Eigenvalue density correlation functions

We define the n-point eigenvalue correlation function as

$$\boxed{R^{(\beta)}_{N,k}(x_1,\dots,x_n) \equiv \frac{N!}{(N-n)!}\int_D dx_{k+1}\dots dx_N\, P^{(\beta)}_N(x_1,\dots,x_N)}$$

where the jpdf is given by

$$P^{(\beta)}_N(x_1,\dots,x_N)=\begin{cases} \frac{\tilde{C}^{(\beta)}_N}{Z^{\beta}_N}\prod_{j=1}^{N} e^{-\overset{=x_j^2+gx^3+\dots}{V(x_j)}}_{\,"w(x_j)} |\Delta_N(\{x\})|^{\beta} & \text{OE/UE/SE } \beta=1,2,4 \text{ for } P^{\beta}_N(H)=e^{-\mathrm{Tr}V(H)},\ H=H^{+} \\ \text{with } D=\mathbb{R} & \\ \frac{\tilde{C}^{(\beta)}_N}{Z^{(\beta)}_N}\prod_{j=1}^{N} \underbrace{x_j^{\frac{\beta}{2}(\nu+1)-1}e^{-\overset{=x_j+gx_j^2+\dots}{V(x_j)}}}_{w_{WL}(x_j)} |\Delta_N(\{x\})|^{\beta} & \text{Wishart-Laguerre-chiral O/U/SE } P^{\beta}_N(H)=e^{-\mathrm{Tr}V(HH^{+})} \\ \text{with } D=\mathbb{R}_+ & \end{cases}$$

we dropped the $C^{(\beta)}_N$'s

H $N\times(N+\nu)$, $\nu\in\mathbb{N}$: real asym. / complex non-Herm. / quat. real non-self dual

6 ensembles of RMT with real eigenvalues

- in the second **3** ensembles we compute either

  singular values $H=U\Lambda V$ $\quad \lambda_i^2=x_i$ for $\lambda_i>0$

  or eigenvalues of $HH^{+}=UXU^{+}$ $\quad x_i\geq 0$

- the Jacobian depends on N,ν for rectangular matrices
- characteristic equations in the 2 cases

$$\det(\lambda \mathbb{1}_N - HH^{+}) = \prod_{j=1}^{N}(\lambda-x_j)$$

vs

$$\det\left(y\mathbb{1}_{2N+\nu}-\underbrace{\begin{pmatrix}0_N & H\\ H^{+} & 0_{N+\nu}\end{pmatrix}}_{\text{Dirac matrix}}\right) = y^{\nu}\prod_{j=1}^{N}(y^2-\lambda_j^2)$$

which can be seen from

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{A^{-1}\text{ exists}}{=} \det\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}\begin{pmatrix} 1 & A^{-1}B \\ 0 & D-CA^{-1}B \end{pmatrix} = \det A \det(D - CA^{-1}B)$$

$$\overset{D^{-1}\text{ ex}}{=} \det D \det(A - BD^{-1}C)$$

<u>Computation of all n-point function for $\beta = 2$:</u>

using $\Delta_N(\{\lambda\}) = \det_{i,j}[P_{j-1}(\lambda_i)]$

and $\det A = \det A^T$, $\det A \cdot \det B = \det A \cdot B$

$$\Delta_N(\{\lambda\})^2 = \det\left[\tilde{P}_{j-1}(\lambda_i)\right]^2 = \prod_{j=0}^{N-1} h_j \det\Big[\underbrace{P_{j-1}(\lambda_i)}_{A_{ij}}\Big]^2$$

$$= \prod_{j=0}^{N-1} h_j \det_{i,k}\Big[\underbrace{\sum_{j=1}^{N} A_{ij}A_{kj}}_{(AA^T)_{ik}}\Big]$$

$$= \prod_{j=0}^{N-1} h_j \det_{1\le i,k\le N}\left[K_N(\lambda_i,\lambda_k) = \sum_{l=0}^{N-1} P_l(\lambda_i)P_l(\lambda_k)\right]$$

- by multiplying the square root of the weights into rows and colums we may write the entire jpdf as a det

$$\mathcal{P}_N^{(\beta=2)} = \prod_{j=0}^{N-1} h_j \det_{1\le i,k\le N}\left[\omega(\lambda_i)^{\frac{1}{2}}\omega(\lambda_k)^{\frac{1}{2}} K_N(\lambda_i,\lambda_k)\right] ,$$

this is called determinantal point process.

Generalisations to det p.p. on $\mathbb{C}$ exist, as well as to Pfaffian point processes (later)

<u>Theorem</u> [Dyson 1970, see Mehta 5.1.4] here $\beta = 2$ only

Let $K(x,y)$ be a complex valued function, with $K(x,y)^* = K(y,x)$ that is integrable in both x and y, s.th.

$$\int K(x,y)\,K(y,z)\,dy = K(x,z) \quad \text{holds}$$

Then
$$\boxed{\int \det_{1\le i,j\le N}\left[K(x_i,x_j)\right] dx_N = (C-N+1) \det_{1\le i,j\le N-1} K_N(x_i,x_j)}$$

with $C = \int K(x,x)\,dx$

<u>Remark:</u> in choosing $K(x,y) = \omega(x)^{\frac{1}{2}}\omega(y)^{\frac{1}{2}} \sum_{j=0}^{N-1} P_j(x) P_j(y)$

this reduces the $(N-k)$-fold integral in $R_N^{(\beta)}(x_1 \ldots x_k)$ to a determinant of size $k \times k$.

<u>Proof</u>:

$$\det_{1..N}\left[K(x_i,x_j)\right] = \sum_\sigma (-)^\sigma K(x_1,x_{\sigma(1)}) \cdots K(x_N,x_{\sigma(N)})$$

- All $N!$ permutations can be reached starting from the identity $(-)^{Id}=+1$ and exchanging pairs of second arguments of K's
- Start with Id and do all $(N-1)!$ permutations keeping the last arg. fixed $K(x_N,x_N)$

$\Rightarrow \int dx_N K(x_N,x_N) = C$, which multiplies by definition

$$\det_{1..N-1}\left[K(x_j,x_e)\right]$$

- to do all remaining $(N-1)\cdot(N-1)!$ perm start again with Id (which we have counted already) and do 1 pair exchange with the second argument of the last kernel

$$K(x_1,x_1)\;\ldots\;K(x_j,x_j)\;\ldots\;K(x_N,x_N)$$

- there are $N-1$ possibilities (positions) to do so, we get a "$(-)$" from the 1 pair exchange

- $\int dx_N\, K(x_j,x_N)\,K(x_N,x_j) = K(x_j,x_j)$

giving $-$ Id of $N-1$ variables. Now we do the remaining $(N-1)!$ permutations of the $(N-1)$ arguments yielding

$\det\limits_{1..N-1} K$

$$\Rightarrow \quad \int dx_N \det_{1..N}[K(x_i,x_j)] = (c-(N-1)) \det_{1..N-1}[K(x_i,x_j)]$$

$\square$

[alternatively see Mehta, Th 5.1.4 p 75, 76]

$\Rightarrow$ compute all $\beta=2$ correlation functions

$$K_{N,k}^{(\beta=2)}(x_1\ldots x_k) = \frac{\cancel{N!}}{(N-k)!}\int dx_{k+1}\ldots dx_N \prod_{j=1}^{N}\omega(x_j)\,\Delta_N(\{x\})^2 \cdot \frac{1}{\cancel{N!}\prod_{j=0}^{N-1}h_j}$$

$$= \frac{\cancel{N!}}{(N-k)!}\,\frac{\int dx_{k+1}\ldots dx_N \cancel{\prod_{j=0}^{N-1}h_j}}{\cancel{N!\prod_{j=0}^{N-1}h_j}} \det\Big[\underbrace{\omega(x_i)^{\frac12}\omega(x_j)^{\frac12}\sum_{\ell=0}^{N-1}P_\ell(x_i)P_\ell(x_j)}_{K(x_i,x_j),\ \text{with } c=N}\Big]$$

- the integrations yield

$$\int dx_N\,(N-(N-1))\det_N = \det_{N-1}$$

$$\int dx_{N-1}\det_{N-1} = (N-(N-1-1))\det_{N-2}$$

$$\int dx_{k+1}\det_{k+1} = (N-(k+1-1))\det_{k}$$

$$\Rightarrow \quad R_{N,k}^{(\beta=2)}(x_1,\dots,x_k) = \frac{(N-k)!}{(N-k)!} \det_{1..k}\left[\omega(x_i)^{\frac{1}{2}}\omega(x_j)^{\frac{1}{2}}\sum_{l=0}^{N-1} P_l(x_i)P_l(x_j)\right]$$

$$= \prod_{e=1}^{k}\omega(x_e) \det_{1..k}\left[K_N(x_i,x_j)\right]$$

some authors define the kernel with or without the weights

You will also find the notation of a

wave-function $\Psi_k(x) = \omega(x)^{\frac{1}{2}} P_k(x)$

as then the Ψ are orthonormal functions

$$\boxed{\int dx\, \Psi_k(x)\Psi_l(x) = \delta_{kl}}$$

$$\Rightarrow \quad R_{N,k}^{(\beta=2)}(x_1,\dots,x_k) = \det_{1\le i,j\le k}\left[\sum_{l=0}^{N-1}\Psi_l(x_i)\Psi_l(x_j)\right]$$

- all correlation functions for $\beta=1,4$ can be expressed as a Pfaffian ($\mathrm{Pf}\,A = \det A^{\frac{1}{2}}$) or Qdet of a quaternion-values (2x2 matrix valued) kernel of skew-orthogonal polynomials

scalar products: $\langle f,g\rangle_2 = \int f(x)g(x)\omega(x)dx \quad \beta=2$ ordinarily

skew " " : $\langle f,g\rangle_4 = \int [f(x)g'(x) - f'(x)g(x)]\omega(x)dx = -\langle g,f\rangle_4 \quad \beta=4$

$$\langle f,g\rangle_1 = \iint f(x)g(y)\,\varepsilon(x-y)\,\omega(x)\omega(y)\,dx\,dy = -\langle g,f\rangle_1 \quad \beta=1$$

where $\varepsilon(x) = \begin{cases} +\frac{1}{2} & x>0 \\ -\frac{1}{2} & x<0 \end{cases}$

Appearance of the scalar product:

$\beta=4$: replace $\Delta_N(\{x\})^4$ in jpdf.

$$\Delta_N^{\,4} = \prod_{j>i}(x_j - x_i)^4 = \det\left[\, x_i^{\,j} \quad j x_i^{\,j-1} \right]_{\substack{1\le i\le N\\ 0\le j\le 2N-1}} = \det\left[Q_j(x_i) \quad Q_j'(x_i)\right]_{\substack{1\le i\le N\\ 0\le j\le 2N-1}}$$

use 2. de Bruijn formula 1955

$$\boxed{\int dx_1 \ldots \int dx_N \det\left[\varphi_i(x_j) \quad \psi_i(x_j)\right]_{\substack{1\le i\le 2N\\ 1\le j\le N}} = N!\, \mathrm{Pf}\left[\int (\varphi_i(x)\psi_j(x) - \varphi_j(x)\psi_i(x))\,dx\right]}$$

$$\Rightarrow Z_N^{(\beta=4)} = \tilde{C}_N^{(\beta)} \int dx_1 \ldots \int dx_N \prod_{i=1}^{N} \omega(x_i) \det\left[Q_j(x_i) \quad Q_j'(x_i)\right]$$

$$= \tilde{C}_N^{\beta} \prod_{j=0}^{N-1} h_j^{(\beta=4)}\; N!\, \mathrm{Pf}\left[\int \frac{Q_i(x)}{\sqrt{h_i^{(4)}}}\frac{Q_j'(x)}{\sqrt{h_j^{(4)}}} - Q_i'(x)\, Q_j(x)\, dx\right]$$

where ($N=2s$ even), A $N\times N$, $A=-A^T$

$$\mathrm{Pf}[A] = \frac{1}{(2^s s!)} \sum_{\substack{P \text{ all}\\ N! \text{ perm}\\ \text{of } N \text{ indices}}} (-)^P A_{i_1 i_2} \ldots A_{i_{2s-1} i_{2s}}$$

$$= \sum_{\substack{P \text{ all perm}\\ \text{s.th. } i_1<i_2,\ i_3<i_4 \ldots i_{2s-1}<i_{2s}}} (-)^P A_{i_1 i_2} \ldots A_{i_{2s-1} i_{2s}}$$

example 2×2 : $\mathrm{Pf}A = A_{12}$

4×4 : $\mathrm{Pf}A = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$

note in $\det A = \sum_\sigma (-)^\sigma A_{1\sigma(1)} \ldots A_{N\sigma(N)}$ every index appears twice, here only once

If we choose the $Q_j(x)$ s.th.

$$\langle Q_{2j}, Q_{2k+1}\rangle_4 = -\langle Q_{2k+1}, Q_{2j}\rangle = h_j^{(\beta=4)}\delta_{kj}$$

$$\langle Q_{2j}, Q_{2k}\rangle_4 = \langle Q_{2j+1}, Q_{2k+1}\rangle = 0$$

we have that $\mathrm{pf}\left[\langle Q_i, Q_j\rangle\right] = 1$ and we can construct a kernel from them

example: Laguerre weight $\omega(x) = x^a e^{-x}$ on $\mathbb{R}_+$
[Mehta ch 19]

$$Q_{2j+1}(x) = -L_{2j+1}^{(a)}(x) + L_{2j}^{(a)}(x) \Rightarrow Q_{2j+1}'(x) = L_{2j}^{(a)}(x)$$

$$Q_{2j}(x) = L_{2j}^{(a)}(x) - L_{2j-1}^{(a)}(x) - \frac{(2j+a-1)}{(2j-1)} Q_{2j-2}(x)$$

$$\Rightarrow Q_{2j}'(x) = -L_{2j-1}^{(a)}(x) - \frac{(2j+a-1)}{(2j-1)} Q_{2j-2}'(x)$$

$$h_j^{(4)} = \frac{\Gamma(2j+a+1)}{(2j)!}$$

- the skew orthogonal polynomials enjoy a determinantal rep:

$$Q_{2j}(x) = \langle \det(x-H)\rangle$$

$$Q_{2j+1}(x) = \langle \det(x-H)\,(\mathrm{Tr}(H) + x + \mathrm{const.})\rangle$$

as the odd polynomials are only defined up $+\ \mathrm{const.}\cdot Q_{2j}(x)$
(skew-product!)

$\underline{\beta = 1}$ is analogous, except we need another integration formula

- replace $\det x_i^{j-1} = \det P_j(x_i)$, remove absolute value of $|\Delta_N|^1$ by ordering eigenvalues

$$Z_N^{(\beta=1)} = \tilde{C}_N^{(\beta=1)} \int dx_1 \dots dx_N \, \omega(x_1) \dots \omega(x_N) \, |\Delta_N(\{x\})|^1$$

$$= N! \, \tilde{C}_N^{(\beta=1)} \underset{x_1 \le x_2 \le \dots \le x_N}{\int dx_1 \dots \int dx_N} \prod_{j=1}^{N} \omega(x_j) \, \det \left[P_j(x_i) \right]$$

1. de Bruijn formula:

$$\boxed{\int_{x_1 \le \dots \le x_N} dx_1 \dots \int dx_N \det_{1 \le i,j \le N} \left[\varphi_i(x_j) \right] = \mathrm{Pf} \left[\iint 2\varepsilon(x-y) \varphi_i(x) \varphi_j(y) \, dx \, dy \right]}$$

$\Rightarrow$ skew-product for $\beta = 1$ $\langle f, g \rangle_1$, same construction of SOP $\langle P_{2j}, P_{2k+1} \rangle_1 = h_j^{(\beta=1)} \delta_{jk}$ others 0,

same rep of P's terms of $\langle \det (\mathrm{Tr} \;) \rangle$