The semi-circle, macroscopic universality and its limits

Consider the density

$$R_{N,1}(\lambda) = \omega(\lambda)\, K_N(\lambda,\lambda) = e^{-2V(\lambda)} \sum_{l=0}^{N-1} P_l(\lambda)^2$$

$$\stackrel{C.D.}{=} e^{-2V(\lambda)} c_N \lim_{\mu\to\lambda} \frac{P_N(\lambda)P_{N-1}(\mu) - P_N(\mu)P_{N-1}(\lambda)}{\lambda-\mu}$$

$$R_{N,1}(\lambda) \stackrel{Taylor}{=} c_N e^{-2V(\lambda)} \left(-P_N(\lambda)P_{N-1}'(\lambda) + P_N'(\lambda)P_{N-1}(\lambda)\right)$$

$$\xrightarrow[N\to\infty]{} \rho(\lambda)^{smooth} + \rho(\lambda)^{oscill}$$

- we have already investigated for $V(\lambda) = \frac{\lambda^2}{2}$ Gauss, zooming into bulk/edge to get the Sine/Airy kernel

<u>today</u>: 1) ρ^{smooth}: semi-circle and generalisations, $=$ macro. lim universal & non-universal parts

2) universal and non-universal perturbations of $V(\lambda)$ for the sine-kernel

- $P_n' = A_n P_{n-1} - B_n P_n$, $\lambda P_{n-1} = c_n P_n - c_{n-1} P_{n-2}$

$$\Rightarrow R_{N,1}(\lambda) = c_N e^{-2V}\left[-P_N\left(A_{N-1}\frac{(c_N P_N - \lambda P_{N-1})}{c_{N-1}} - B_{N-1}P_{N-1}\right) + (A_N P_{N-1} - B_N P_N)P_{N-1}\right]$$

$$= c_N e^{-2V}\left[A_N P_{N-1}^2 + \frac{c_N}{c_{N-1}} A_{N-1} P_N^2 - \left(B_N - B_{N-1} + \frac{\lambda}{c_{N-1}} A_{N-1}\right) P_N P_{N-1}\right]$$

- for V polynomial A_N, B_N are polynomials of finite degree ($\deg V$) that remain smooth

$\Rightarrow$ we have to separate the smooth and oscill part of $P_{N(-1)}^2$, $P_N P_{N-1}$

we can do so by projecting

$$\int d\lambda\, f(\lambda)\, R_{N,1}(\lambda) = \int d\lambda\, f(\lambda)\, R^{\text{smooth}}_{N,1}$$

with smooth test functions $f(\lambda)$, i.e. moments

ditto

$$\Lambda^{(N)}_{2m} = \int d\lambda\, e^{-2V(\lambda)} P_N(\lambda)^2 t^{2m} \overset{\text{parity}}{=} \int d\lambda\, \Psi_N(t)^2 t^{2m} \simeq \int d\lambda\, \overline{\Psi_N(t)^2}\, t^{2m}$$

smooth part

$$\Gamma^{(N)}_{2m+1} = \int d\lambda\, e^{-2V} P_N P_{N-1} t^{2m+1} = \int d\lambda\, \Psi_N \Psi_{N-1} t^{2m+1}$$

<u>Assumption</u>: $\boxed{C_{N+q} \approx C_N}$ approaching a smooth funct for $q = O(1)$

- this can be made rigorous using the Riemann-Hilbert method
- as we will see this is equivalent to postulate a <u>single interval</u> support of R^{smooth} for a given V

$$\Rightarrow \quad \lambda P_N(\lambda) \approx C_N \left(P_{N+1}(\lambda) + P_{N-1}(\lambda)\right)$$

$$\Rightarrow \quad \lambda^\ell P_N(\lambda) \approx C_N^\ell \sum_{j=0}^{\ell} \binom{\ell}{j} P_{N+2j-\ell}(\lambda) \qquad \ell \geq 1$$

for the smooth part we thus get

$$\Lambda^{(N)}_{2m} \approx C_N^{2m} \sum_{j=0}^{2m} \binom{2m}{j} \underbrace{\int d\lambda\, e^{-2V} P_N P_{N+2j-2m}}_{\delta_{2j,2m} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{2i\theta(j-m)}}$$

$$= C_N^{2m} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(e^{-i\theta}\right)^{2m} \left(1 + e^{2i\theta}\right)^{2m} = \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2C_N \cos\theta\right)^{2m}$$

$$= 2\int_0^{\pi} \quad \text{periodic}$$

$$t = 2C_N \cos\theta, \quad \frac{dt}{d\theta} = -2C_N \sin\theta = -\sqrt{(2C_N)^2 - (2C_N\cos\theta)^2}$$

$$\Longrightarrow \Lambda_{2m}^{(N)} \approx \frac{1}{\pi} \int_{-2c_N}^{2c_N} dt \, \frac{t^{2m}}{\sqrt{4c_N^2 - t^2}}$$

which implies $\overline{\Psi_N(t)}^2 = \frac{1}{\pi} \frac{1}{\sqrt{4c_N^2 - t^2}} \underbrace{\Theta(4c_N^2 - t^2)}_{\text{support of } R_{N,1}^{\text{smooth}}}$

- the zeros of the P_N become dense inside σ, outside they vanish exponentially.

$\sigma = [-2c_N, +2c_N]$

- Similarly one can show that

$$\Gamma_{2m+1}^{(N)} \approx \frac{1}{\pi 2c_N} \int_{-2c_N}^{2c_N} dt \, t^{2m+1} \frac{\cdot \, t}{\sqrt{4c_N^2 - t^2}}$$

implying $\overline{\Psi_N \Psi_{N-1}(t)}^2 = \frac{1}{\pi 2c_N} \frac{t}{\sqrt{4c_N^2 - t^2}} \Theta(4c_N^2 - t^2)$

We thus obtain for the limiting smooth density

$$R_{N,1}^{\text{smooth}}(\lambda) \approx c_N A_N(\lambda) \left[2 \overline{\Psi_N}^2 - \frac{\lambda}{c_N} \overline{\Psi_N \Psi_{N-1}}^2 \right]$$

$$= \frac{2c_N A_N(\lambda)}{\pi} \Theta(4c_N^2 - t^2) \frac{1 - \left(\frac{\lambda}{2c_N}\right)^2}{\sqrt{4c_N^2 - \lambda^2}}$$

$$= \frac{A_N(\lambda)}{2c_N \pi} \sqrt{4c_N^2 - \lambda^2} \, \Theta(\quad)$$

Examples

1) $V(\lambda) = \frac{\lambda^2}{2} \Rightarrow A_N(\lambda) = 2c_N \Rightarrow R_{N,1}^{\text{smooth}}(\lambda) \simeq \frac{1}{\pi} \sqrt{4c_N^2 - \lambda^2} \, \Theta(\ldots)$

Gauss

$$c_N^2 = \frac{N}{2}$$

$\rightarrow$ rescaling $R_{N,1}(N^{\frac{1}{2}} \lambda) \frac{1}{N} \xrightarrow[N \to \infty]{} \boxed{\rho(\lambda) = \frac{1}{\pi} \sqrt{4 - \lambda^2} \, \Theta(4 - \lambda^2)}$ Semi-circle

2) $V_4(\lambda) = \frac{\lambda^2}{2} + g\frac{\lambda^4}{4} \Rightarrow V'(\lambda) = \lambda + g\lambda^3$

$$\Rightarrow A_N(\lambda) = 2c_N \int dt\, e^{-2V} \underbrace{\frac{(\overbrace{t+gt^3}-\lambda-g\lambda^3)}{t-\lambda}}_{= 1+g(t^2+\lambda^2+\cancel{t\lambda})} p_N^2(t)$$

$$= 2c_N\left(1+g\lambda^2+g\left(\int dt\, e^{-2V} c_N^2 (p_{N+1}(t)+p_{N-1}(t))^2\right)\right)$$

$$= 2c_N\left(1+gc_N^2+gc_N^2\lambda^2\right)$$

$$\Rightarrow R^{\text{smooth}}_{N,1}(\lambda) \underset{V_4 \text{ quartic}}{=} h_N(\lambda)\sqrt{4c_N^2-\lambda^2} \neq \text{semi circle}$$

non-universal perturbation for R^{smooth}!

- eq for c_N also modified (-> see below)

(replacing V Gaus with iid r.v. still leads to semi-circle
V Gaus with $\beta = 1,4$ " " " " ")

Q: are there correlation functions in the macroscopic large-N limit that remain universal under such perturbations?

Yes: connected correlation functions:

$$R_{N,2}(\lambda,\mu) = \underbrace{\omega(\lambda)\omega(\mu)\left(K_N(\lambda,\lambda)K_N(\mu,\mu)\right.}_{\substack{R_{N,1}(\lambda)\cdot R_{N,1}(\mu) \\ \text{disconnected part} \\ \text{non universal}}} \underbrace{- K_N(\lambda,\mu)^2}_{\substack{R^{\text{conn}}_{N,2}(\lambda,\mu) \\ \text{connected part} \\ \text{universal}}}\left.\right)$$

- in the macroscopic large-N limit the factorising part is $\angle O$ (large-N factorisation), the subdominant part remains universal for higher orders too
- in the microscopic limit in contrast the factorised and connected part remain of the same order, both are universal (the kernel itself is)

$$R^{conn}_{N,2}(\lambda,\mu) = -\frac{c_N^2}{(\lambda-\mu)^2}\left[\psi_N(\lambda)\psi_{N-1}(\mu) - \psi_{N-1}(\lambda)\psi_N(\mu)\right]^2$$

smooth limit $\rightarrow -\frac{c_N^2}{(\lambda-\mu)^2}\Big[\overline{\psi_N(\lambda)}^2\,\overline{\psi_{N-1}(\mu)}^2 - 2\,\overline{\psi_N(\lambda)\psi_{N-1}(\lambda)}\cdot\overline{\psi_N(\mu)\psi_{N-1}(\mu)}$

$$+\overline{\psi_{N-1}(\lambda)}^2\,\overline{\psi_N(\mu)}^2\Big]$$

$$R^{conn}_{N,2}(\lambda,\mu) = \frac{-1}{2\pi^2(\lambda-\mu)^2}\cdot\frac{(4c_N^2-\lambda\mu)}{\sqrt{4c_N^2-\lambda^2}\sqrt{4c_N^2-\mu^2}}$$

- this and higher connected k-point densities remain universal for non-Gaussian potentials $V(\lambda)$ [first in Ambjørn, Jurkiewicz, Makeenko, Phys. Lett. B251 (1990) 517–524]
- the universal parameter $2c_N$ = endpoint of the support encodes the influence of $V(\lambda)$: it is determined by the string eq: ($\hat{P}_n(\lambda) = \lambda^n + ..$ monic $\Rightarrow \tilde{P}_n' = n\lambda^{n-1} + ..$)

$$\Rightarrow \int dt\, e^{-2V(t)}\,\hat{P}_n'(t)\,\tilde{P}_{n-1}(t) = n\int dt\, e^{-2V(t)}(t^{n-1} + \cancel{\ldots})\tilde{P}_{n-1}(t) = n\, h_{n-1}$$

or int by parts $\quad = \int dt\, e^{-2V(t)}(2V'(t))\,\tilde{P}_n(t)\,\tilde{P}_{n-1}(t)$

<u>example V_4 quartic</u>:

$$n\, h_{n-1} = 2\int dt\, e^{-2V}(t+gt^3)\,\hat{P}_n(t)\,\tilde{P}_{n-1}(t)\,,\quad \text{use recurrence}$$

$$t\tilde{P}_n = \tilde{P}_{n+1} + \frac{h_n}{h_{n-1}}\tilde{P}_{n-1}$$

term $\sim g$: $t^3 \tilde{P}_n \tilde{P}_{n-1} = \left(\tilde{P}_{n+1} + \frac{h_n}{h_{n-1}} \tilde{P}_{n-1}\right) t \left(\tilde{P}_n + \frac{h_{n-1}}{h_{n-2}} \tilde{P}_{n-2}\right)$

$$\Rightarrow \int dt\, e^{-2V} g t^3 \tilde{P}_n \tilde{P}_{n-1} = g\left(h_{n+1} + \left(\frac{h_n}{h_{n-1}}\right)^2 \cdot h_{n-1} + \frac{h_n}{h_{n-1}} \cdot \frac{h_{n-1}}{h_{n-2}} \cdot h_{n-1} \right)$$

$$\Rightarrow \boxed{ n = 2\frac{h_n}{h_{n-1}} + 2g\left(\frac{h_{n+1}}{h_n} \cdot \frac{h_n}{h_{n-1}} + \left(\frac{h_n}{h_{n-1}}\right)^2 + \frac{h_n}{h_{n-1}} \frac{h_{n-1}}{h_{n-2}} \right) }$$

with $c_n = \sqrt{\frac{h_n}{h_{n-1}}}$ and our ansatz $c_{N\pm 1} \to c_N$ this defines

$$\to \underline{N = 2\, c_N^2 + 2 \cdot 3 \cdot g\, c_N^4}$$

- for higher order potentials we get higher order alg. eqs., but $R^{conn}_{N,2}$ remains the same

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Microscopic universality - example sine-kernel

<u>recall</u> $\psi_n''(\lambda) - F_n(\lambda)\psi_n'(\lambda) + G_n(\lambda)\psi_n(\lambda) = 0$

with F_n, G_n given in terms of A_n, B_n, V which are polynomials. As we saw already these remain smooth funct. when $c_{N\pm 1} \to c_N$

- $B_N(\lambda) + B_{N-1}(\lambda) - \frac{\lambda}{c_{N-1}} A_{N-1}(\lambda) + 2V'(\lambda) = 0$

$$\to 2B_N - \frac{\lambda A_N}{c_N} + 2V' = 0$$

$$\Rightarrow G_N = \frac{c_N}{c_{N-1}} A_N A_{N-1} - (B_N + V')^2 + (B_N + V')' - \frac{A_N'}{A_N}(B_N + V')$$

$$\to \underbrace{A_N^2 - \frac{\lambda^2}{4c_N^2} A_N^2}_{\sim\, R^{\text{smooth}}_{N,1}(\lambda)^2} + \left(\frac{\lambda A_N}{2c_N}\right)' - A_N' \frac{\lambda}{2c_N}$$

$$F_N = \frac{A_N'}{A_N} = \frac{d}{d\lambda} \log A_N(\lambda) \text{ remains}$$

- doing the same rescalings as on p. 38 for the bulk limit (where in our quartic V_4 we also need to do $g \to \frac{1}{N}\hat{g}$ with $\lambda \to \sqrt{N}\lambda$) we obtain the same asymptot eq

$$\frac{d^2}{dz^2}\Psi_N\left(\lambda - \sqrt{N}\hat{\lambda}_0 + \frac{z}{\sqrt{N}}\right) + \pi^2 \rho(\lambda_0)^2 \Psi_N(\;) = 0$$

where the universal parameter $\rho(\lambda_0)$ (= non-universal ρ in macro-limit) encodes all influence of the non-Gauss'V.

$\to$ we get the same universal Sine-Kernel as before [first in Brézin, Zee, Nucl. Phys. B402 (1993) 613]

Q: Non universal deformations of $V(\lambda)$?

consider $\omega(\lambda) = |\lambda|^{2\alpha} e^{-2V(\lambda)} \equiv e^{-2V_\alpha(\lambda)}$, $V_\alpha(\lambda) = V(\lambda) - \alpha \log|\lambda|$, $\alpha \in \mathbb{R}_+$, V polynomial

$\Rightarrow$ this adds an oscillatory part to B_N:

$$B_N(\lambda) \to B_N(\lambda) + (1-(-1)^N)\frac{\alpha}{\lambda}, \quad \lambda \neq 0$$

$\Rightarrow$ the resulting 2. order diff. eq get modified at $\lambda = 0$, due to the singularity:

$$\boxed{\frac{d^2\Psi_N}{dz^2} + \left(\pi^2\rho(0)^2 + \frac{(-1)^N\alpha - \alpha^2}{z^2}\right)\Psi_N = 0}$$

with solutions $\quad \Psi_{\substack{2N \\ 2N+1}} \sim \sqrt{z}\, J_{\alpha \mp \frac{1}{2}}(\pi\rho(0) z)$

- the resulting Bessel-Kernel is still universal $\forall$ polyn. $V(\lambda)$ [G.A., Damgaard, Magnea, Nishigaki Nucl. Phys. B487 (1997) 721]

the parameter α labels different universality classes.

* other deformations: fine-tune V s.th. $R_{N+1}(\lambda)(A_N(\lambda)) = 0$ inside σ: multi-critical