Winter term 2020/21	G. Akemann, T. Würfel	24.11.2020

[Submit your solutions up to 30.11. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 5

Exercise 5.1: (4 Points)

The Galilei group Gal(3) is the set of all possible transformations between the space-time coordinates of two inertial reference frames in non-relativistic physics, with spatial coordinates measured in righthanded orthonormal systems and a constant direction of time. That is, Gal(3) consists of space and time translations, three-dimensional rotations, and of proper Galilei transformations (or boosts) - where the latter corresponds to the case of inertial frames in uniform linear motion with respect to each other - and their compositions.

(i) To refresh your knowledge from a past Classical Mechanics lecture, write down a (non-trivial!) example of transformation $(t, x, y, z) \to (t', x', y', z')$ for each of the four classes of Galilei transformations listed above.

(ii) Check that the Galilei group with the composition of transformation...fulfills the group axioms!

(iii) Show that the translations form a normal subgroup of Gal(3).

(iv) Give examples showing that neither the subgroup of three-dimensional rotations nor that of Galilei boosts is normal in Gal(3).

Exercise 5.2: (4 Points)

Let G be a group. The *center* of G is the set of elements $g \in G$ that commute with every element of G, i.e.

$$Z(G) = \{g \in G \mid gg' = g'g \quad \forall g' \in G\} \tag{1}$$

If S denotes a subset of G, the *centralizer* $C_G(S)$ of S is the set of elements $g \in G$ that commute with each element of S, i.e.

$$C_G(S) = \{g \in G \mid gg' = g'g \quad \forall g' \in S\} \tag{2}$$

The *normalizer* $N_G(S)$ of S is defined by

$$N_G(S) = \{g \in G \mid gS = Sg\} \tag{3}$$

(i) Show that $Z(G)$, $C_G(S)$ and $N_G(S)$ are subgroups of G. How are $C_G(S)$ and $N_G(S)$ related, when $S = \{a\}$ (S contains only one element)?

(ii) Show that $Z(G)$ is Abelian and normal in G, and $C_G(S)$ is normal in $N_G(S)$. Note that the centralizer is not necessarily normal in G, nor Abelian.

(iii) Show that if H is a subgroup of G, then H is a normal subgroup of its normalizer $N_G(H)$.

(iv) Consider the centralizer of a single element a. Find a bijection between $G/C_G(a)$ and the conjugacy class (a) of a.

Exercise 5.3: (4 Points)

Let us consider the set of Pauli matrices and the two-dimensional unit-matrix,

$$\Sigma = \{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \qquad (4)$$

(i) Why is the set Σ not a group with respect to matrix multiplication? Extend Σ to a group G by closing the matrix multiplication.

(ii) Calculate the center of G identify it with a discrete group you already know.

(iii) Calculate all conjugacy classes of G.

(iv) Calculate the centralizer of all elements of G.