Lecture 5

Reminder: $$Z = \int dH\, e^{-2\,\mathrm{Tr}\, V(H)} \qquad \text{partition function}$$

$H=H^{+}$ $\beta=2$ Unitary Ensemble
$$= \int dU \int \prod_{j=1}^{N} d\lambda_j\, e^{-2V(\lambda_i)}\, \Delta_N(\{\lambda\})^2 \,, \qquad \Delta_N(\{\lambda\}) = \prod_{j>k}^{N}(\lambda_j - \lambda_k)$$
Vandermonde
$$= \int dU \cdot N! \prod_{j=0}^{N-1} h_j$$

where $P_j(\lambda) = \frac{1}{\sqrt{h_j}}\tilde{P}_j(\lambda) = \frac{1}{\sqrt{h_j}}\lambda^j + \dots$ orthogonal polynomials

$$\int d\lambda\, \omega(\lambda)\, P_j(\lambda) P_k(\lambda) = \delta_{jk} \quad \text{wrt weight } \omega(\lambda) = e^{-2V(\lambda)}$$

$$R_{N,k}(\lambda_1, \dots, \lambda_k) = \prod_{j=1}^{k}\omega(\lambda_j) \det_{l,n=1..k}\left[K_N(\lambda_l, \lambda_n)\right] \qquad k\text{-point density}$$

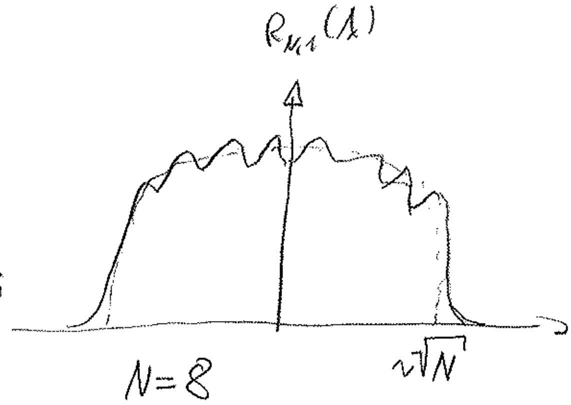
where $$K_N(\lambda,\mu) = \sum_{l=0}^{N-1} P_l(\lambda) P_l(\mu) = c_N \frac{P_N(\lambda)P_{N-1}(\mu) - P_N(\mu)P_{N-1}(\lambda)}{\lambda - \mu} \qquad \text{Christoffel-Darboux}$$

due to $$\lambda P_N(\lambda) = c_{N+1}P_{N+1}(\lambda) + a_N P_N(\lambda) + c_N P_{N-1}(\lambda) \,, \quad c_N = \sqrt{\frac{h_N}{h_{N-1}}}$$

today:
- differential eq for wave function $\Psi_k(\lambda) = \omega(\lambda)^{\frac{1}{2}} P_k(\lambda)$, valid for finite k and for arbitrary weight with polynomial potential V
- for simplicity we shall assume $V(\lambda) = V(-\lambda)$ even
  $\Rightarrow a_N = 0\,, \quad P_k(\lambda) = (-)^k P_k(-\lambda)$
- we'll first derive results for $N \to \infty$ for the GUE with Gaussian weight, and then come to
- universality: indep of result at $N\to\infty$ of details of weight

large-N limit – heuristics

GUE $\quad R_{N,1}(\lambda) = e^{-\lambda^2} \sum_{n=0}^{N-1} \frac{1}{\sqrt{\pi}\, 2^n n!} H_n(\lambda)^2$



– the density has a smooth & oscillatory part

$$R_{N,1}(\lambda) = R^{smooth}_{N,1}(\lambda) + R^{oscill}_{N,1}(\lambda)$$

Semi-circle: $\quad R^{smooth}_{N,1}(\lambda) = \frac{1}{\pi}\sqrt{2N - \lambda^2} \qquad \lambda^2 \leq N$

$\rightsquigarrow$ in order to get a compact support we need to rescale $\lambda \to \sqrt{N}\lambda$

– displays some degree of universality : same for i.i.d. rand. var.

Oscillatory part:

we need to zoom into a specific part of the spectrum λ_0

$$N^{\delta}(\lambda - \lambda_0)$$

where δ depends on the location : $\begin{cases} \underline{\text{bulk}} & \delta = 1 \\ \underline{\text{edge}} & \delta = \frac{1}{6} \end{cases}$

* the oscill. part is universal under a much larger class of deformations V even polynomial ($V(\lambda) \neq \lambda^2$ destroys semi-circle)

for invariant deform. and iid r.v

for non " "

* the same orth. polynomials, e.g. Hermite $H_n(\lambda)$ can show different limiting behaviours :

$H_n \sim$ sine, cosine in the bulk

$\sim$ Airy at the edge (vicinity of spectral support) exact differential eq. for $\psi_n(\lambda)$ with even $V(\lambda)$ [Kanzieper, Freilikher cond-mat/9809365]

multiplication $\lambda P_n(\lambda) = c_{n+1} P_{n+1}(\lambda) + c_n P_{n-1}(\lambda)$ (same for $\psi_n = P_n e^{-V}$)

differentiation: $P_n(\lambda)' = A_n(\lambda) P_{n-1}(\lambda) - B_n(\lambda) P_n(\lambda)$

$$\left(= \frac{n \lambda^{n-1}}{\sqrt{h_n}} + \dots \right)$$

even though $P_n'(\lambda)$ is of degree $n-1$ it is convenient to def the polynomials

$$\boxed{\begin{aligned} A_n(\lambda) &= c_n \int dt\, e^{-2V(t)} \, 2\frac{(V'(t) - V'(\lambda))}{t-\lambda} P_n^2(t), & \deg A_n &= (\deg V) - 2 \\ B_n(\lambda) &= c_n \int dt\, e^{-2V(t)} \, 2\frac{(V'(t) - V'(\lambda))}{t-\lambda} P_n(t) P_{n-1}(t), & \deg B_n &= \deg V - 3 \end{aligned}}$$

- given $V(\lambda)$ the functions A_n, B_n can be computed using multipl.

→ using mult. in $P_n'(\lambda)$ all terms of $O(\lambda^{\geq n-1})$ will cancel.

→ the depth of P' depends on $\deg V = d$: $P_n' = \dots + \text{const}\, P_{n-1-(d-2)}$

example: $V(\lambda) = \frac{\lambda^2}{2} \Rightarrow A_n(\lambda) = 2c_n$, $B_n(\lambda) = 0$

$h_n = \sqrt{\pi}\, 2^{-n} n!$ $\Rightarrow P_n'(\lambda) = 2c_n P_{n-1}(\lambda) = 2\sqrt{\frac{n}{2}}\, P_{n-1}(\lambda)$

$c_n = \sqrt{\frac{n}{2}}$

proof for A_n, B_n:

$P_n'(\lambda)$ is a polynomial of degree $n-1$, express through complete set of basis functions:

$$P_n'(\lambda) = \sum_{l=0}^{n-1} P_l(\lambda)\, \alpha_l, \quad \alpha_l = \int d\lambda\, w(\lambda)\, P_l(\lambda) \cdot P_n'(\lambda)$$

$$= \int dt\, \underbrace{w(t)}_{e^{-2V(t)}} P_n'(t) \underbrace{\sum_{l=0}^{n-1} P_l(\lambda) P_l(t)}_{K_n(\lambda,t) \text{ kernel}}$$

integrating by parts and using $\int dt\, \omega(t)\, P_n(t)\, P_\ell'(t) = 0$ for $\ell = 0, \ldots, n-1$

we have (for vanishing boundary terms)

$$P_n'(\lambda) = \int dt\, \omega(t) \Big(2V'(t) \underset{\substack{\uparrow \\ \text{add as it gives zero}}}{- 2V'(\lambda)} \Big) P_n(t) \sum_{\ell=0}^{n-1} P_\ell(\lambda) P_\ell(t)$$

Christoffel-Darboux

$$= \int dt\, \omega(t)\, 2c_n \frac{(V'(t) - V'(\lambda))}{t - \lambda} P_n(t) \Big(P_n(t) P_{n-1}(\lambda) - P_n(\lambda) P_{n-1}(t) \Big)$$

$$\leadsto A_n(\lambda) \qquad \leadsto B_n(\lambda) \qquad \square$$

Using multiplication, differentiation of $\Psi_n(\lambda) = e^{-V(\lambda)} P_n(\lambda)$

and the following identity

$$(*) \qquad B_n(\lambda) + B_{n-1}(\lambda) - \frac{\lambda}{c_{n-1}} A_{n-1}(\lambda) + 2V'(\lambda) = 0$$

we arrive at exact 2. order diff. eq. for arbitrary $V(\lambda)$:

$$\boxed{\Psi_n(\lambda)'' - F_n(\lambda)\, \Psi_n'(\lambda) + G_n(\lambda)\, \Psi_n(\lambda) = 0}$$

where

$$F_n(\lambda) = \frac{1}{A_n(\lambda)} A_n'(\lambda)$$

$$G_n(\lambda) = B_n'(\lambda) + \frac{c_n}{c_{n-1}} A_n(\lambda) A_{n-1}(\lambda) - B_n(\lambda) \Big(B_n(\lambda) + 2V'(\lambda) + \frac{1}{A_n(\lambda)} A_n'(\lambda) \Big)$$
$$+ V''(\lambda) - (V'(\lambda))^2 - \frac{1}{A_n(\lambda)} A_n'(\lambda) V'(\lambda)$$

proof of $(*)$:

$$B_n(\lambda) + B_{n-1}(\lambda) = 2 \int dt\, \omega(t) \frac{V'(t) - V'(\lambda)}{t - \lambda} P_{n-1}(t) \underbrace{\Big(c_n P_n(t) + c_{n-1} P_{n-2}(t) \Big)}_{(-\lambda + \lambda + t) P_{n-1}(t)}$$

$$= \frac{+\lambda}{c_{n-1}} A_n(\lambda) + 2 \int dt\, \omega(t) \underbrace{(V'(t) - V'(\lambda))}_{\text{odd}} \underbrace{P_{n-1}^2(t)}_{\text{orthonormal}}$$

$\Rightarrow$ diff. eq. by direct differentiation example large-N limit: the sine-kernel:

GUE $V(\lambda)=\frac{\lambda^2}{2}$: $A_n(\lambda)=2c_n$, $B_n(\lambda)=0$, $c_n=\sqrt{\frac{n}{2}}$ $\Rightarrow$ $f_n=0$, $g_n=2n+1-\lambda^2$

$$\frac{d^2}{d\lambda^2}\psi_N(\lambda)+(2N+1-\lambda^2)\psi_N(\lambda)=0$$

1. rescaling: compact support: $\boxed{\lambda=\sqrt{N}\hat{\lambda}}$, $e^{-\frac{\hat{\lambda}^2 N}{2}}$

(later: $\Rightarrow$ $\lim_{N\to\infty}\frac{1}{N}R_{N,1}(\lambda=\sqrt{N}\hat{\lambda})=\begin{cases}\frac{1}{\pi}\sqrt{2-\hat{\lambda}^2}\equiv\rho(\hat{\lambda})\text{ semi-circle}\\ 0 \quad \hat{\lambda}^2>2\end{cases}$)

$$\Rightarrow \frac{1}{N}\frac{d^2}{d\hat{\lambda}^2}\psi_N(\lambda=\sqrt{N}\hat{\lambda})+(2N+1-N\hat{\lambda}^2)\psi_N(\lambda=\sqrt{N}\hat{\lambda})=0$$

2. rescaling: $\boxed{N(\hat{\lambda}-\hat{\lambda}_0)=\xi}$, $\lambda_0\in\text{support}=[-\sqrt{2},\sqrt{2}]$

$\Leftrightarrow$ $\hat{\lambda}=\hat{\lambda}_0+\frac{\xi}{N}$

$$\Rightarrow \frac{N^{\cancel{2}}}{\cancel{N}}\frac{d^2}{d\xi^2}\psi_N\left(\lambda=\sqrt{N}\left(\hat{\lambda}_0+\frac{\xi}{N}\right)\right)+N\left(2-\hat{\lambda}_0^2-2\frac{\hat{\lambda}_0\xi}{N}-\frac{\xi^2}{N^2}+1\right)\psi_N(\;)=0$$

L.O. $\Rightarrow$

$$\boxed{\frac{d^2}{d\xi^2}\psi_N\left(\lambda=\sqrt{N}\hat{\lambda}_0+\frac{\xi}{\sqrt{N}}\right)+\pi^2\rho(\lambda_0)^2\psi_N\left(\lambda=\sqrt{N}\hat{\lambda}_0+\frac{\xi}{\sqrt{N}}\right)=0}$$

$$\Rightarrow \psi_N=\psi_N(\xi)=\begin{cases}c_1\cos(\pi\rho(\lambda_0)\xi) & \text{for } N \text{ even}\\ c_2\sin(\pi\rho(\lambda_0)\xi) & \text{for } N \text{ odd}\end{cases}$$

$$\Rightarrow \frac{1}{N\pi\rho(\lambda_0)}\,w^{\frac12}(\lambda)w^{\frac12}(\mu)K_N(\lambda,\mu)=\underbrace{c_N}_{\sqrt{\frac{N}{2}}}\,\frac{\psi_N(\lambda)\psi_{N-1}(\mu)-\psi_N(\mu)\psi_{N-1}(\lambda)}{\underbrace{\lambda-\mu}_{=\sqrt{N}(\hat{\lambda}-\hat{\mu})=\sqrt{N}\left(\frac{\xi_1}{N}-\frac{\xi_2}{N}\right)}}\cdot\frac{1}{N\pi\rho(\lambda_0)}$$

$$=\frac{c_1c_2}{\sqrt{2}}\,\frac{\sin((\xi_1-\xi_2)\pi\rho(\lambda_0))}{(\xi_1-\xi_2)\pi\rho(\lambda_0)}\equiv K_{\text{sine}}(\xi_1,\xi_2)$$

normalisation $=1$

All local = oscillatory k-point density correl. are

given by the sine-kernel for all λ_0 inside the support (= "bulk")

$$\frac{1}{(N\pi\rho(\lambda_0))^k} R_{N,k} \to S_k(\xi_1 \ldots \xi_k) = \det_{1\le i,j\le k}\left[K_{sine}(\xi_i, \xi_j)\right]$$

in particular

$$\rho_1(\xi) = \lim_{\xi_1 \to \xi_2} K_{sine}(\xi_1, \xi_2) = 1$$

$$S_2(\xi_1, \xi_2) = 1 - \left(\frac{\sin \pi\rho(\lambda_0)(\xi_1 - \xi_2)}{(\text{—''—})}\right)^2$$

Comparing to standard asymptotic : [i.e. Gradshteyn, Ryzhik 8.955]

asymptotic expansion for Hermite polynomials:

$$e^{-\frac{x^2}{2}} H_{2n=N}(x) \sim \left(\cos(\sqrt{2N+1}\,x) + O\left(\frac{1}{\sqrt{N}}\right)\right)$$

$$e^{-\frac{x^2}{2}} H_{2n+1=N}(x) \sim \left(\sin(\sqrt{2N+1}\,x) + \text{—''—}\right)$$

for $N = 2n$ even $\to\infty$ and $x = O(1)$

gives the correct asymptotic only for $\lambda_0 = 0$:

argument of P_N: $\lambda = \sqrt{N}\underset{\overset{\shortparallel}{0}}{\hat{\lambda}_0} + \frac{\xi}{\sqrt{N}}$

$$\Rightarrow \quad \Psi_n(\lambda = \ldots) \sim \begin{cases} \cos(\sqrt{2}\,\xi) \\ \sin(\sqrt{2}\,\xi) \end{cases}$$

with $\pi\rho(\lambda_0 = 0) = \frac{\pi}{\pi}\sqrt{2 - 0^2} = \sqrt{2}$

Here we get the same universal spectral statistics throughout

the support!

Large-N limit: the Airy kernel

$$\frac{d^2}{d\lambda^2}\psi_N(\lambda) + (2N+1-\lambda^2)\,\psi_N(\lambda) = 0$$

has another non-trivial scaling limit around the edge $\lambda_c = \sqrt{2N}$

$$\lambda = \lambda_c + s\,(2N)^{\delta} \quad \Longrightarrow \quad \frac{d^2}{d\lambda^2} = \frac{1}{(2N)^{2\delta}}\frac{d^2}{ds^2}$$

$$\Rightarrow (2N+1-\lambda^2) = 2N+1-\left(2N + 2\sqrt{2N}\,s\,(2N)^{\delta} + s^2\,(2N)^{2\delta}\right)$$

$$= 1 - 2sd\,(2N)^{\frac{1}{2}+\delta} + s^2 d^2 (2N)^{2\delta} \qquad \text{sub leading}$$

we need $\quad -2\delta = \frac{1}{2}+\delta \iff \delta = \frac{1}{6}, \quad \frac{1}{2}+\delta = \frac{4}{6} = \frac{2}{3}$

$$\Longrightarrow \frac{d^2\psi_N\left(\lambda = \sqrt{2N} + s\,(2N)^{\frac{1}{6}}\right)}{ds^2} - s\,\psi_N(\ldots) = 0$$

solutions are given by the Airy function,

$Ai(s)$, $\quad Bi(s)$, $\quad$ after taking into account $\psi_N(\lambda \to \infty) \to 0$

we get $\boxed{K_N(\lambda,\mu) \to \dfrac{Ai(s_1)\,Ai'(s_2) - Ai(s_2)\,Ai'(s_1)}{s_1 - s_2} \cdot \text{const}}$

which is the Airy kernel $\Downarrow$

with $\rho_1(s) = \left(Ai'(s)\right)^2 - s\left(Ai(s)\right)^2$