We are interested in spectral properties, e.g. distr. of indiv. eigenvalue, density correlation functions

<u>Q:</u> What is the probability distr. fct. of the eigenvalues?

1.3 Computation of j.p.d.f. – Eigenvalue basis

e.g. $H = O \Lambda O^T$ is a change of variables

$\Rightarrow$ Jacobian $\left[\det \left| \dfrac{\partial H_{ij}}{\partial(\lambda_n, O_{kl})} \right| \right]^{\frac{1}{2}}$

Important: Keep the same # of dof on lhs and rhs!

later

$\beta = 1$:

- $H = H^T$, $H_{ij} \in \mathbb{R}$ $\Rightarrow$ $N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$ dof
- $\Lambda = \mathrm{diag}\,\lambda_i$, $\lambda_j \in \mathbb{R} \Rightarrow$ N dof
- $OO^T = 1 = O^T O$ N^2 var with cond. $\sum_{j=1}^{N} O_{ij} O_{kj} = \delta_{ik} = \delta_{ki}$ $N + \frac{N(N-1)}{2}$ cond $\Rightarrow N^2 - N - \frac{N(N-1)}{2}$ dof

$\Sigma = \frac{N(N+1)}{2}$ ✓

trick to compute Jacobian: if we can write

$\mathrm{Tr}(dH\,dH) = \begin{pmatrix} d\Lambda \\ dO \end{pmatrix} g \begin{pmatrix} d\Lambda \\ dO \end{pmatrix}$ we have $g = \dfrac{\partial H_{ij}}{\partial(\Lambda, O)}$

inv. volume el.

- g will be different for $\beta = 1, 2, 4$

$\rightarrow$ resulting Jacobian is indep. of O, U, B and only depends on β = # dof per matrix element! Vandermonde$^\beta$

β general:

consider $H = S \Lambda S^\dagger$, $SS^\dagger = S^\dagger S = \mathbb{1}$ $\forall \beta = 1, 2, \cancel{4}$

$$\Rightarrow \quad 0 = d(SS^\dagger) = dS\,S^\dagger + S\,dS^\dagger \;, \quad S^\dagger dS + dS^\dagger S = 0$$

$$dH = dS \Lambda S^\dagger + S\, d\Lambda\, S^\dagger + S \Lambda\, dS^\dagger$$

$$= S\left(S^\dagger dS \Lambda + d\Lambda + \Lambda \underbrace{dS^\dagger S}_{= -S^\dagger dS} \right) S^\dagger$$

$$\boxed{dH = S\left(d\Lambda + [S^\dagger dS, \Lambda]\right) S^\dagger} \;, \quad [A,B] = AB - BA$$

commutator

$H = H^\dagger$

$$\Rightarrow \quad \mathrm{Tr}\left[dH\, dH^\dagger\right] = \mathrm{Tr}\left[\left(d\Lambda + [S^\dagger dS, \Lambda]\right)^2 \right]$$

$\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$

$$= \mathrm{Tr}\Big[d\Lambda^2 + \underbrace{2 d\Lambda [S^\dagger dS, \Lambda]}_{2 d\Lambda S^\dagger dS \Lambda - 2 d\Lambda \Lambda S^\dagger dS} + [S^\dagger dS, \Lambda]^2 \Big]$$

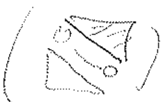
$2 d\Lambda\, S^\dagger dS \Lambda - 2 d\Lambda \Lambda\, S^\dagger dS \quad \overset{\mathrm{Tr\ cyclic}}{\Longrightarrow}$ cancel

diag matrices commute

- $\mathrm{Tr}([A,B]^2) = \mathrm{Tr}((AB - BA)^2) = \mathrm{Tr}(ABAB - 2ABBA + BABA)$

$$\boxed{\mathrm{Tr}\left[dH\, dH^\dagger\right] = \mathrm{Tr}\left[d\Lambda^2 + 2 (S^\dagger dS) \Lambda (S^\dagger dS) \Lambda - 2 \Lambda^2 (S^\dagger dS)^2\right]}$$

def $dt = S^\dagger dS$, $dt^\dagger = -dt$ anti hermitian (up-1 anti symm)



write out

$$= \sum_{i=1}^{N} d\lambda_i^2 + 2 \sum_{ij=1}^{N} \left(dt_{ij} \lambda_j\, dt_{ji} \lambda_i - \lambda_i^2\, dt_{ij}\, dt_{ji} \right)$$

$i = j$ gives 0

$\underline{\beta = 1}$: $\quad S = O \quad , \quad dt = O^T dO \quad O \in O(N)$

<u>indep var</u> lhs dH_{ij} $\quad i \leq j \quad \frac{N(N+1)}{2}$

rhs $d\lambda_i$ N, $\quad dt_{ij}$ $\quad i<j \quad \frac{N(N-1)}{2}$, $\quad dt_{ij}\, dt_{ji} = -(dt_{ij})^2$

$$\Rightarrow \mathrm{Tr}\left[dH\, dH^T\right] = \sum_{i=1}^{N} d\lambda_i + 2 \sum_{i<j} \underbrace{\left(\lambda_i^2 + \lambda_j^2 - \lambda_j \lambda_i - \lambda_i \lambda_j\right)}_{(\lambda_i - \lambda_j)^2} dt_{ij}^2$$

diagonal in indep variables !

$$\Rightarrow \mathcal{G}^{\beta=1}_{\frac{N(N+1)}{2} \times \frac{N(N+1)}{2}} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 2(\lambda_1-\lambda_2)^2 & & & \\ & & & & 2(\lambda_1-\lambda_3)^2 & & \\ & & & & & \ddots & \\ & & & & & & 2(\lambda_{N-1}-\lambda_N)^2 \end{pmatrix}$$

$\underbrace{}_{N} \quad \underbrace{}_{\frac{N(N-1)}{2}}$

$$\Rightarrow \sqrt{\det \mathcal{G}} = 2^{\frac{N(N-1)}{4}} \prod_{i<j}^{N} |\lambda_i - \lambda_j|^1 \qquad \text{Jacobian } \beta = 1$$

$$\Delta_N(\{\lambda\}) = \prod_{i>j} (\lambda_i - \lambda_j) \qquad \text{Vandermonde det}$$

alternative: from $dH = S(d\Lambda + [\ ])S^T$ compute

$$\frac{\partial\, dH_{i\leq j}}{\partial(d\lambda_i, dt_{i<j})} = \mathcal{G}^{\beta=1} \rightarrow \text{Mehta ch. 3}$$

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$\underline{\beta = 2}$ $\quad S = U \quad dt = U^\dagger dU \quad U \in U(N) \quad H = U \Lambda U^\dagger$

- $H = H^\dagger$ complex hermitian $\Rightarrow N + 2\frac{(N-1)N}{2} = N^2$ indep var rhs $\underline{}$ $dH_{ij},\ i \leq j \begin{cases} i=j \in \mathbb{R} \\ i \neq j \in \mathbb{C} \end{cases}$
- $\Lambda = \mathrm{diag}\, \lambda_i, \lambda_j \in \mathbb{R} \Rightarrow N$ dof.
- $UU^\dagger = U^\dagger U = 1$ unitary $\quad \sum_j U_{ij} U^*_{kj} = \delta_{ik}$:

$2N^2 - N^2$ cond $= N^2$ dof

$\Rightarrow$ rhs has $N^2 + N$ dof ?

wayout: take dt anti-hermitian $\Rightarrow N^2 - N$ as indep d.o.f.

(why: $H = U \Lambda U^\dagger$ is invariant under $U \to U\hat{U}$

with $\hat{U} = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_N} \end{pmatrix}$, ev λ_i unchanged

so we only need $U \in U(N) / U(1)^N$)

$$\Rightarrow \mathrm{Tr}(dH dH^\dagger) = \sum_{i=1}^N d\lambda_i^2 + 2 \sum_{i<j} (\lambda_i^2 + \lambda_j^2 - \lambda_i\lambda_j - \lambda_j\lambda_i)\, dt_{ij}\, dt^*_{ij}$$

again diag is indep d.o.f.

$$G = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 2(\lambda_1-\lambda_2)^2 & \\ & & & & 2(\lambda_1-\lambda_2)^2 \\ & & & & & \ddots \end{pmatrix} \begin{matrix} \leftarrow \mathrm{Re}\, dt_{ij} \\ \leftarrow \mathrm{Im}\, dt_{ij} \end{matrix}$$

$\underbrace{}_{N}$ $\underbrace{}_{N^2-N}$

$$\Rightarrow \boxed{\sqrt{\det G}} = 2^{\frac{N(N-1)}{2}} \prod_{i<j}^N |\lambda_i - \lambda_j|^2 \sim |\Delta_N(\{\lambda\})|^2 \text{ Jacobian}$$

$\beta = 2$

$\underline{\beta = 4}$: dH (and H) has $2N^2 - N$ d.o.f.

$$d\Lambda = \mathrm{diag}\left(\begin{pmatrix} d\theta_j & 0 \\ 0 & d\theta_j \end{pmatrix}\right) \qquad N \text{ d.o.f.}$$

$$dt_{i<j} = \text{need } 2N^2 - 2N = 2N(N-1) = 4\,\frac{N(N-1)}{2}$$

4 per upper triang. 2×2 matrix

✓

$\Rightarrow$ we get $(\lambda_1 - \lambda_2)^2$ 4 times from $dt_{12}^{(j=0,1,2,3)}$

$$\Rightarrow \quad \text{Jacobian} = 2^{N(N-1)} \prod_{i<j}^{N} |\lambda_i - \lambda_j|^{4 \leftarrow \beta}$$

- $\mathrm{Tr}\, V(H)$ is invariant under rotations

$$\Rightarrow \quad Z_N^\beta = \int dH\, e^{-\mathrm{Tr}\, V(H)}$$

$\int dH \equiv \prod \int_{\mathbb{R}} \frac{dH_{ij}}{\sqrt{2\pi}}$ — induced natural Haar

same Jacobian

$$\boxed{Z_N^\beta = 2^{\frac{\beta N(N-1)}{4}}\, C_N^{(\beta)} \underbrace{\left(\prod_{i=1}^{N} \int_{\mathbb{R}} d\lambda_i\right) e^{-\sum_{i=1}^{N} V(\lambda_i)} |\Delta_N(\{\lambda\})|^\beta}}$$

this will be computed later

with $C_N^{(\beta)} = \begin{cases} \beta = 1 & \int dO \quad \text{volume of } O(N) \\ \beta = 2 & \int dU \quad \text{vol of } U(N)/U(1)^N \\ \beta = 4 & \int dB \quad \text{vol of } Sp(N)/U(1)^N \end{cases}$

- these constants will drop out when we compute invariant expectation values, e.g. of $\mathrm{Tr}(H^k)$

$$\mathbb{E}\left(\mathrm{Tr}(H^k)\right) \equiv \frac{1}{Z_N^\beta} \int dH\, \mathrm{Tr}(H^k)\, e^{-\mathrm{Tr}\, V(H)}$$

* we will compute Z_N^β and all eigenvalue density correlation functions ("marginals") using Orthogonal polynomials!

Orthogonal Polynomials of real variables

- Let $\omega(\lambda)>0$ be a measurable weight on D such that for all moments $m_k \equiv \int_D d\lambda\, \omega(\lambda)\, \lambda^k < \infty$

- Consider $\tilde{P}_k(\lambda) = \lambda^k + d_k \lambda^{k-1} + \dots$ monic polynomial of degree k

It is called <u>orthogonal polynomial (OP)</u>

if it satisfies orthogonality (scalar product)

$$\boxed{\int_D d\lambda\, \omega(\lambda)\, \tilde{P}_k(\lambda)\, \tilde{P}_\ell(\lambda) = \delta_{k\ell}\, h_k} \quad \forall k,\ell \in \mathbb{N}$$

- the $h_k > 0$ are the squared norms $\|\tilde{P}_\ell\|^2$

<u>Note</u> $\tilde{P}_k(\lambda)$ can be constructed recursively using Gram-Schmidt

$$\tilde{P}_k(\lambda) \sim \det \begin{vmatrix} m_0 & m_1 & & m_k \\ m_1 & m_2 & & \\ \vdots & & & \\ m_{k-1} & m_k & & m_{2k-1} \\ 1 & \lambda & \dots & \lambda^k \end{vmatrix}$$

(→ normalisation is $\det(m_{i+j})$)

- the OP satisfy a 3-step recurrence relation

$$\boxed{\lambda \tilde{P}_k(\lambda) = \tilde{P}_{k+1}(\lambda) + a_k \tilde{P}_k(\lambda) + b_k \tilde{P}_{k-1}(\lambda)}$$

proof: consider $\lambda \tilde{P}_k(\lambda) = \sum_{\ell=0}^{k+1} \alpha_k^\ell \tilde{P}_\ell(\lambda)$, $\int d\lambda\, \omega\, \lambda \tilde{P}_k \tilde{P}_j$

- the coefficients a_k, b_k can be expressed in terms of norms h_k and second coeff d_k (see e.g. Abramowitz-Stegun, chapter 22)
- $\tilde{P}_k'(\lambda)$ is also expressible, later

<u>example Hermite polynomials → GUE</u>

$$\omega(\lambda) = e^{-\lambda^2} \, , \quad \tilde{P}_n(\lambda) = 2^{-n} H_n(\lambda)$$

$$\Rightarrow \quad h_n = \sqrt{\pi}\, 2^{-n} n! \quad , \; a_k = 0 \text{ from parity}$$

- for an even weight on an even interval D we have $a_k = 0 \;\; \forall k \in \mathbb{N}$ from parity $\tilde{P}_k(-\lambda) = (-)^k \tilde{P}_k(\lambda)$

example $D = \mathbb{R}$, $\omega(\lambda) = e^{-V(\lambda)}$, $V(\lambda) = V(-\lambda)$

- we can define <u>orthonormal polynomials</u> as

$$P_k(\lambda) \equiv \frac{1}{\sqrt{h_k}} \tilde{P}_k(\lambda) \quad \Rightarrow \quad \boxed{\int_D d\lambda \, \omega(\lambda) P_k(\lambda) P_\ell(\lambda) = \delta_{k\ell}}$$

⇒ recurrence is symmetric

$$\boxed{\begin{aligned} \lambda P_k(\lambda) &= c_{k+1} P_{k+1}(\lambda) + a_k P_k(\lambda) + c_k P_{k-1}(\lambda) \\ \text{with} \quad c_k &= \sqrt{\frac{h_k}{h_{k-1}}} \end{aligned}}$$

- define the kernel of orthonormal $P_k(\lambda)$

$$\boxed{K_N(x,y) \equiv \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)}$$

<u>Christoffel-Darboux:</u> for $x \neq y$

exercise: proof

$$\boxed{K_N(x,y) = c_N \frac{P_N(x) P_{N-1}(y) - P_N(y) P_{N-1}(x)}{x-y}}$$

- <u>Contraction properties:</u> $\displaystyle\int_D d\lambda \, \omega(\lambda) K_N(\lambda,\lambda) = N$

$$\int_D d\lambda \, \omega(\lambda) K(\lambda, x) = 1 \quad ; \quad \int_D d\lambda \, \omega(\lambda) K_N(x,\lambda) K_N(\lambda,y) = K_N(x,y)$$

Computation of the partition function for $\beta=2$

Consider $Z_N^{\beta=2} = \underbrace{2^{\frac{N(N-1)}{2}} C_N^{(2)}}_{\tilde{C}_N^{(2)}} \prod_{j=1}^{N} \int_D d\lambda_j \, e^{-V(\lambda_j)} \quad |\Delta_N(\{\lambda\})|^2$

it holds for the Vandermonde

$$\Delta_N(\{\lambda\}) = \prod_{1\le i<j\le N} (\lambda_j - \lambda_i) = \det_{1\le i,j\le N} [\lambda_i^{j-1}]$$

$$= \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \ldots \lambda_1^{N-1} \\ \vdots & & \\ 1 & \lambda_N & \ldots \quad \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} \tilde{P}_0(\lambda_1) & \tilde{P}_1(\lambda_1) & \ldots & \tilde{P}_{N-1}(\lambda_1) \\ \\ \tilde{P}_0(\lambda_N) & \tilde{P}_1(\lambda_N) & \ldots & \tilde{P}_{N-1}(\lambda_N) \end{vmatrix}$$

invariance of det under adding rows

for any monic polynomials. Here we choose those OP wrt. weight $\omega(\lambda) = e^{-V(\lambda)}$:

$$\int_D d\lambda\, \omega(\lambda) \tilde{P}_k(\lambda) \tilde{P}_\ell(\lambda) = h_k \delta_{k\ell}$$

- by def $\det_{1\le i,j\le N}[\tilde{P}_{j-1}(\lambda_i)] = \sum_\sigma (-)^\sigma \tilde{P}_{\sigma(1)}(\lambda_1) \tilde{P}_{\sigma(2)}(\lambda_2) \ldots \tilde{P}_{\sigma(N)}(\lambda_N)$

sum over all $N!$ permutations — sign of perm σ — here $\sigma(1)\ldots\sigma(N) \in \{0,1,\ldots,N-1\}$

expand det's $\Rightarrow$ $Z_N^{\beta=2} = 2^{\frac{N(N-1)}{2}} C_N^{(2)} \prod_{j=1}^N \int_D d\lambda_j\, e^{-V(\lambda_j)} \sum_{\sigma,\sigma'} (-)^\sigma (-)^{\sigma'} \tilde{P}_{\sigma(1)}(\lambda_1) \ldots \tilde{P}_{\sigma(N)}(\lambda_N) \tilde{P}_{\sigma'(1)}(\lambda_1) \ldots \tilde{P}_{\sigma'(N)}(\lambda_N)$

$\delta_{\sigma(1)\sigma'(1)} h_{\sigma(1)}$ ...

$$= 2^{\frac{N(N-1)}{2}} C_N^{(2)} \sum_\sigma \underbrace{h_{\sigma(1)} h_{\sigma(2)} \ldots h_{\sigma(N)}}_{= \prod_{j=0}^{N-1} h_j \text{ always}} = N!\, 2^{\frac{N(N-1)}{2}} C_N^{(2)} \prod_{j=0}^{N-1} h_j$$

- the partition function (normalisation) is expressed in terms of $\prod$ norms.

(This is also true for $\beta=1,4$, from skew OP wrt antisym. (skew) product.) see [Mehta], later

Relating OP and characteristic polynomials for $\beta=2$

consider $\mathbb{E}(\det(\lambda - H)) \equiv \boxed{\langle \det(\lambda-H)\rangle_N \overset{!}{=} \tilde{P}_N(\lambda)}$

proof: def is in

$$= \frac{1}{Z_N^{\beta=2}} C_N^{(2)} \prod_{i=1}^N \int d\lambda_i \, e^{-V(\lambda_i)} \underbrace{\prod_{j=1}^N (\lambda - \lambda_j) \prod_{k>l}^N (\lambda_k - \lambda_l)} \prod_{m>n}^N (\lambda_m - \lambda_n)$$

defining $\lambda \equiv \lambda_{N+1}$ this is $\Delta_{N+1}(\{\lambda_i\})$

$$= \frac{1}{N! \prod_{j=0}^{N-1} h_j} \prod_{i=1}^N \int d\lambda_i \, e^{-V(\lambda_i)} \det_{1..N+1} \tilde{P}_{j-1}(\lambda_i) \det_{1..N} \tilde{P}_{j-1}(\lambda_i)$$

$$\sum_{\sigma,\sigma'} (-1)^\sigma (-1)^{\sigma'} \tilde{P}_{\sigma(1)}(\lambda_1) \ldots \tilde{P}_{\sigma(N)}(\lambda_N) \tilde{P}_{\sigma(N+1)}(\lambda_{N+1}) \tilde{P}_{\sigma'(1)}(\lambda_1) \ldots \tilde{P}_{\sigma'(N)}(\lambda_N)$$

δ δ

- the var λ_{N+1} is unintegrated. Only $\sigma(N+1) = N$ contributes, as otherwise $\tilde{P}_N(\lambda_{j<N+1})$ gives zero from orthogonality with $\tilde{P}(\lambda_i)$, $0 \le \sigma'(j) \le N-1$
- if $\sigma'(N) = N$ the perm σ, σ' will be the same otherwise, from δ's $\Rightarrow$ no sign

$$= \frac{1}{N! \prod_{j=0}^{N-1} h_j} \cdot \sum_{\sigma | \sigma(N)=N} \tilde{P}_N(\lambda_{N+1}) \prod_{j=1}^N h_{\sigma(j)} = \underline{\tilde{P}_N(\lambda)}$$

Heine formula 19th century

<u>Check:</u> $\lambda \gg 1$: $\langle \det(\lambda - H)\rangle_N \to \langle \lambda^N \rangle + \ldots$
$= \lambda^N + \ldots$

vs. $\tilde{P}_N(\lambda) = \lambda^N + \ldots$

- this is an N-fold integral rep for arbitrary OP of degree N for an arbitrary weight $w(\lambda)$! = alternative to Gram-Schmidt
- example Hermite: $\exists$ single integral rep

1×1 matrix

$$H_N(x) = e^{x^2} \frac{2^{N+1}}{\sqrt{\pi}} \int dt \, e^{-t^2} t^N \cos\left(2xt - \frac{N\pi}{2}\right) = \ldots = \langle \det(\lambda - H)^N \rangle_{N=1}$$

duality

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- for Gaussian weights this duality is part of a more general relation

$$\Big\langle \det(\underset{N\times N}{\lambda - H})^M \Big\rangle_N \sim \Big\langle \det(\lambda - \underset{M\times M}{H})^N \Big\rangle_M$$

[Mehta-Normand 2001]

Kernel - characteristic polyn. relation:

$$\boxed{\Big\langle \det(\lambda - H)\det(\mu - H)\Big\rangle_N = h_N\, K_{N+1}(\lambda,\mu)}$$

[P. Zinn-Justin 1997]

proof: as for the OP relation we can include the 2 det's in the 2 $\Delta_N \to \Delta_{N+1}$:

$$\frac{1}{N!\prod_{j=0}^{N-1} h_j}\prod_{i=1}^{N}\int_0 d\lambda_i\,\omega(\lambda_i)\underset{\substack{\text{all perm of}\\ N+1\text{ var}}}{\sum_{\sigma,\sigma'}}(-)^{\sigma}(-)^{\sigma'}\tilde{P}_{\sigma(1)}(\lambda_1)\ldots\tilde{P}_{\sigma(N)}(\lambda_N)\cdot\tilde{P}_{\sigma(N+1)}(\lambda)\,\tilde{P}_{\sigma'(1)}(\lambda_1)\ldots\tilde{P}_{\sigma'(N)}(\lambda_N)\,\tilde{P}_{\sigma'(N+1)}(\mu)$$

- from the orthogonality and δ's we get $\sigma(N+1)=\sigma'(N+1)$ and whence perm $\sigma=\sigma'$
- now the index of the polyn. of the integrated variables $\lambda_1,\ldots,\lambda_N$ can take the value N $\tilde{P}_N(\lambda_j)$
- σ contains $(N+1)! = (N+1)\,N!$ perm. These are labelled by $\sigma(N+1)$ taking values $0,\ldots,N$:

$$= \frac{1}{N!\prod_{j=0}^{N-1}h_j}\underset{\substack{\sigma\text{ all}\\ \text{perm of }N\text{ var}}}{\sum}\sum_{j=0}^{N}\tilde{P}_j(\lambda)\tilde{P}_j(\mu)\prod_{\substack{l=1\\ \sigma(l)\neq j}}^{N} h_{\sigma(l)}$$

$$= \frac{1}{N!\prod_{j=0}^{N-1}h_j}\,N!\,\prod_{j=0}^{N}h_j\sum_{j=0}^{N}\frac{\tilde{P}_j(\lambda)\tilde{P}_j(\mu)}{h_j} = h_N\sum_{j=0}^{N}P_j(\lambda)P_j(\mu)$$

we automatically get the kernel of orthonormal polynomials, $\forall$ weights!

- Formulae for $\Big\langle \prod_{k=1}^{K}\det(\lambda_k - H)\Big\rangle_N$ later