Winter term 2020/21	G. Akemann, T. Würfel	19.01.2021

[Submit your solutions up to 25.01. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 12

Exercise 12.1: (3 Points)

Recall that the Cartan-subalgebra $\mathcal{C}$ is an Abelian algebra of a Lie-algebra $\mathcal{L}$ fulfilling the condition: $Y \in \mathcal{L} \text{ with } [Y,X]_- = 0\ \forall Y \in \mathcal{C} \Rightarrow X \in \mathcal{C}$. Hence the Cartan-subalgebra and, thus, the Cartan-subgroup are not unique at all. However for a simply connected, compact group G different Cartan-subalgebras, $\mathcal{C}_1$ and $\mathcal{C}_2$, are similar, in particular there is a fixed group element $g \in G$ such that $g\mathcal{C}_1 g^{-1} = \mathcal{C}_2$.

(i) Calculate the Cartan-subalgebras and their corresponding Cartan-subgroups of the Lie-groups $\mathrm{U}(N)$, $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$. For this purpose show that one can find Cartan-sybalgebras of $\mathrm{U}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$ as subsets of the diagonal matrices and for $\mathrm{SO}(N)$ as subsets of 2×2 block-diagonal matrices.

(ii) Let us consider the non-compact group $\mathrm{U}(1,1) = \{A \in \mathrm{Gl}(2) | A\,\mathrm{diag}(1,-1)A^\dagger = \mathrm{diag}(1,-1)\}$. Show that there are different Cartan-subalgebras and Cartan-subgroups which are not similar to each other. There are all in all three Cartan-subalgebras. Enlist them all.

(iii) The Weyl-reflection group is another important subgroup of a Lie-group. Let Cart_G be the Cartan subgroup of the Lie-Group G. Then the Weyl-reflection group is $\mathrm{Weyl}_G = \{g \in G | g\mathrm{Cart}_G g^{-1} = \mathrm{Cart}_G\}$. Calculate this group for $\mathrm{U}(N)$, $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$.
Hint: You have to distinguish the groups $\mathrm{SO}(2N)$ and $\mathrm{SO}(2N+1)$ since you have always to check if g is in the considered group.

Exercise 12.2: (3 Points)

Let Δ be the set of roots. Given $\alpha \in \Delta$, define σ_α to be the linear transformation implementing reflection in the hyperplane orthogonal to α.

(i) Show that for a root β this projection explicitly reads

$$\sigma_\alpha(\beta) = \beta - \frac{2\alpha \cdot \beta}{|\alpha|^2}\alpha. \tag{1}$$

(ii) Show that $\sigma_\alpha(\beta) \in \Delta$ with help of the Cartan-Weyl basis introduced in the lecture and of the antisymmetry of any generator in the adjoint representation.
Background: The reflections σ_α are induced by the Weyl-reflections introduced in Exercise 12.1(iii) and are, therefore, the Weyl-reflections in the root space. This can be seen by choosing an orthonormal basis in the Cartan-subalgebra $H = (H_1, \ldots, H_r)$ and the remaining basis vectors of the Lie-algebra $\{E_\alpha\}_{\alpha \in \Delta}$. Let $H_\alpha = 2\alpha \cdot H/|\alpha|^2$ with $\alpha \in \Delta$. Then a Weyl reflection in the Cartan-subalgebra $H \to WH$ is a linear operation (W is an orthogonal $r \times r$ matrix which is the adjoint representation of $g \in \mathrm{Weyl}_G$ acting only on Cart_G) and carries over to one on the root space by $\alpha \cdot (WH) = (W^T\alpha) \cdot H$.

(iii) For two distinct roots, α and β, note that $\alpha \cdot \beta \neq 0$ and the quantization of the roots implies $2\alpha \cdot \beta/|\alpha|^2 = \pm 1$ or $2\alpha \cdot \beta/|\beta|^2 = \pm 1$. Show with help of the quantization of the roots and the Weyl-reflections that

$$\begin{aligned} &\alpha \cdot \beta < 0 \Rightarrow \alpha + \beta \in \Delta, \\ &\alpha \cdot \beta > 0 \Rightarrow \alpha - \beta \in \Delta. \end{aligned} \tag{2}$$

Exercise 12.3: (4 Points)

Let us consider the three groups $\mathrm{SO}(n) \subset \mathrm{U}(n)$ (special orthogonal group), $\mathrm{SU}(n) \subset \mathrm{U}(n)$ (special unitary group), and $\mathrm{USp}(2n) \subset \mathrm{U}(2n)$ (unitary symplectic group) denoted by the Dyson index $\beta = 1, 2, 4$, respectively. All three groups are defined in their fundamental representations by

$$\begin{cases} UU^T = \mathbb{1}_n, & \beta = 1, \\ UU^\dagger = \mathbb{1}_n, & \beta = 2, \\ U \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} U^T = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, & \beta = 4, \end{cases} \tag{3}$$

and the uni modularity

$$\det U = 1. \tag{4}$$

Recall that $(.)^T$ and $(.)^\dagger$ are the transpose and the adjoint of a matrix, respectively.

(i) Show that the corresponding Lie algebras, $U = \exp[\imath A]$, in the fundamental representations are given by

$$\begin{cases} A = A^\dagger = -A^T, & \beta = 1, \\ A = A^\dagger, & \beta = 2, \\ A = A^\dagger = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, & \beta = 4, \end{cases} \tag{5}$$

and the traceless condition

$$\mathrm{tr} A = 1. \tag{6}$$

What is the dimension d of the three kinds of Lie-groups in terms of the dimension N of the fundamental representation of the Lie groups?

(ii) Let $\mathbf{e}_j$ be the normalized n-dimensional vector with a 1 at the j'th position and otherwise zero and $\{\sigma_1, \sigma_2, \sigma_3\}$ the three Pauli matrices. Then show that the matrix sets

$$\bigcup_{1\leq a<b\leq n} \left\{ \frac{\mathbf{e}_a \otimes \mathbf{e}_b - \mathbf{e}_b \otimes \mathbf{e}_a}{\sqrt{2}\imath} \right\}, \tag{7}$$

for $\beta = 1$,

$$\left[\bigcup_{1\leq a\leq n-1} \left\{ \frac{1}{\sqrt{a(a+1)}} \left(\sum_{j=1}^{a} \mathbf{e}_j \otimes \mathbf{e}_j - a\, \mathbf{e}_{a+1} \otimes \mathbf{e}_{a+1} \right) \right\} \right] \cup \left[\bigcup_{1\leq a<b\leq n} \left\{ \frac{\mathbf{e}_a \otimes \mathbf{e}_b - \mathbf{e}_b \otimes \mathbf{e}_a}{\sqrt{2}\imath}, \frac{\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a}{\sqrt{2}} \right\} \right] \tag{8}$$

for $\beta = 2$, and

$$\begin{aligned} &\left[\bigcup_{1\leq a\leq n} \left\{ \frac{\mathbf{e}_a \otimes \mathbf{e}_a \otimes \sigma_1}{\sqrt{2}}, \frac{\mathbf{e}_a \otimes \mathbf{e}_a \otimes \sigma_2}{\sqrt{2}}, \frac{\mathbf{e}_a \otimes \mathbf{e}_a \otimes \sigma_3}{\sqrt{2}} \right\} \right] \\ \cup &\left[\bigcup_{1\leq a<b\leq n} \left\{ \frac{(\mathbf{e}_a \otimes \mathbf{e}_b - \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \mathbb{1}_2}{2\imath}, \frac{(\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \sigma_3}{2} \right\} \right] \\ \cup &\left[\bigcup_{1\leq a<b\leq n} \left\{ \frac{(\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \sigma_1}{2}, \frac{(\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \sigma_2}{2} \right\} \right] \end{aligned} \tag{9}$$

for $\beta = 4$ are orthonormal bases of the corresponding Lie algebras.

(iii) Calculate the completeness relation for the three Lie algebras with help of the given bases, in particular calculate $(\sum_{j=1}^d T^j_{ab} T^j_{cd})/d$ where T^j are the generators (7), (8), and (9). For this purpose it is simpler to calculate the sum $[\sum_{j=1}^d \mathrm{tr}(M_1 T^j) \mathrm{tr}(M_2 T^j)]/d$ for two arbitrary matrices M_1 and M_2. Why do we not obtain the unit matrix, only? Interpret the results via the symmetries of the elements of the Lie algebra.

(iv) Calculate the structure constants of the three Lie-Algebras.

(v) Show that all three groups are semi-simple. For this purpose calculate the Cartan metric. Why is $\mathrm{U}(n)$ not semi-simple? For what dimension n are the three groups also simple?

Preparation for Tutorial: (0 Points)

Read the given material regaring Young Diagrams provided in LernRaumPlus as a preparation for the Tutorial session on 25.01.2021, where Tutorial exercises based on the material will be discussed.