Winter term 2020/21	G. Akemann, T. Würfel	08.12.2020

[Submit your solutions up to 14.12. *before* the Tutorial in LernraumPlus]

Symmetries in Physics - Homework Sheet No. 7

Exercise 7.1: (3 Points)

Let $D(g)$ be a (matrix) representation of a group G. For every $g \in G$ $D^\star(g)$ denotes the complex conjugate of $D(g)$.

(i) Show that $D^\star(g)$ is also a representation.

(ii) If D and $D^\star$ are equivalent representations with $D^\star(g) = C^{-1}D(g)C$ for all $g \in G$, show that if D is a irreducible then $CC^\star = \lambda\mathbb{I}$ with $\lambda \in \mathbb{C}$.

(iii) If D is unitary show also that $CC^\dagger = \mu\mathbb{I}$ with $\mu \in \mathbb{C}$. In this case show that C may be rescaled to be either symmetric or anti-symmetric.

Exercise 7.2: (3 Points)

(i) Let $(\mathbf{L}_j)_{k\ell} = -i\epsilon_{jk\ell}$. Show that the following two representations of a rotation $\mathbf{R}(\hat{\boldsymbol{n}}, \varphi)$ are equivalent:

$$\mathbf{R}(\hat{\boldsymbol{n}}, \varphi) = \exp(-i(\hat{\boldsymbol{n}} \cdot \underline{\mathbf{L}})\varphi) = \cos\varphi\, \mathbb{1} + (1 - \cos\varphi)\hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}} + \sin\varphi\, ({}^*\hat{\boldsymbol{n}}).$$

Here, ${}^*\hat{\boldsymbol{n}}$ is the antisymmetric matrix with entries $({}^*\hat{\boldsymbol{n}})_{ij} = -\epsilon_{ijk}\hat{n}_k$.
Hint: Compute the Taylor series of both sides, and compare the terms at each order.

(ii) Decompose a vector $\boldsymbol{x}$ into components parallel and perpendicular to $\hat{\boldsymbol{n}}$, and study how $\mathbf{R}(\hat{\boldsymbol{n}}, \varphi)$ acts on these components.

Exercise 7.3: (2 Points)

Let G be a group and D an irreducible representation of G. Show that $D \otimes D$ is an irreducible representation of the product group $G \times G$.

Exercise 7.4: (2 Points)

For a group action on a set Ω define the *orbit of* x to be $\mathrm{Orb}(x) = \{g \cdot x \in \Omega : g \in G\}$ and define the *stabiliser of* x to be $\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$.

(i) Show that $\mathrm{Stab}(x)$ is a subgroup.

(ii) By finding a bijection from $G/\mathrm{Stab}(x)$ to $\mathrm{Orb}(x)$ show that $|G| = |\mathrm{Orb}(x)||\mathrm{Stab}(x)|$