Lecture 1 (+2)

1.1. Motivation: why is RMT interesting for < Physicists / Mathematicians

1.2. Matrices & prob. distr. (G)O/U/SE, Wishart

1.3. Eigenvalue basis, Jacobian, j.p.d.f. for eigenvalues

1.1. Motivation: Physics

- take an $N\times N$ matrix $H = H^T$ real symm, with elements H_{ij} normal Gaussian random var $N(0,1)$
- diagonalise and order N eigenvalues $\lambda_1 > \lambda_2 .. > \lambda_N$
- sample many times (Matlab) and determine the empirical prob. dist. p_E that say

$\lambda_{\frac{N}{2}+1} - \lambda_N = s$ normalised s.th. $\int_0^\infty ds\, p_E(s) = 1 = \int_0^\infty ds\, s\, p_E(s)$

You will find that it agrees very well with

$$p_E(s) \approx \boxed{P_{WS}(s) = a_\beta s^\beta e^{-b_\beta s^2}}$$

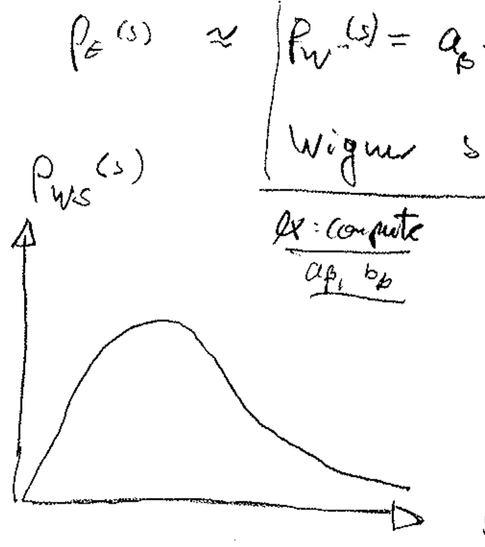
Wigner surmise

, $a, b > 0$ const. s.th. $P_{WS}(s)$ is norm.

approx for $N \gg 1$! $\beta = 1$

ex: compute a_β, b_β

$P_{WS}(s)$



- works for other spacings too (away from λ_1, λ_N)
- extremely simple
- take spacing from Eigenergies of various phys. syst (→ plot), or parking cars, bus schedule etc ✓

- simple to solve RM problem describes a variety of complicated physics problems
- $\exists$ other sym. classes of RM with $H_{ij} \in \mathbb{C}, \in \mathbb{H}$ $\boxed{\Rightarrow \beta = 2, 4}$

  or other combinations, e.g. $H \neq H^{+} (= H^{T*})$ complex non-Hermitian [Ginibre]

  or consider $W = HH^{+}$ positive compl. Herm. [Wishart]

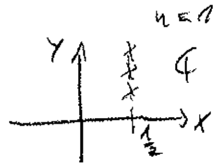
Q: what is the distribution of the smallest ev of W $\lambda_1 > 0$?

$\rightarrow$ application in particle phys, theorie of strong interactions

Motivation: Mathematics

- Random Matrix Theory combines a variety of fields:
  - Linear algebra
  - classical Orthogonal Polynomials
  - asymptot. analysis of OP $\rightarrow$ Riemann-Hilbert problem, Complex analysis
  - probability theory
  - Graph theory
  - Number theory: Riemann Hypothesis $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ (Re s > 1)

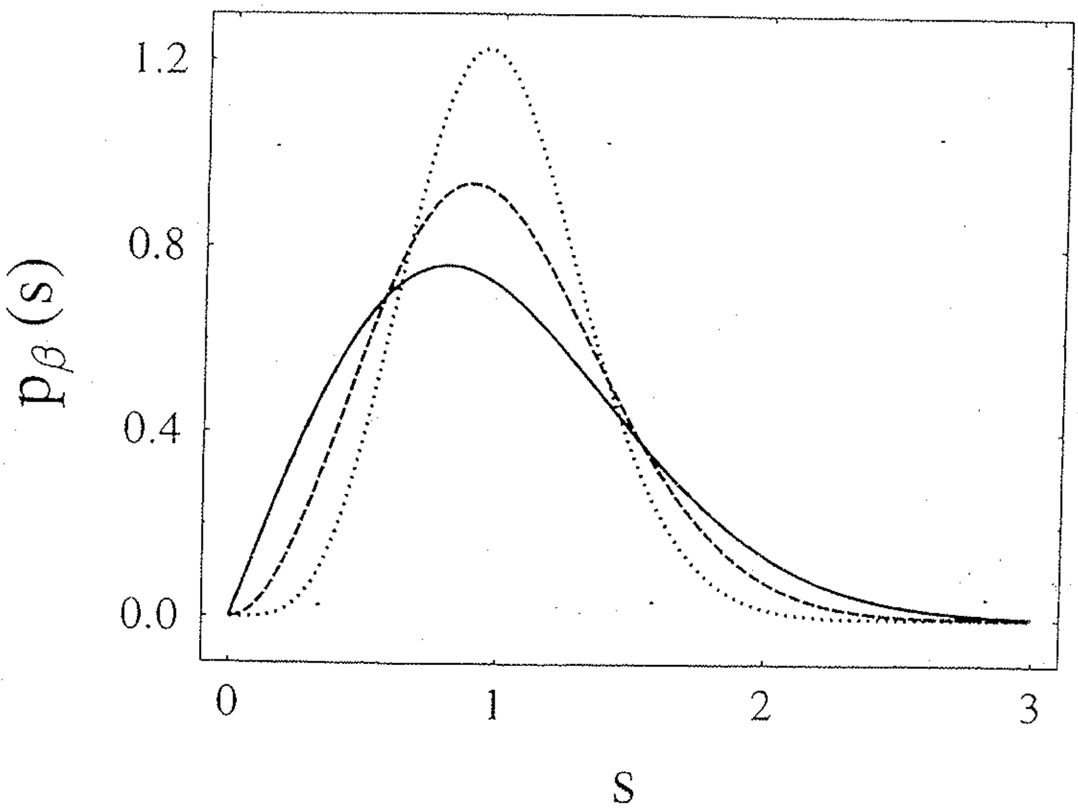
    has nontrivial 0's on

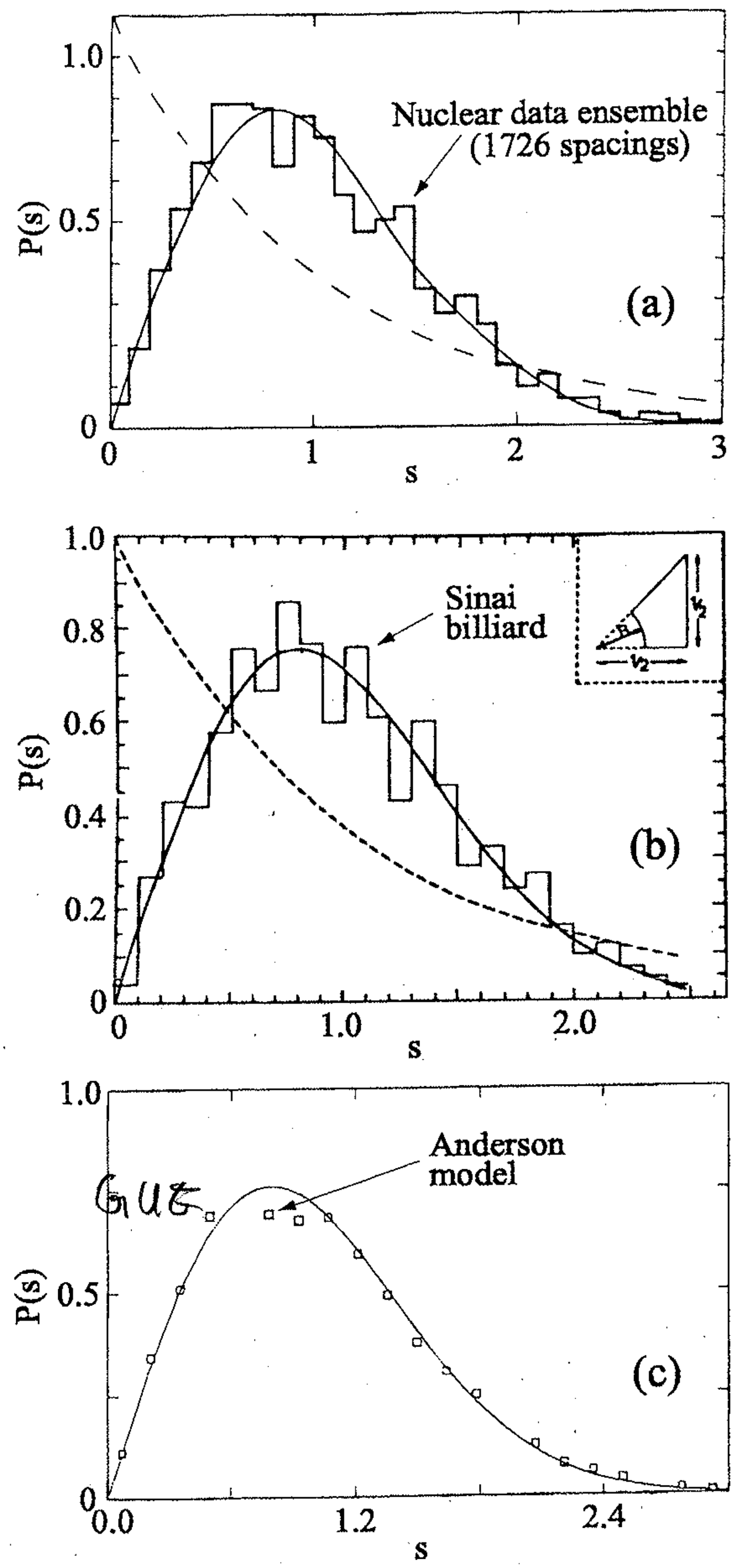
  $\rightarrow$ Correlation of zero's follows RM statistics

$$p_\beta(s) \sim s^\beta e^{-cs^2}$$

$\beta = 1$ GOE
2 GUE
4 GSE



ises $p_\beta(s)$ for the nearest-neighbor spacing distribution. The solid line is
l), the dashed line is the result for unitary symmetry ($\beta = 2$) and the dotted line
l). We notice the importance of the repulsion law s^β for small spacings.



Nuclear data ensemble
(1726 spacings)
(a)
P(s)
s
Sinai
billiard
(b)
P(s)
s
Anderson
model
Gut
(c)
P(s)
s

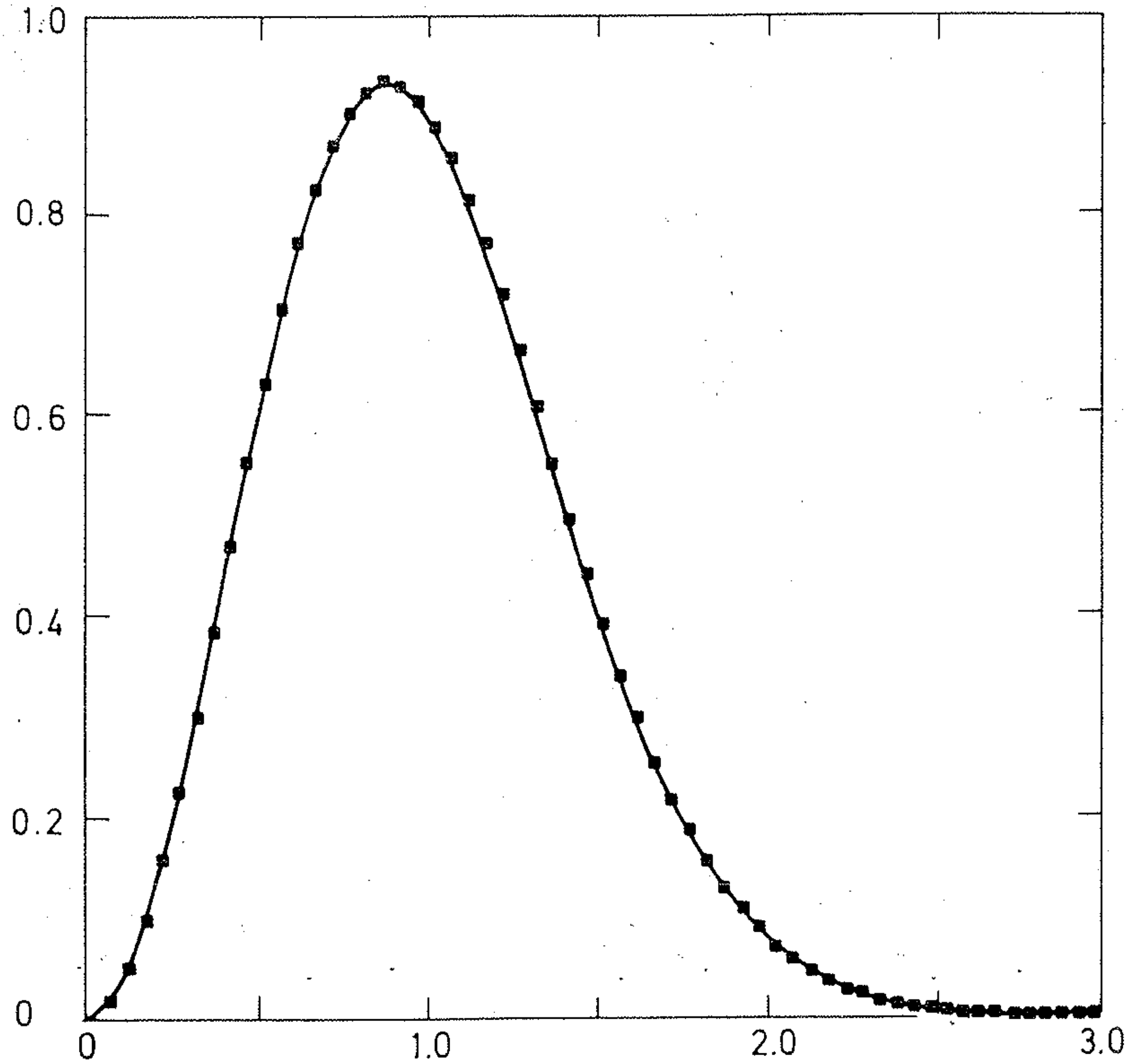


FIG. 1.14. The same as Figure 1.12 but for the 79 million zeros around the 10^{20}th zero. From Odlyzko (1989). Copyright ©, 1989, American Telephone and Telegraph Company reprinted with permission.

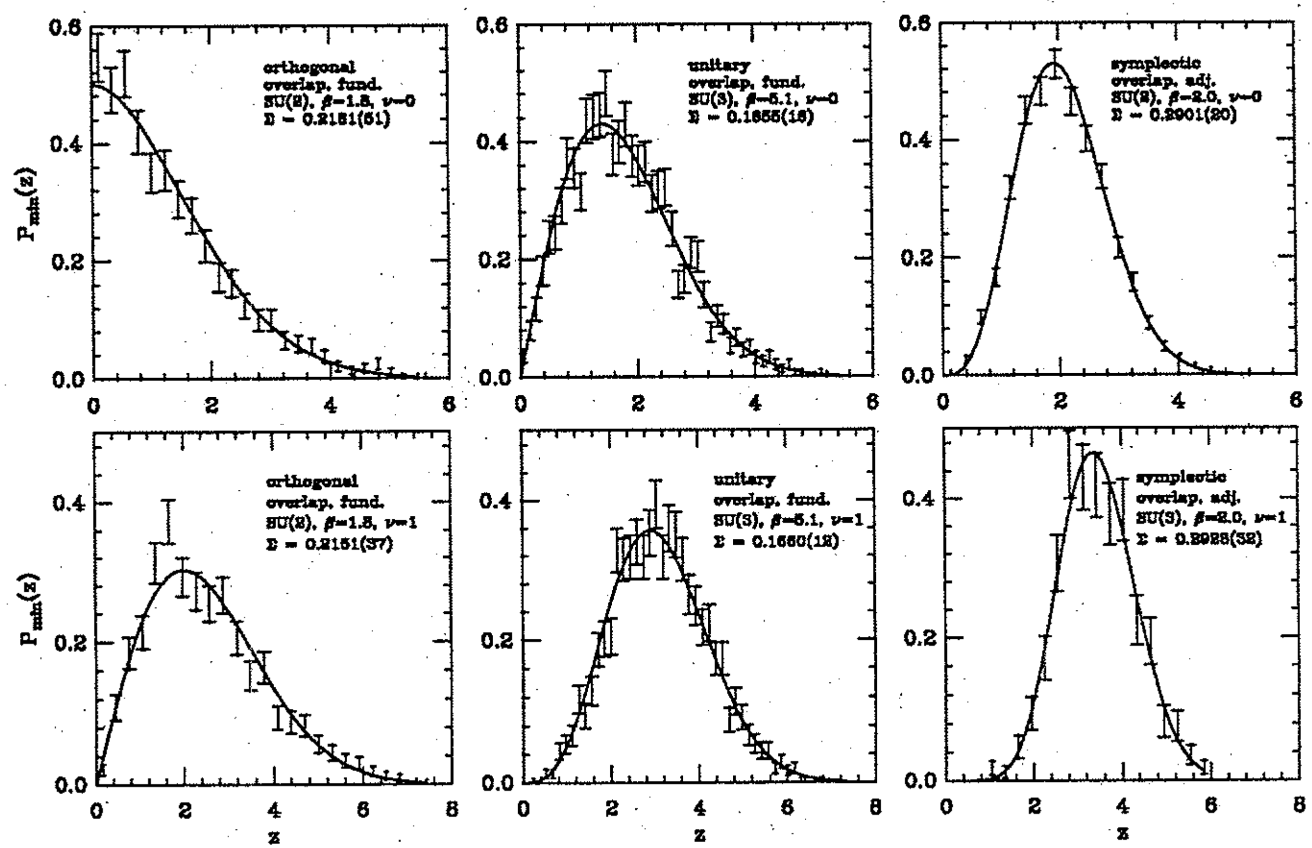


Figure 4. Plots of the distribution of the lowest eigenvalue for all three ensemble in the lowest two topological sectors. The curves are fits to the predictions from random matrix theory.

1.2 Matrices and probability distribution

- We consider 3 types of matrix elements:

<u>real</u>: $H_{kl} \in \mathbb{R}$ Dyson index $\beta = 1$ (# of indep var per matrix elem.)

<u>complex</u>: $H_{kl} = \mathrm{Re}(H_{kl}) + i\,\mathrm{Im}(H_{kl}) \in \mathbb{C}$ Dyson index $\beta = 2$

<u>real quaternions</u>:

recall a quaternion $q \in \mathbb{H}$ has a rep. as a 2×2 matrix:

$$q = q^{(0)} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{1}_2 \equiv e_0} + q^{(1)} \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{e_1} + q^{(2)} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{e_2} + q^{(3)} \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{e_3} \equiv q^{(0)} + \vec{q}\cdot\vec{e}$$

it is called real quat. if all 4 $q^{(j)} \in \mathbb{R}$ $j = 0, 1, 2, 3$

(in general the $q^{(j)}$ may be complex)

it holds $e_1^2 = e_2^2 = e_3^2 = -\mathbb{1}_2$, $\mathrm{tr}\, e_i = 0$ $e_1 e_2 = e_3$ + cyclic

<u>quat. real</u> $H_{kl} = \sum_{j=0,1,2,3} H_{kl}^{(j)} e_j \in \mathbb{H}$ Dyson index $\beta = 4$

- conjugate quaternion of q:

$$\bar{q} \equiv q^{(0)} - \vec{q}\cdot\vec{e} \qquad (\Rightarrow q = \bar{q} \text{ is a scalar})$$

- complex conjugate of q:

$$q^* \equiv q^{(0)*} + \vec{q}^{\,*}\cdot\vec{e} \qquad (\Rightarrow q = q^* \text{ is quaternion real})$$

- Hermitian conjugate: combine

$$q^\dagger \equiv \bar{q}^* = q^{(0)*} - \vec{q}^{\,*}\vec{e} \qquad (= \text{Hermitian conj of the } 2\times 2 \text{ matrix})$$

Given H has matrix elements $H_{ij} \in \mathbb{H}$

We define the Hermitian conjugate $(H^\dagger)_{kj} = H^\dagger_{jk}$

and the dual $(H^R)_{kj} = \bar{H}_{jk}$

$\Rightarrow$ a matrix H satisfying

$H^R = H = H^\dagger$ (Hermitian self dual) is quaternion real

it has $N + 4\,\frac{N(N-1)}{2} = N(2N-1)$ indep. parameters (degrees of freedom dof)

We define a non singular matrix U to be symplectic if it satisfies

$$\boxed{U^R = U^\dagger = U^{-1}}$$

• Probability distribution of matrix elements:

$\beta = 1$: consider $H = H^T$ real symmetric, $H_{ij} = H_{ji} \in \mathbb{R}$

$\Rightarrow$ H has $N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$ d.o.f.

$$\Rightarrow \mathrm{Tr}(H^2) = \mathrm{Tr}\, HH^T = \sum_{i,j=1}^{N} H_{ij} H^T_{ji} = \sum_{i,j} H_{ij}^2$$

$$= \sum_i H_{ii}^2 + 2\sum_{i<j} H_{ij}$$

← these are all indep matrix elements

def $P_N^{(\beta=1)}(H) \equiv \frac{1}{(2\pi)^{\frac{N(N-1)}{2}\cdot\frac{1}{2}}}\, e^{-\frac{1}{4}\mathrm{Tr}\, H^2}$ jpdf of matrix elem.

def $Z_N^{(\beta=1)} \equiv \prod_{i=1}^{N} \int_{\mathbb{R}} \frac{dH_{ii}}{\sqrt{2\pi}} \prod_{i<j} \int_{\mathbb{R}} \frac{dH_{ij}}{\sqrt{\pi}}\, e^{-\frac{1}{4}\mathrm{Tr}\, H^2} \equiv \int dH\, P$ partition function

Lebesgue measure

- the Ensemble of matrices with $P_N^{(\beta=1)}(H)$ is called

Gaussian Orthogonal Ensemble GOE

because Orthogonal trafo diag. $H=H^T$

- in other words if $\{\xi_{ij}\}$ are $\frac{N(N+1)}{2}$ are iid. Gaussian random variables $\mathbb{E}(\xi_{ij})=0$, $\mathbb{E}(\xi_{ij}^2)=1$ $(\mathcal{N}(0,1))$

we have for
$$H=\begin{pmatrix} \sqrt{2}\,\xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{12} & \sqrt{2}\,\xi_{22} & \xi_{23} \\ & & \xi_{33} \end{pmatrix}$$

- generalisations:
  - consider $\{\xi_{ij}\}$ iid real random variables $\mathbb{E}(\xi)=0$ $\mathbb{E}(\xi^2)=1$ s.th. all moments exist. $H_{ij}=\xi_{ij}$ is called Wigner matrix (non-invariant)
  - $\boxed{P_N^{(\beta=1)}(H) \sim e^{-\frac{1}{4}\mathrm{Tr}\,V(H)}}$ with potential (polynom or analy funct.) $V(H)=H^2+\sum_{j=2}^{k} g_{2j} H^{2j}$

is called OE, it is inv under trafo $H\to OHO^T$, $O\in O(N)$

$\beta=2$: Consider $H=H^+$ complex Hermitian $H_{ij}=H_{ji}^*\in\mathbb{C}$

$\Rightarrow$ H has $N+2\cdot\frac{N(N-1)}{2}=N^2$ d.o.f.

$$\Rightarrow \mathrm{Tr}\,H^2=\mathrm{Tr}\,HH^+=\sum_{i,j}H_{ij}H_{ij}^*=\sum_i \overset{\in\mathbb{R}}{H_{ii}^2}+2\sum_{i<j}\underbrace{|H_{ij}|^2}_{(\mathrm{Re}\,H_{ij})^2+(\mathrm{Im}\,H_{ij})^2}$$

$$P_N^{\beta=2}(H)=\frac{1}{(2\pi)^{\frac{N^2}{2}}}e^{-\frac{1}{2}\mathrm{Tr}\,H^2}$$

$$Z_N^{\beta=2}=\prod_{i=1}^{N}\int_{\mathbb{R}}\frac{dH_{ii}}{\sqrt{2\pi}}\prod_{i<j}^{N}\int_{\mathbb{R}}\frac{d\,\mathrm{Re}\,H_{ij}}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d\,\mathrm{Im}\,H_{ij}}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\mathrm{Tr}\,H^2}$$ def GUE

Gaussian Unitary Ensemble, UE corresp

$$H = \begin{pmatrix} z_{11} & \frac{z_{12}+i\eta_{12}}{\sqrt{2}} & \\ \frac{z_{12}-i\eta_{12}}{\sqrt{2}} & z_{22} & \\ & & \ddots \end{pmatrix}$$ with $\{z_{ij}, \eta_{ij}\}$ $\mathcal{N}(0,1)$ iid.

- the normalisation $\frac{1}{\sqrt{2\pi}^N} e^{-\frac{1}{2}\mathrm{Tr}H^2}$ is somewhat arb.

($\beta = 4$:) analogous construction for $H = H^R = H^\dagger$

$\Rightarrow \ldots \Rightarrow \mathrm{Tr}\, H^2 = \sum_{i=1}^{N} H_{ii}^2 + 2 \sum_{i<j} \sum_{k=1,2,3} \left(H_{ij}^{(k)}\right)^2$ likewise

$P_N^{\beta=4}$, $Z_N^{\beta=4}$ Gaussian Symplectic Ensemble GSE and SE

* Some facts from Linear Algebra [Mehta Appendix 1.3]

- $H = H^T$ $N\times N$ real sym. can be diagonalized by orth. real matrix O
  $O \in O(N)$, $OO^T = O^T O = 1$, $H = O \Lambda O^T$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$
  with real eigenvalues $\lambda_{j=1..N} \in \mathbb{R}$

- $H = H^\dagger$ $N\times N$ complex Hermitian matrix is diag. by Unitary U
  $U \in U(N)$, $UU^\dagger = U^\dagger U = 1$ $H = U \Lambda U^\dagger$

- $H = H^\dagger = H^R$ quaternion real self dual $N\times N$ is diag. by symplectic matrix $B \in Sp(N)$

$$H = B\,\Theta\, B^{-1} = B\,\Theta\, B^R \quad \text{, with } \Theta = \mathrm{diag}\begin{pmatrix} \theta_j & 0 \\ 0 & \theta_j \end{pmatrix} = \theta_j e_0$$

$\theta_j \in \mathbb{R}$