Winter term 2020/21	G. Akemann, T. Würfel	02.02.2021

[Submit your solutions up to 08.02. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 14

Exercise 14.1: (5 Points)

(i) Show that the Young tableaux dimension formula gives $(\mu_1+1)(\mu_2+1)(\mu_1+\mu_2+2)/2$ for $\mathrm{SU}(3)$, where μ_1 is the difference in length between the first and second row and μ_2 is the length of the second row.

(ii) Use the Young tableaux method to show that $3 \otimes \bar{3} = 8 \oplus 1$. Consider the tensor $\psi_a \phi_{bc} = -\psi_a \phi_{cb}$, which is a representation of $3 \otimes \bar{3}$ and can be interpreted as a quark and an anti-quark. Construct the two tensors corresponding to the two irreducible representations 8 and 1 which can be identified with the meson resonances measured in experiments. What are the transformation properties of these two tensors under the transformation $\psi_a \to U_a^b \psi_b$ and $\phi_{bc} \to U_b^{b'} U_c^{c'} \phi_{b'c'}$ with $U \in \mathrm{SU}(3)$?

(iii) Use the Young tableaux method to show that $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$. Consider the tensor $\psi_a^{(1)} \psi_b^{(2)} \psi_c^{(3)}$ which represents three quarks. Construct the four tensors corresponding to the irreducible representations 10, 8, 8, and 1. What are the transformation properties of these four tensors under the transformation $\psi_a^{(j)} \to U_a^b \psi_b^{(j)}$ with $U \in \mathrm{SU}(3)$?

Exercise 14.2: (4 Points)

(i) Show that the operator $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ satisfies,

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i(\eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\nu\sigma} L_{\mu\rho}). \tag{1}$$

(ii) Show that $P_\mu = i\partial_\mu$ satisfies,

$$[P_\mu, L_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \tag{2}$$

(iii) Show $[P^2, L_{\rho\sigma}] = 0$

(iv) Defining $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} P^\sigma$, show $[W^2, P_\mu] = 0$

Exercise 14.3: (3 Points)

Define the γ matrices by the anti-commutation relation $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.

(i) Show that $L_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ form a representation of the Lorentz algebra.

(ii) One representation of the gamma matrices is,

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \qquad \gamma_0 = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{3}$$

where σ_i are the Pauli matrices. Find the form of the generators J_i and K_i and comment on their form.

(iii) Check that the definitions

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{4}$$

give another possible choice for the γ matrices.