Gap-probabilities and individual eigenvalue distr.

- So far we have only studied density correlation functions, $R^{(\beta)}_{N,k}$, i.e. the probability to find an eigenvalue at x: $R^{(\beta)}_{N,k=1}(x)$, or the probability to find 1 ev at x and 1 at y, $R^{(\beta)}_{N,k=2}(x,y)$ etc

- Now we will consider the probability that an interval I (e.g. $I=(-s,s)$) is empty of eigenvalues, also called gap-probability. First $\beta=2$:

$$A_I = \prod_{j=1}^{N} \int_{\mathbb{R}-I} dx_j \, P_N^{(\beta=2)}(x_1,\dots,x_N)$$

include to make orthonormal

$$= \frac{1}{N! \prod_{j=0}^{N-1} h_j} \prod_{j=1}^{N} \underbrace{\int_{\mathbb{R}-I} dx_j\, w(x_j)}_{= \int_{\mathbb{R}} dx_j \underbrace{(1-\chi_I(x_j))\, w(x_j)}_{\tilde w(x_j)}} \det_{1\le i,j\le N}\left[\tilde P_{j-1}(x_i)\right]^2$$

↑ char. funct on I

$$= \frac{1}{N!} \prod_{j=1}^{N} \int_{\mathbb{R}} dx_j \det_{1\le i,j\le N}\left[\varphi_j(x_i)\right]^2, \quad \varphi_j(x) = \tilde w(x)^{\frac{1}{2}} P_{j-1}(x)$$

Andréief formula $\varphi=\psi$ →

$$= \det_{1\le i,j\le N}\left[\int_{\mathbb{R}} dx\, \varphi_i(x)\varphi_j(x)\right] = \det_{1\le i,j\le N}\left[\int_{\mathbb{R}} dx\,(1-\chi_I(x))\, w(x)\, P_i(x) P_j(x)\right]$$

$$\boxed{A_I = \det_{1\le i,j\le N}\left[\delta_{ij} - \int_I dx\, w(x)\, P_i(x) P_j(x)\right]}$$

→ Fredholm determinant

Andréief formula (integral identity)

Let $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2, \dots N$ be 2 sets of integrable functions

Then $$\int dx_1 \dots \int dx_N \det_{1\le i,j\le N}[\varphi_i(x_j)] \det_{1\le i,j\le N}[\psi_i(x_j)] = N! \det_{1\le i,j\le N}\left[\int dx\, \varphi_i(x)\psi_j(x)\right]$$

Proof: • induction: $N=1$ is trivial

• " assumption: holds for N

$N+1$: expand both det's on l.h.s. w.r.t. column containing x_{N+1}:

$$= \int dx_1 \dots \int dx_N \int dx_{N+1} \sum_{i=1}^{N+1} (-)^{N+1+i} \varphi_i(x_{N+1}) \underbrace{\det_{\substack{j\neq N+1\\ k\neq i}}\left[\varphi_k(x_j)\right]}_{N \text{ funct. } \varphi_1 \dots \cancel{\varphi_k} \dots \varphi_{N+1}} \sum_{l=1}^{N+1} (-)^{N+1+l} \psi_l(x_{N+1}) \underbrace{\det_{\substack{m\neq N+1\\ n\neq l}}[\psi_n(x_m)]}_{N \text{ funct.}}$$

induction assumpt. for N

$$= \int dx_{N+1}\, N! \sum_{i,l=1}^{N+1} (-)^{i+l} \varphi_i(x_{N+1}) \psi_l(x_{N+1}) \det_{\substack{k\neq i\\ n\neq l}}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$$

• for fixed i: $\sum_{l=1}^{N+1} (-)^{i+l} \int dx\, \varphi_i(x)\psi_l(x) \det_{\substack{k\neq i\\ n\neq l}}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$

is the Laplace expansion wrt row "i" of

of $\det_{1\le k,n\le N+1}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$

$\Rightarrow$ we sum over $(N+1)$ such expansions

$$= (N+1)\, N! \det_{1\le k,n\le N+1}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$$

$\square$

$$A_I = \prod_{j=0}^{N} (1-\lambda_j)\,, \quad \lambda_j \text{ eigenvalues of matrix}$$

$$M_{ij} = \int_I dx\, w(x)\, P_i(x)\, P_j(x)$$

<u>remark:</u>

The eigenvalues λ_j are the same as the solutions of the integral eq.

$$\int_I dy\; w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} K_N(x,y)\, f(y) = \lambda f(x) \qquad [\text{see Mehta}]$$

k-th gap probability and k-th eigenvalue: for $\beta = 1, 2, 4$

$$\boxed{E_k^{(\beta)}(s) \equiv \frac{N!}{(N-k)!\,k!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_k \int_s^{\infty} d\lambda_{k+1} \ldots d\lambda_N\; P_N^{(\beta)}(\lambda_1 \ldots \lambda_N) \quad \text{kth gap prob.}}$$

<u>example:</u> $E_{k=0}(s) = \int_s^{\infty} d\lambda_1 \ldots d\lambda_N\, P_N^{\beta}$ prob. that all ev are $\geq s$, $E_{k=0}(s=-\infty)=1$

general: E_k " " k ev $\leq s$, $N-k \geq s$

$$\boxed{P_k^{(\beta)}(s) \equiv \frac{k\cdot N!}{k!(N-k)!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_{k-1} \int_s^{+\infty} d\lambda_{k+1} \ldots d\lambda_N\; P_N^{(\beta)}(\lambda_1, \ldots \lambda_{k-1}, \lambda_k = s, \lambda_{k+1} \ldots \lambda_N)}$$

if we order $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

$P_k^{(\beta)}(s)$ is the <u>probability to find the k-th eigenvalue at $\lambda_k = s$</u>

- it holds $\quad \frac{\partial}{\partial s} E_0^{(\beta)}(s) = -P_0^{(\beta)}(s)$

and

$$\frac{\partial}{\partial s} E_k^{(\beta)}(s) = \frac{N!}{(N-k)!}\frac{k}{k!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_{k-1} \int_s^{\infty} d\lambda_{k+1} \ldots d\lambda_N\, P_N^{(\beta)}\Big|_{\lambda_k = s}$$
$$- \frac{N!}{(N-k)!}\frac{(N-k)}{k!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_k \int_s^{\infty} d\lambda_{k+2} \ldots d\lambda_N\, P_N\Big|_{\lambda_{k+1} = s}$$

$$\Rightarrow \quad \frac{\partial}{\partial s} E_k^{(\beta)}(s) = p_k^{(\beta)}(s) - p_{k+1}^{(\beta)}(s)$$

$$\Leftrightarrow \quad \boxed{p_k^{(\beta)}(s) = -\sum_{\ell=0}^{k-1} \frac{\partial}{\partial s} E_\ell^{(\beta)}(s)}$$

- if we know all gaps we know all indiv. eigenvalue distributions
- normalisation: $\forall_k \quad \boxed{\int_{-\infty}^{\infty} dx\, p_k^{(\beta)}(s) = 1}$ exercise or [arXiv 0311171 App. A with $w(x)$ on $\mathbb{R}_+$ $E_k \to k!E_k$]

we will now compute $E_k^{(\beta)}(s)$ in two ways:

i) expressed through $R_{N,k}^{(\beta)}$'s in general

ii) expressed through $\langle \Pi \det$'s$\rangle$ for $w(x) = x^\alpha e^{-x}\Theta(x)$

i) idea: $(a-b)^j = \sum_{\ell=0}^{j} (-)^\ell \binom{j}{\ell} a^{j-\ell} b^\ell$

with $a = \int_{-\infty}^{\infty} dx$, $b = \int_{-\infty}^{s} dx \Rightarrow a - b = \int_{s}^{+\infty} dx$

using the symmetry of $\mathcal{P}_N^{(\beta)}$ under permutations of arguments:

$$\Rightarrow E_k^{(\beta)}(s) = \binom{N}{k} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_k \sum_{\ell=0}^{N-k} (-)^\ell \binom{N-k}{\ell} \left(\int_{-\infty}^{\infty}\right)^{N-k-\ell} \left(\int_{-\infty}^{s}\right)^{\ell} d\lambda_{k+1} \dots d\lambda_N \mathcal{P}_N^{(\beta)}$$

$$= \sum_{\ell=0}^{N-k} (-)^\ell \frac{N!}{k!\,\ell!\,(N-k-\ell)!} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_{k+\ell} \int_{-\infty}^{\infty} d\lambda_{k+\ell+1} \dots d\lambda_N \mathcal{P}_N^{(\beta)}$$

$$E_k^{(\beta)}(s) = \sum_{\ell=0}^{N-k} (-)^\ell \frac{1}{k!\,\ell!} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_{k+\ell}\, R_{N,k+\ell}^{(\beta)}(\lambda_1, \dots, \lambda_{k+\ell})$$

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here we define $R_{N,0}^{(\beta)} \equiv 1 \quad \left(= \frac{Z_N^{(\beta)}}{Z_N^{(\beta)}}\right)$

e.g. $E_0^{(\beta)}(s) = 1 - \int_{-\infty}^{s} dx_1 \, R_{N,1}^{(\beta)}(x_1) + \frac{1}{2} \int_{-\infty}^{s} dx_1 dx_2 \, R_{N,2}^{(\beta)}(x_1,x_2) + \dots$

- numerically this sum converges very rapidly
- in the limit $N\to\infty$ this is the def. of a Fredholm det., alternatively to p. 25

remark: we may also introduce a generating functional

$$E^{(\beta)}(s;\xi) \equiv \sum_{\ell=0}^{N} (-\xi)^\ell \frac{1}{\ell!} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_\ell \, R_{N,\ell}^{(\beta)}(\lambda_1 \dots \lambda_\ell)$$

$$\Rightarrow \quad E_k^{(\beta)}(s) = \frac{(-)^k}{k!} \frac{\partial^k}{\partial \xi^k} E(s;\xi)\Big|_{\xi=1} \quad \text{for } k=0,1,\dots,N$$

ii) let us choose the chiral or Wishart-Laguerre ensembles ($N\times N+\nu$ matrices)

with weight $w_\nu^{(\beta)}(x) = x^{\frac{\beta}{2}(\nu+1)-1} e^{-x}$ on $\mathbb{R}_+$ $\Rightarrow$ all lower $\int$-bounds are at 0

consider $E_{0,\nu}^{(\beta)}(s)$: display ν-dep

$$E_{0,\nu}^{(\beta)}(s) = \frac{1}{N! \prod_{j=0}^{N-1} h_j^{(\beta)}} \int_s^\infty \prod_{\ell=1}^{N} dx_\ell \, x_\ell^{\frac{\beta}{2}(\nu+1)-1} e^{-x_\ell} \prod_{n>m}^{N} |x_n - x_m|^\beta$$

shift $x_k - s = y_k$

$$= \frac{1}{N! \prod_{j=0}^{N-1} h_j^{(\beta)}} \int_0^\infty \prod_{\ell=1}^{N} dy_\ell \, (y_\ell+s)^{\frac{\beta}{2}(\nu+1)-1} e^{-(y_\ell+s)} \prod_{n>m} |y_n - y_m|^\beta$$

if we compare to

$$\left\langle \prod_{f=1}^{L} \det(z_f - HH^\dagger) \right\rangle_{N,\nu} = \frac{1}{N! \prod_{j=0}^{N-1} h_j^{(\alpha)}} \int_0^\infty \prod_{\ell=1}^{N} d\lambda_\ell \, \lambda_\ell^{\frac{\beta}{2}(\nu+1)-1} e^{-\lambda_\ell} \prod_{f=1}^{L} (z_f - \lambda_\ell) |\Delta_N|^{\beta}_{\beta=1,2,4}$$

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- we have that $E_{0,\nu}^{(\beta)}(s)$ is the ave. of $\frac{\beta}{2}(\nu+1)-1$ char. polynomials with argument $z_1 = \dots = z_\nu = -s$ (for $\beta=4$ because of the degeneracy $\Theta_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix}$ the eigenvalues always appear in pairs)

- for this interpretation we need $\frac{\beta}{2}(\nu+1)-1 \in \mathbb{N}$ (and even for $\beta=4$)

$\underline{\beta=1}$: $\quad \frac{1}{2}(\nu+1)-1 = \frac{(\nu-1)}{2} \in \mathbb{N} \Leftrightarrow \nu = 2\ell+1,\ \ell\in\mathbb{N}$

$\underline{\beta=4}$: $\quad \frac{4}{2}(\nu+1)-1 = 2\nu+1$ even $(\Leftrightarrow \nu = n-\frac{1}{2})$

$\underline{\beta=2}$: $\quad \frac{2}{2}(\nu+1)-1 = \nu \in \mathbb{N}$ ✓ for all $\nu = 0,1,2,\dots$

- in $E_{0,\nu}^{(\beta)}(s)$ the "effective" ν_{eff} leads to $\frac{\beta}{2}(\nu_{eff}+1)-1 = 0$

$$\Leftrightarrow \nu_{eff} = \frac{2}{\beta} - 1 \in \mathbb{N} = \begin{cases} 1 & \beta=1 \\ 0 & \beta=2 \\ -\frac{1}{2} & \beta=4 \end{cases}$$

$$E_{0,\nu}^{(\beta=2)}(s) = \left\langle \prod_{\ell=1}^{\nu} \det(-s-HH^\dagger) \right\rangle_{N,\nu_{eff}=0} (-)^{\nu N} e^{-Ns}$$

$$E_{0,\nu=2\ell+1}^{(\beta=1)}(s) = \left\langle \prod_{j=1}^{\ell} \det(-s-HH^T) \right\rangle_{N,\nu_{eff}=1} e^{-Ns} (-)^{\ell N}$$

example: $\beta=2,\ \nu=0$ or $\beta=1,\ \ell=0$ ($\nu=1$): $\boxed{E_0(s) = e^{-Ns}}$

- the computation of $E_{0,\nu=2\ell}^{\beta=1}$, $E_0^{\beta=4}$ is an open problem (see however Edelman, J. Matrix Anal. Appl. 9 (88) for a recursion)
- the two problems are related, solving one solves the other
- E_k can also be expressed in terms of $\langle \prod \det \rangle$:

$$E_k^{(\beta)} \sim \underbrace{\int_0^s d\lambda_1 \dots d\lambda_k}_{\text{need}} \quad \text{shift} \quad \underbrace{\int_s^\infty d\lambda_{k+1} \dots d\lambda_N P_N}_{\hookrightarrow \langle \det's \rangle_{N-k}}$$

Computation of averages of characteristic polynomials

- we will focus again on $\beta=2$, for $\beta=1,4$ see [Mehta, Normand arXiv 0101469 cond-mat]

Theorem Brézin-Hikami [CMP 214 (2000) 111] as done de Bruijn

$$\boxed{\left\langle \prod_{\ell=1}^{L} \det(x_\ell - H)\right\rangle_N = \frac{1}{\Delta_L(\{x\})} \begin{vmatrix} \tilde{p}_N(x_1) & \cdots & \tilde{p}_{N+L-1}(x_1) \\ \tilde{p}_N(x_L) & & \tilde{p}_{N+L-1}(x_L) \end{vmatrix}}$$

where $\tilde{p}_k(x)$ are the monic orthogonal polynomials wrt weight $\omega(x)$ $\int dx\, \omega(x)\, \tilde{p}_k(x)\, \tilde{p}_\ell(x) = h_\ell\, \delta_{k\ell}$ ((Note: for wishart $\prod \det(x_\ell - HH^\dagger)$))

Proof [Baik, Deift, Strahov arXiv 0304016 math-ph]

Consider the weight $\omega^{[\ell]}(x) = \left(\prod_{j=1}^{\ell}(x_j - x)\right)\omega(x)$ $(0 \le \ell \le L$

Then $\tilde{p}^{[\ell]}_n(t) \equiv \dfrac{1}{(t-x_1)\cdots(t-x_\ell)} \dfrac{\begin{vmatrix} \tilde{p}_n(x_1) & \cdots & \tilde{p}_{n+\ell}(x_1) \\ \tilde{p}_n(x_\ell) & \cdots & \tilde{p}_{n+\ell}(x_\ell) \\ \tilde{p}_n(t) & & \tilde{p}_{n+\ell}(t) \end{vmatrix}}{\begin{vmatrix} \tilde{p}_n(x_1) & \cdots & \tilde{p}_{n+\ell-1}(x_1) \\ \tilde{p}_n(x_\ell) & \cdots & \tilde{p}_{n+\ell-1}(x_\ell) \end{vmatrix}}$

are the monic orthogonal polyn. wrt $\omega^{[\ell]}(x)$ (($\ell=1$ Szegö)):

Take $q^{[\ell]}_n(t)$ to be the det in the num

It holds $q^{[\ell]}_n(x_j) = 0$ for $j = 1,\ldots,\ell$, polyn of deg $\le n+\ell$

$\Rightarrow$ $\dfrac{q^{[\ell]}_n(t)}{(t-x_1)\cdots(t-x_\ell)}$ is a poly of deg $\le n$

It holds $\displaystyle\int t^j\, q^{[\ell]}_n(t)\, \omega(t) = \int t^j \underbrace{\frac{q^{[\ell]}_n(t)}{(t-x_1)\cdots(t-x_\ell)}}_{\sim\, \tilde{p}^{[\ell]}_n(t)}\, \omega^{(\ell)}(t) = 0$ for $j = 0,\ldots,n-1$

The normalisation follows from looking at $t \gg 1$ (it can be shown that det in denom $\neq 0$, see Baik, Deift, Strahov)

Theorem (2-point funct.) [BDS]

$$\left\langle \prod_{j=1}^{k} \det(x_j - H)\det(y_j - H) \right\rangle_N = \frac{\prod_{l=N}^{N+k-1} h_l}{\Delta_k(\{x\})\Delta_k(\{y\})} \det_{1..k}\left[K_{N+k}(x_i, y_j)\right]$$

Proof see BDS (who state this using Christoffel-Darboux)

Note that using the previous theorem this also equals a det of size $2k+2k$ of polynomials only. In fact a more general Th holds

Theorem [AV arXiv 0212051 [hep-th]]

Let $k \geq L$:
$$\left\langle \prod_{i=1}^{k} \det(x_i - H) \prod_{j=1}^{L} \det(y_j - H) \right\rangle_N = \frac{\prod_{i=N}^{N+k-1} h_i^{\frac{1}{2}} \prod_{j=N}^{N+L-1} h_j^{\frac{1}{2}}}{\Delta_k(\{x\})\Delta_L(\{y\})} \cdot$$

$$\cdot \det_{k\times k} \begin{vmatrix} K_{N+L}(x_l, y_m) \\ P_{N+m-1}(x_l) \end{vmatrix} \begin{matrix} m = 1..L \\ m = L+1..k \end{matrix}$$

Proof: for non-Herm RMT, valid in Herm. limit! $\qquad l = 1..k$

Back to gap probabilities!

- for $\beta = 2$ and $w(x) = x^{\nu} e^{-x}$ we have $\tilde{P}_n(x) = (-)^n n! L_n^{(\nu)}(x)$

$$E_{0,\nu}^{(\beta=2)}(s) = \lim_{\substack{x_a \to s \\ a=1..\nu}} \frac{(-1)^{\nu N}}{\Delta_\nu(\{x\})} \det \begin{vmatrix} (-)^N N! L_N^{(0)}(-x_1) & \dots & (-)^{N+\nu-1}(N+\nu-1)! L_{N+\nu-1}^{(0)}(-x_1) \\ \\ (-)^N N! L_N^{(0)}(-x_\nu) & \dots & \end{vmatrix}$$

index $= \nu_{\text{eff}} = 0$

this limit can be computed by doing a Taylor expansion.

Alternatively Mehta, Normand computed this average of a char. polynomial to some power directly:

The Brézin Mikami th (see also Mehta-Normand)

Define $\langle \det(x-H)^m \rangle_N = \frac{I_2(N,m;x)}{I_2(N,0;x)}$ for $\beta=2$

$I_2(N,0;x) = Z_N^{(\beta=2)}$

it holds

$$I_2(N,m;x) = \frac{N!}{\prod_{\ell=0}^{m-1} \ell!} \prod_{j=0}^{N-1} h_j \det_{0\le k,\ell \le m-1} \left[\tilde{p}_{N+k}^{(\ell)}(x) \right]$$

where $\tilde{p}_n^{(\ell)}(x)$ is the ℓ-th derivative of the monic OP

For an explicit expression for $w(x) = x^\nu e^{-x}$ for $E_{0,\nu}^{(\beta)}(s)$ see

- Forrester, Hughes JMP 35 (1994) 6736 $\beta=2$
- Nagao, Forrester Nucl. Phys. B 509 (1998) 561 $\beta=1$ ν odd only

There $E_0(s)$ is given by a Pfaffian of Laguerre polynomials

→ See also arXive 1103.5617 for summarising what is known on individual eigenvalue distributions

Duality: Mehta-Normand

$$I_2(N,m;x) = \text{const} \cdot I_2(m,N;ix)$$

exchanging $N \leftrightarrow m$ matrix dimension and # of determinants