Winter term 2020/21	G. Akemann, T. Würfel	12.01.2021

[Submit your solutions up to 18.01. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 11

Exercise 11.1: (3 Points)

(i) Let G be a matrix Lie group and $\mathfrak{g}$ the space of matrices such that $M \in \mathfrak{g}$ implies $e^M \in G$. Show that

   (a) $\mathfrak{gl}(N)$, associated with the group GL(N), consists of arbitrary $N \times N$ matrices.

   (b) $\mathfrak{sl}(N)$, associated with the group SL(N), consists of traceless $N \times N$ matrices.

   (c) $\mathfrak{o}(N) = \mathfrak{so}(N)$, associated with the group O(N), or SO(N), consists of anti-symmetric $N \times N$ matrices.

   (d) $\mathfrak{u}(N)$, associated with the group U(N), consists of anti-hermitian $N \times N$ matrices.

   (e) $\mathfrak{su}(N)$, associated with the group SU(N), consists of traceless anti-hermitian $N \times N$ matrices.

(ii) For each of the above cases, check that the characteristic property of the matrices of $\mathfrak{g}$ is preserved by the commutator, thereby ensuring that $\mathfrak{g}$ is a Lie algebra.

Exercise 11.2: (4 Points)

The Lie algebra $\mathfrak{so}(N)$ of the special orthogonal group SO(N) is the space of traceless anti-symmetric $N \times N$ matrices, where $N \geq 2$, which we have seen also in the previous exercise.

(i) Check that a basis of generators (in the physicists' convention) consists of matrices T^{ab}, with $1 \leq a < b \leq N$, whose ij-th entry is $(T^{ab})_{ij} = -i(\delta_i^a \delta_j^b - \delta_j^a \delta_i^b)$, where δ_l^k is the usual Kronecker symbol. Show that the Lie bracket of two such generators is

$$[T^{ab}, T^{cd}] = -i(\delta^{bc}T^{ad} + \delta^{ad}T^{bc} - \delta^{bd}T^{ac} - \delta^{ac}T^{bd}) \tag{1}$$

(ii) Denote X a vector with N real components x^j. Given a matrix $O \in$ SO(N) close to the identity matrix $\mathbb{1}_N$, the transformation $X \to OX \equiv X'$ yields $X' = X + \delta X$ where the components δx^j of δX are small. Show that one may write

$$\delta x^j = -\frac{i}{2} w_{ab} T^{ab} x^j \quad \text{with} \quad T^{ab} \equiv -i\left(x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a}\right) \tag{2}$$

and with w_{ab} antisymmetric (and traceless).

(iii) Take now $N = 4$ and define (beware the signs!)

$$A_1 \equiv \frac{1}{2}(T^{12} - T^{34}) \quad A_2 \equiv \frac{1}{2}(T^{13} + T^{24}) \quad A_3 \equiv \frac{1}{2}(T^{14} - T^{23}) \tag{3}$$

and

$$B_1 \equiv \frac{1}{2}(T^{12} + T^{34}) \quad B_2 \equiv \frac{1}{2}(-T^{13} + T^{24}) \quad B_3 \equiv \frac{1}{2}(T^{14} + T^{23}) \tag{4}$$

Compute the commutators $[A_i, A_j]$, $[B_i, B_j]$ and $[A_i, B_j]$. How would you be tempted to interpret your findings?

Exercise 11.3: (3 Points)

Let G be a simply connected, compact Lie-group of dimension d (it has d linearly independent generators and each element $g \in G$ can be generated by $g = \exp[\imath T]$ with $T \in \mathrm{Lie}\, G$) and $D_G^{(\mathrm{adj})}$ be the adjoint representation of G.

(i) Prove that $D_G^{(\mathrm{adj})}$ is a subgroup of $D_{\mathrm{SO}(d)}^{(\mathrm{fund})}$ which is the fundamental representation of the special orthogonal group $\mathrm{SO}(d)$.

(ii) Let $\{T_1, \ldots, T_d\}$ be an orthonormal basis of the Lie-algebra, $\mathrm{Lie}\, D_G^{(\mathrm{fund})}$, of G in the fundamental representation, i.e. $\mathrm{tr}(T_a T_b) = \delta_{ab}$, and $D_G^{(\mathrm{fund})}$ its fundamental representation. Show that the adjoint representation of the group G is equal to

$$\begin{array}{rcl} D_G^{(\mathrm{fund})} \times \mathrm{Lie}\, D_G^{(\mathrm{fund})} & \longrightarrow & \mathrm{Lie}\, D_G^{(\mathrm{fund})} \\ A \times \sum_{c=1}^{d} \alpha_c T_c & \longmapsto & \sum_{c=1}^{d} \alpha_c A T_c A^{-1} \end{array} \tag{5}$$

with some real coefficients $\alpha_c \in \mathbb{R}$. In particular show that the map

$$\begin{array}{rcl} D_G^{(\mathrm{fund})} & \longrightarrow & D_G^{(\mathrm{adj})} \\ A & \longmapsto & B = \{\mathrm{tr}(T_a A T_b A^{-1})\}_{1 \le a,b \le d} \end{array} \tag{6}$$

gives a coordinate representation for $D_G^{(\mathrm{adj})}$, meaning that it is surjective and is exactly the adjoint representation. To prove this show that the structure constants generate the algebra of the representation (6).