Winter term 2020/21	G. Akemann, T. Würfel	26.01.2021

[Submit your solutions up to 01.02. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 13

Exercise 13.1: (4 Points)

In this exercise we construct all Casimir operators of a representation of a Lie-algebra $\mathrm{Lie}\, D_G^{(\mathrm{o})}$ for a d-dimensional Lie-group G. For this purpose we denote the fundamental representation of G by $D_G^{(\mathrm{fund})}$ and the corresponding orthonormal basis ($\mathrm{Tr} T_a^{(\mathrm{f})} T_b^{(\mathrm{f})} = \delta_{ab}$) of the Lie-algebra by $\mathrm{span}\{T_1^{(\mathrm{f})}, \ldots, T_d^{(\mathrm{f})}\} = \mathrm{Lie}\, D_G^{(\mathrm{fund})}$. The basis of the representation $\mathrm{Lie}\, D_G^{(\mathrm{o})}$, namely $\{T_1^{(\mathrm{o})}, \ldots, T_d^{(\mathrm{o})}\}$, is given by the mapping $T_j^{(\mathrm{o})} = \phi(T_j^{(\mathrm{f})})$ of the homomorphism $\phi : D_G^{(\mathrm{fund})} \to D_G^{(\mathrm{o})}$, i.e. $[T_a^{(\mathrm{f})}, T_b^{(\mathrm{f})}]_- = \imath \sum_{c=1}^d f_{abc} T_c^{(\mathrm{f})} \Rightarrow [T_a^{(\mathrm{o})}, T_b^{(\mathrm{o})}]_- = \imath \sum_{c=1}^d f_{abc} T_c^{(\mathrm{o})}$. Recall that an operator A in the operator algebra (this can be matrices but also the algebra of differential operators as in quantum mechanics) closing the representation $\mathrm{Lie}\, D_G^{(\mathrm{o})}$ under matrix multiplication (it is up to now only closed under the Lie-bracket and addition) is a Casimir operator if and only if $[A, T_j^{(\mathrm{o})}]_- = 0$ for all $j = 1, \ldots, d$. We will construct such operators via a larger operator

$$C = \sum_{j=1}^{d} T_j^{(\mathrm{o})} \otimes T_j^{(\mathrm{f})} \tag{1}$$

which lives in the tensor space $\mathrm{Lie}\, D_G^{(\mathrm{o})} \otimes \mathrm{Lie}\, D_G^{(\mathrm{f})}$. Recall that a product of two operators $A_1 \otimes B_1$ and $A_2 \otimes B_2$ is given by $(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$.

(i) Show that for any $m \in \mathbb{N}$ the operator

$$C_m^{(\mathrm{o})} = \mathrm{Tr}_{\mathrm{fund}} C^m = \sum_{j_1, \ldots, j_m = 1}^{d} T_{j_1}^{(\mathrm{o})} \cdots T_{j_m}^{(\mathrm{o})} \mathrm{Tr}(T_{j_1}^{(\mathrm{f})} \cdots T_{j_m}^{(\mathrm{f})}) \tag{2}$$

is a Casimir operator of the representation $\mathrm{Lie}\, D_G^{(\mathrm{o})}$.
Hint: Proof first that $[C, T_b^{(\mathrm{o})}]_- = [T_b^{(\mathrm{f})}, C]_-$ and recall that the commutator of a product of operators is $[A_1 \cdots A_m, B]_- = \sum_{j=1}^m A_1 \cdots A_{j-1} [A_j, B]_- A_{j+1} \cdots A_m$.

(ii) The operators $C_1^{(\mathrm{o})}$ and $C_2^{(\mathrm{o})}$ are of particular interest in physics. What is there significance in physics for the groups $\mathrm{U}(1)$, $\mathrm{SU}(2)$, and $\mathrm{SO}(3)$?
What is $C_1^{(\mathrm{o})}$ and $C_2^{(\mathrm{o})}$ for the Lie-groups $\mathrm{U}(N)$, $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$? Calculate them explicitly for the fundamental and the adjoint representation. Calculate hereby also $\mathrm{Tr} T_a^{(\mathrm{adj})} T_b^{(\mathrm{adj})}$.
What is $C_{2m+1}^{(\mathrm{o})}$ for $\mathrm{SO}(N)$?
Calculate the explicit expression of C_3 for $\mathrm{SU}(3)$ in the fundamental as well as in the adjoint representation.

Exercise 13.2: (4 Points)

In this exercise we explicitly construct the Cartan-Weyl bases, the Chevalley bases and the Dynkin diagrams for $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$.

(i) Take the Cartan subalgebras in Homework Exercise 12.1 and employ the basis of the three Lie algebras from Homework Exercise 12.3. Show that the eigenvalue equation $[H_j, E_\alpha]_- = \alpha_j E_\alpha$ reduce to decoupled eigenvalue equations for 2×2 matrices for $\mathrm{SU}(N)$ and for 4×4 matrices for $\mathrm{SO}(N)$ and $\mathrm{USp}(2N)$.

(ii) Diagonalize these matrix equations and calculate the set of root vectors Δ and the eigenvectors E_α.

(iii) Calculate the angles between the root vectors and the relations between length of the root vectors. In particular, identify a set of positive root vectors, the corresponding simple root vectors α and their corresponding H_α and $E_{\pm\alpha}$. A more general definition of the set of positive root vectors Δ_+ than the one in the lecture is a set which contains one root vector of each pair of root vectors $\pm\alpha$ such that if $\alpha, \beta \in \Delta_+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_+$. A simple root vector is then an element $\gamma \in \Delta_+$ such that there is no $\alpha, \beta \in \Delta_+$ with $\gamma = \alpha + \beta$.

(iv) Calculate the Cartan matrix for all three groups and construct the Dynkin-diagrams.

Exercise 13.3: (2 Points)

We consider the group $\mathrm{SU}(3)$. This group serves as the flavor symmetry when QCD is composed of three quarks (u,d,s), only. This model is a good approximation in certain energy regimes. Some irreducible representations of this flavor group can explain the spectra. Construct all states of the following irreducible representations given by the highest weight states, $(\mu_1, \mu_2)_W$:

(i) $(1,1)_W$ (the adjoint representation corresponding to a subset of mesons (quark-antiquark) and can be found for the baryons (three quarks) as well),

(ii) $(3,0)_W$ (the representation corresponding to a subset of baryons). To which particles does the representation $(0,3)_W$ correspond?

What is the dimension of the three representations? What are the eigenvalues of the states with respect to operators in the Cartan subalgebra?