[Tutorials on Thursdays: 8-10 in C01-148 and 16-18 in U2-135]

Exercise 4.1: Hamilton operator of the free complex Klein-Gordon field
Starting from the Lagrange density $\mathcal{L}_{KG} = \partial_\mu \phi(x)^* \partial^\mu \phi(x) - m^2 \phi(x)^* \phi(x)$, one finds after dividing $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$ up into two real fields ϕ_1 and ϕ_2 and performing two Legendre transforms for each real degree of freedom the following expression for the Hamilton operator:

$$\hat{H} = \int d^3x \left[\partial_0 \hat{\phi}(x) \partial_0 \hat{\phi}(x)^\dagger + \nabla \hat{\phi}(x)^\dagger \cdot \nabla \hat{\phi}(x) + m^2 \hat{\phi}(x)^\dagger \hat{\phi}(x)\right].$$

Express it in terms of lowering and raising operators with the help of the solution $\hat{\phi}(x)$ given in lecture:

$$\hat{H} = \int d^3p E_{\vec{p}} \left[\frac{1}{2}(\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \hat{a}_{\vec{p}} \hat{a}^\dagger_{\vec{p}}) + \frac{1}{2}(\hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} + \hat{b}_{\vec{p}} \hat{b}^\dagger_{\vec{p}})\right] = \int d^3p E_{\vec{p}} \left[\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} + \delta^{(3)}(\vec{0})\right].$$

Furthermore, check that the conserved charge $Q(t) = \int d^3x \, i[\phi(x)^* \partial^0 \phi(x) - \phi(x) \partial^0 \phi(x)^*]$ can be written as follows:

$$\hat{Q} = \int d^3p \left[\frac{1}{2}(\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \hat{a}_{\vec{p}} \hat{a}^\dagger_{\vec{p}}) - \frac{1}{2}(\hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} + \hat{b}_{\vec{p}} \hat{b}^\dagger_{\vec{p}})\right] = \int d^3p \left[\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}}\right].$$

Exercise 4.2:
Show the following regularisation of the delta function with finite volume V:

$$\delta^{(3)}(\vec{0}) = \frac{V}{(2\pi)^3}.$$

What is the physical meaning for this part of the Hamilton operator?

Exercise 4.3:
Prove the following relation for the polarisation vectors appearing in the solution to Maxwell's equations in Coulomb gauge:

$$\sum_{\lambda=1,2} \varepsilon^i_{(\lambda)}(\vec{p}) \varepsilon^j_{(\lambda)}(\vec{p}) = \delta^{ij} - \frac{p^i p^j}{(\vec{p})^2}.$$

Exercise 4.4:
Consider the following spinors from the solution to the Dirac equation:

$$u(\vec{p}, s) = C(\gamma_\mu p^\mu + m) \begin{pmatrix} \xi_s \\ 0 \end{pmatrix}, \quad v(\vec{p}, s) = C'(\gamma_\mu p^\mu - m) \begin{pmatrix} 0 \\ \xi_{-s} \end{pmatrix},$$

with $s = \pm$ and $\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Verify that the normalisation

$$\bar{u}(\vec{p}, s) u(\vec{p}, s') = 2m\delta_{ss'}, \quad \bar{v}(\vec{p}, s) v(\vec{p}, s') = -2m\delta_{ss'}$$

is satisfied through $C = -C' = 1/\sqrt{E_{\vec{p}} + m}$. Compute $u(\vec{p}, s)^\dagger u(\vec{p}, s')$ and $v(\vec{p}, s)^\dagger v(\vec{p}, s')$.