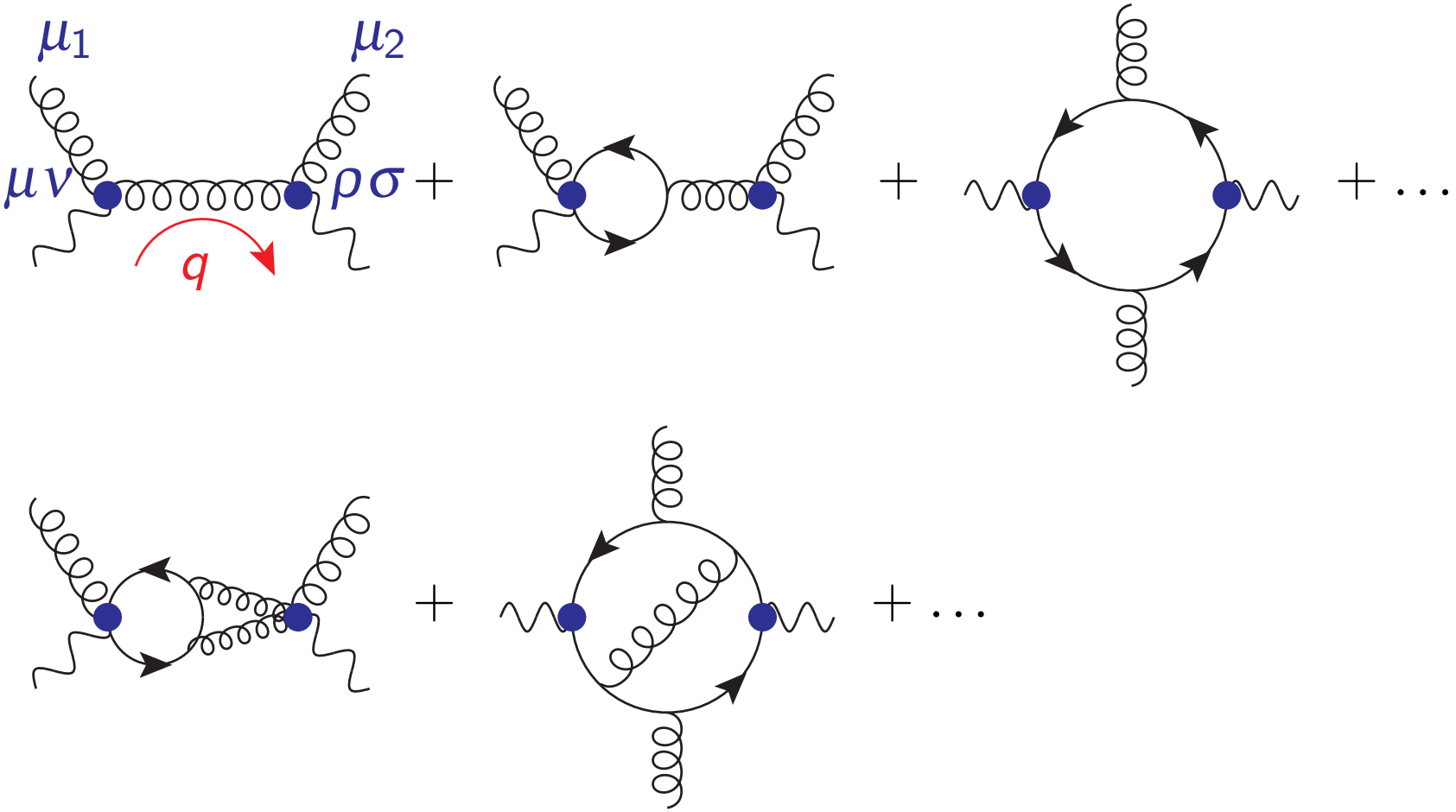
Projector onto $O_1 = -\frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu}$:

$$C_{1,B}^{\mu\nu;\rho\sigma}(q) =$$

$$\frac{\delta^{ab}}{n_g} \frac{g^{\mu_1\mu_2}}{(D-1)} \frac{1}{D} \frac{\partial}{\partial k_1} \cdot \frac{\partial}{\partial k_2}$$



Calculation of C_1



OPE of the correlator of two $T^{\mu\nu}$

2 loop result for C_1 (Coefficient in front of $[O_1] = [-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}]$) in the $\overline{MS}$-scheme:

$$C_1^{\mu\nu;\rho\sigma}(q) = C_1^{(S)}(Q^2)\frac{1}{(Q^2)^2}\, t_S^{\mu\nu\rho\sigma}(q) + C_1^{(T)}(Q^2)\frac{1}{(Q^2)^2}\, t_T^{\mu\nu\rho\sigma}(q)$$

Tensor structures: $\eta^{\mu\nu}(q) = q^2 g^{\mu\nu} - q^\mu q^\nu$

$$t_S^{\mu\nu\rho\sigma}(q) = \eta^{\mu\nu}\eta^{\rho\sigma}$$

$$t_T^{\mu\nu\rho\sigma}(q) = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{D-1}\eta^{\mu\nu}\eta^{\rho\sigma} \qquad \text{(traceless)}$$

$$\begin{aligned}
C_1^{(S)} =& a_s \left\{ \frac{22}{27} C_A - \frac{8}{27} n_f T_F \right\} \\
&+ a_s^2 \left\{ \frac{83}{324} C_A^2 - \frac{2}{9} n_f T_F C_F - \frac{8}{81} n_f C_A T_F - \frac{4}{81} n_f^2 T_F^2 \right\}, \\
C_1^{(T)} =& a_s \left\{ -\frac{5}{18} C_A - \frac{5}{72} n_f T_F \right\} \\
&+ a_s^2 \left\{ -\frac{83}{432} C_A^2 + \frac{43}{96} n_f T_F C_F + \frac{41}{432} n_f C_A T_F - \frac{1}{216} n_f^2 T_F^2 \right\}
\end{aligned}$$

$\alpha_s = \frac{g_s^2}{4\pi}$ $\qquad a_s = \frac{\alpha_s}{\pi}$

Structure of C_1

$$\boxed{\text{RGI: } T^{\mu}_{\mu} = -2\beta(a_s)[O_1] \Longrightarrow O_1^{RGI} \equiv \hat{\beta}(a_s)[O_1]}$$

$\hat{\beta}(a_s) = \frac{-\beta(a_s)}{\beta_0} = a_s\left(1 + \sum_{i\geq 1} \frac{\beta_i}{\beta_0} a_s^i\right)$

Construct RGI coefficients: $\boxed{C_{1,RGI}^{(S,T)} \equiv C_1^{(S,T)}/\hat{\beta}(a_s)}$

with $C_{1,RGI}^{(S,T)} O_1^{RGI} = C_1^{(S,T)}[O_1]$

$$\begin{aligned}
C_{1,RGI}^{(S)} =& \frac{22}{27} C_A - \frac{8}{27} n_f T_F - \frac{a_s}{324}(11 C_A - 4 n_f T_F)^2 \\
C_{1,RGI}^{(T)} =& -\frac{5}{72}(4 C_A + n_f T_F) \\
&+ \frac{a_s}{864(11 C_A - 4 n_f T_F)}\big(214 C_A^3 + 876 C_A^2 n_f T_F \\
&+ 3537 C_A C_F n_f T_F - 672 C_A n_f^2 T_F^2 \\
&- 1728 C_F n_f^2 T_F^2 + 16 n_f^3 T_F^3\big)
\end{aligned}$$

Structure of C_1

$$C_{1,RGI}^{(S,T)} = (a_1 + b_1\, l_{\mu q}) + a_s(a_2 + b_2\, l_{\mu q} + c_2\, {l_{\mu q}}^2) + a_s^2(a_3 + b_3\, l_{\mu q} + c_3\, {l_{\mu q}}^2 + d_3\, {l_{\mu q}}^3) + \mathscr{O}(a_s^3)$$

scale independent coefficients $a_i, b_i, c_i,\ d_i$ and $l_{\mu q} = \log\left(\frac{\mu^2}{Q^2}\right)$

The derivative with respect to μ^2 must vanish:

$$\mu^2 \frac{d}{d\mu^2} C_{1,RGI}^{(S,T)} = b_1 + a_s(b_2 + 2c_2\, l_{\mu q}) + a_s\beta(a_s)(a_2 + b_2\, l_{\mu q} + c_2\, {l_{\mu q}}^2) + a_s^2(b_3 + 2c_3\, l_{\mu q} + 3d_3\, {l_{\mu q}}^2) + \mathscr{O}(a_s^3) = 0 \quad \forall \mu^2$$

Structure of C_1

$$C_{1,RGI}^{(S,T)} = (a_1 + b_1\, l_{\mu q}) + a_s(a_2 + b_2\, l_{\mu q} + c_2\, l_{\mu q}{}^2) + a_s^2(a_3 + b_3\, l_{\mu q} + c_3\, l_{\mu q}{}^2 + d_3\, l_{\mu q}{}^3) + \mathscr{O}(a_s^3)$$

scale independent coefficients $a_i, b_i, c_i,\ d_i$ and $l_{\mu q} = \log\left(\frac{\mu^2}{Q^2}\right)$

The derivative with respect to μ^2 must vanish:

$$\begin{aligned}
\mu^2 \frac{d}{d\mu^2} C_{1,RGI}^{(S,T)} &= b_1 + a_s(b_2 + 2c_2\, l_{\mu q}) \\
&\quad + a_s \beta(a_s)(a_2 + b_2\, l_{\mu q} + c_2\, l_{\mu q}{}^2) \\
&\quad + a_s^2(b_3 + 2c_3\, l_{\mu q} + 3d_3\, l_{\mu q}{}^2) + \mathscr{O}(a_s^3) = 0 \quad \forall \mu^2 \\
&\Rightarrow b_1 = 0 \\
&\Rightarrow b_2 = 0,\ c_2 = 0 \\
&\Rightarrow b_3 = \beta_0 a_2,\ c_3 = 0,\ d_3 = 0.
\end{aligned}$$

Summary, further results and applications to high-temperature QCD

- For $T^{\mu\nu;\rho\sigma}(q)$ in massless QCD $\Longrightarrow$ C_0, C_1
- C_1 (2 loop) for the correlator $i\int d^4x e^{iqx}\, T\left[O_1(x)O_1(0)\right]$
- C_0 (4 loop) available [Baikov, Chetyrkin]
- Extensions of some results from
  - "The Ultraviolet limit and sum rule for the shear correlator in hot Yang-Mills theory"
    [Schröder, Vepsäläinen, Vuorinen, Zhu]
    arxiv: 1109.6548, JHEP 1112 (2011) 035
  - "Ultraviolet asymptotics of scalar and pseudoscalar correlators in hot Yang-Mills theory"
    [Laine, Vepsäläinen, Vuorinen]
    arxiv: 1008.3263, JHEP (2010) 010

  for Trace anomaly and Shear stress correlators to higher orders.