[Submit your solutions up to 07.12. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 6

Exercise 6.1: (3 Points)

Consider the group of rotations, SO(3), composed of the matrices;

$$R(\theta) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in [0, 2\pi) \right\}. \tag{1}$$

By changing basis from $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$ to $(\frac{1}{\sqrt{2}}(-\mathbf{e_x} + \mathbf{ie_y}), \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{e_x} + \mathbf{ie_y}), \mathbf{e_z})$ show that this representation is completely reducible.

Compute the characters for each representation and comment on their value.

Exercise 6.2: (3 Points)

In a previous exercise you already encountered the Galilei group Gal(3).

(i) Show that a 5-dimensional linear representation of Gal(3) consists of the matrices of the type

$$\begin{pmatrix} \underline{\mathbf{R}} & \vec{v} & \vec{a} \\ \vec{0}^T & 1 & \tau \\ \vec{0}^T & 0 & 1 \end{pmatrix} \tag{2}$$

where $\vec{0}^T$ denotes a row vector with three zero entries, while $\vec{v}$ and $\vec{a}$ are two column vectors with three arbitrary real entries and $\underline{\mathbf{R}}$ is a three-dimensional rotation matrix, $\underline{\mathbf{R}} \in$ SO(3).

(ii) Is this representation faithful? Is it completely reducible?

Exercise 6.3: (4 Points)

Let us consider the $m \times m$ matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 \\ & 0 & 1 & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \\ 0 & & & & 0 & 1 \\ 1 & 0 & & \cdots & & 0 \end{pmatrix}, \tag{3}$$

i.e. $T_{ij} = 1$ for all $j - i = 1$ and for $i = m$ and $j = 1$ and otherwise $T_{ij} = 0$.

(i) Construct a group, $\widetilde{G}$, from T via matrix multiplication. The matrix group $\widetilde{G}$ is a representation of a finite group, G, you already know. Which group is this?

(ii) With help of Maschke's theorem we know that the representation $\widetilde{G}$ is completely reducible. What are all irreducible representations of $\widetilde{G}$? Calculate for this purpose the eigenspaces of T.