Lecture 7

- The Tracy-Widom distribution of the largest GUE eigenvalue

[Tracy, Widom, Comm. Math. Phys. 159 (94) 151
arXiv hep-th/9211141]

- is universal for non-Gauss $V(\lambda)$

- appears in many different problems, i.e. growth processes, random walks, long. increasing subseq. of random partitions, ...

* in the derivation presented here we will not use the original Fredholm det. approach by Tracy-Widom, but asymptot. OP by Nadal, Majumdar arXiv 1102.0738 (for GUE (it is universal!))

the cumulative distr. of the maximal GUE eigenvalue reads

$$\mathbb{P}_N(\lambda_{max} \le y) = \frac{Z_N(y,\alpha)}{Z_N(\infty,\alpha)} \quad \text{with } Z_N(y,\alpha) = \frac{1}{N!}\prod_{i=1}^{N}\int_{-\infty}^{y} d\lambda_i \, e^{-\alpha\lambda_i^2} \Delta_N(\{\lambda\})^2$$

$$\Rightarrow \boxed{F_2(x) = \exp\left[-\int_x^\infty ds\,(s-x)\, q^2(s)\right],} \qquad q(s) \xrightarrow[s\to\infty]{} Ai(s) \quad \text{Hastings-McLeod sol.}$$

of Painlevé II $q''(x) = 2q^3(x) + xq(x)$

- task: OP wrt asym. weight $w(\lambda) = e^{-\alpha\lambda^2}\Theta(y-\lambda)$ for $Z(y,\alpha)$

we know for $\tilde{p}_n(\lambda) = \lambda^n + \dots$ monic

$$\lambda\tilde{p}_n = \tilde{p}_{n+1} + a_n\tilde{p}_n + b_n\tilde{p}_{n-1}, \qquad h_n = h_n(y,\alpha) = \int_{-\infty}^{y} dx\, e^{-\alpha x^2}\tilde{p}_n(x)^2$$

determine $a_n h_n = \int_{-\infty}^{y} d\lambda\, \underbrace{e^{-\alpha\lambda^2}\lambda}_{-\frac{1}{2\alpha}\partial_\lambda\left(e^{-\alpha\lambda^2}\right)}\tilde{p}_n(\lambda)^2$

as $\frac{\partial}{\partial y}\left(\tilde{p}_n(x) = \lambda^n + \dots\right)$ polynom. of $O(\lambda^{n-1})$

int. by parts $\Rightarrow$ $a_n h_n = -\frac{1}{2\alpha} e^{-\alpha y^2}\tilde{p}_n(y)^2 = -\frac{1}{2\alpha}\frac{\partial}{\partial y} h_n(y,\alpha)$

so $$\boxed{b_n = \frac{h_n}{h_{n-1}} \quad , \quad a_n = -\frac{1}{2\alpha}\,\partial_y \ln[h_n]}$$

We had shown $$Z_N(y,\alpha) = \frac{N!}{N!}\prod_{j=0}^{N-1} h_j = h_0 \prod_{n=1}^{N} b_n$$

- we shall now derive a system of diff. eqs. for a_n, b_n that leads to Painlevé $\underline{\mathrm{IV}}$:

it holds $$\frac{\partial}{\partial\alpha} h_n = \int_{-\infty}^{y} d\lambda\,(-\lambda^2 e^{-\alpha\lambda^2})\,\tilde{p}_n(\lambda)^2 + \underbrace{\int_{-\infty}^{y} d\lambda\, e^{-\alpha\lambda^2}\, 2\tilde{p}_n\, \underbrace{\partial_\alpha \tilde{p}_n(\lambda)}_{O(\lambda^{n-1})}}_{=0}$$

$$= -\int_{-\infty}^{y} d\lambda\,(\tilde{p}_{n+1} + a_n\tilde{p}_n + b_n\tilde{p}_{n-1})^2$$

$$= -(h_{n+1} + a_n^2 h_n + b_n^2 h_{n-1})$$

$$\Leftrightarrow \quad \underline{-\partial_\alpha \ln h_n = b_{n+1} + a_n^2 + b_n} \qquad \circledast$$

$$\Rightarrow \boxed{\begin{array}{ll} (\mathrm{I}) & -\partial_\alpha \ln b_n = b_{n+1} - b_{n-1} + a_n^2 - a_{n-1}^2 \\ (\mathrm{II}) & -\dfrac{\partial_y \ln b_n}{2\alpha} = a_n - a_{n-1} \end{array}}$$

recurrence depth 2 for b_n

depth 1 for a_n

boundary conditions:

- $\lim y \to \infty$: rescaled Hermite $h_n(y=\infty,\alpha) = \sqrt{\frac{\pi}{\alpha}}\,\frac{n!}{(2\alpha)^n}$

$$\Rightarrow \underline{b_n(y=\infty,\alpha) = \frac{n}{2\alpha}} \, , \quad \underline{a_n(y=\infty,\alpha) = 0} \text{ from symmetry}$$

- $\underline{n = 0, 1}$: $\boxed{\tilde{p}_0(\lambda) = 1}$ by def.

$$\Rightarrow \boxed{\underline{h_0(y,\alpha)} = \int_{-\infty}^{y} d\lambda\, e^{-\alpha\lambda^2} = \underline{\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\left(1 + \mathrm{erf}(y\sqrt{\alpha})\right)}}$$

$\lambda\tilde{p}_0 = \tilde{p}_1 + a_0\tilde{p}_0$, setting $\boxed{b_0 = 0}$, $\Rightarrow$ $\underline{\tilde{p}_1(\lambda) = \lambda - a_0}$

$$\Longrightarrow \boxed{a_0 = -\frac{1}{2\alpha}\frac{\partial_y h_0}{h_0} = \frac{-e^{-\alpha y^2}}{\sqrt{\pi\alpha}\,(1+\mathrm{erf}(y\sqrt{\alpha}))}}$$

(*) for $n=0$: $$-\frac{\partial_\alpha h_0}{h_0} = b_1 + a_0^2 \qquad , \mathrm{erf}(y\sqrt{\alpha}) = \frac{2}{\sqrt{\pi}}\int_0^{y\sqrt{\alpha}} dt\, e^{-t^2}$$

$$\frac{+\frac{1}{2\alpha}h_0 - \frac{y}{2\sqrt{\alpha}}\frac{2}{\sqrt{\pi}}e^{-\alpha y^2}\cdot\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}}{h_0} = b_1 + a_0^2$$

$$\Leftrightarrow \qquad \boxed{b_1 = \frac{1}{2\alpha} + y\,a_0 - a_0^2}$$

- scaling limit :

in deriving the Airy-kernel we had rescaled around the edge $\sqrt{2N}$

$$(y - \sqrt{2N}) = s\,(2N)^{-\frac{1}{6}} \qquad \text{or, after rescaling by } \alpha$$

$$x \equiv \sqrt{2\alpha}\,N^{+\frac{1}{6}}\left(y - \sqrt{\frac{2N}{\alpha}}\right)$$

→ we expect $\underline{\ln \mathbb{P}_N(\lambda_{max} \leq y) \to f(x)}$ a funct of the scaling var. x

on the other hand $$b_N(y,\alpha) = \frac{h_N(y,\alpha)}{h_{N-1}(y,\alpha)} = \frac{Z_{N+1}(y,\alpha)}{Z_N(y,\alpha)}\cdot\frac{Z_{N-1}(y,\alpha)}{Z_N(y,\alpha)}$$

and thus after normalising

$$\mathbb{P}_N = \frac{Z_N(y,\alpha)}{Z_N(\infty,\alpha)} \Rightarrow \ln\left[\frac{\mathbb{P}_{N+1}\cdot\mathbb{P}_{N-1}}{\mathbb{P}_N^2}\right] = \ln\left[\frac{b_N(y,\alpha)}{b_N(\infty,\alpha)}\right]$$

we Taylor expand :

$$\ln \mathbb{P}_{N\pm1}(\lambda_{max} \leq y) = f\left(\sqrt{2\alpha}\,(N\pm1)^{\frac{1}{6}}\left(y - \sqrt{\frac{2(N\pm1)}{\alpha}}\right)\right)$$

$$= f\left(x \mp N^{-\frac{1}{3}} \pm \frac{x}{6N} + O(N^{-4/3})\right)$$

$$= f(x) \mp N^{-\frac{1}{3}} f'(x) + \frac{1}{2}N^{-\frac{2}{3}} f'' + O(N^{-1})$$

$\Rightarrow$ $\ln P_{N+1} + \ln P_{N-1} - 2\ln P_N = N^{-\frac{2}{3}} f''(x) + O(N^{-1}) \approx \ln\left[\frac{b_N(y,x)}{N/2\alpha}\right]$

$\Rightarrow$ we get an expansion $b_N(y,\alpha) = \frac{N}{2\alpha}\left(1 + N^{-\frac{2}{3}} v_1(x) + N^{-1} v_2(x) + N^{-\frac{4}{3}} v_3(x) + \dots\right)$

with $\underline{f''(x) = v_1(x)}$

* The scaling function $f(x)$ can be determined using our syst. of diff. eq.

$\underline{II}$: $\frac{\partial}{\partial y}\ln b_N = \frac{\partial x}{\partial y}\frac{\partial}{\partial x}\ln b_N = \sqrt{2\alpha}\,N^{\frac{1}{6}} \cdot v_1'(x)\,N^{-\frac{2}{3}} + \dots$

leads to a scaling of a_N as $a_N - a_{N-1} \equiv N^{-\frac{1}{2}}\frac{s_1'(x)}{\sqrt{2\alpha}} + \dots$

and thus $\underline{v_1(x) = s_1(x)}$ (the const is fixed by $a_N(\infty, x) = 0$)

$\underline{I}$: after expanding to third order and comparing lhs with rhs (v_2, v_3 drop out) we get (details see 1102.0738)

$$x v_1' - 2 v_1 v_1' = -\frac{2}{3}v_1 - \frac{x}{3}v_1' + \frac{1}{3}v_1''' + 2 s_1 s_1'$$

$$\Rightarrow \quad \underline{2 x v_1' + v_1 = \frac{1}{2}v_1''' + 6 v_1 v_1'}$$

substituting $\boxed{f''(x) = v_1(x) \equiv -u^2(x)}$, $w(x) \equiv u'' - xu - 2u^3$ we have

$\Leftrightarrow$ $u(x)\frac{dw(x)}{dx} = -3u'(x)\,w(x)$ solved by $\underline{w(x) = \frac{A}{u(x)^3}}$

- from $\lim\limits_{x\to\infty} v_1 = -u^2 = 0$ we get $A = 0$

$\Rightarrow$ $u(x)$ satisfies $\boxed{u''(x) = x u(x) - 2u(x)^3}$ $P\,\underline{II}$

- the b.c. of f determine those of the sol $u(x)$ to $\uparrow$ (see also $g(x) = (Ai')^2 - x Ai^2$ for b.c.)

and we have $\boxed{f(x) = -\int_x^\infty ds\,(s-x)\,u^2(s)}$

Multi-matrix Models, Harish-Chandra-Itzykson-Zuber Integral and Brownian motion

[see P. Forrester's book, ch 11
Pandey, Mehta, CMP 89(1983) 449]

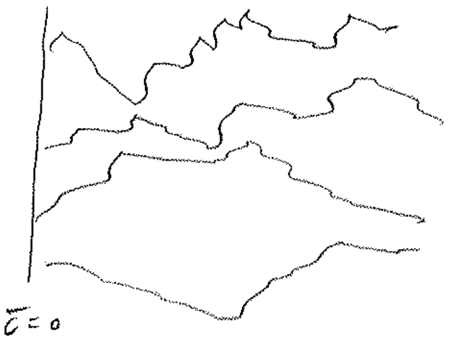
- suppose we want to add some time dependence to our random matrix H, i.e. by starting from a fixed matrix at $\tau=0$ (↖Hamiltonian) and study the time evolution of ev $\lambda_j(\tau)$:

result: brownian motion

$$\boxed{\underline{H=}\ \sqrt{1-e^{2\tau}}\,X+e^{-\tau}H^{(0)}}\equiv\boxed{A+B}$$

random matrix

matrix H at $\tau=0$ can be fixed or random



non-intersect.

2-matrix model and eigenvalue basis

suppose both X and $H^{(0)}$ belong to the GUE (or $H^{(0)}$ to GOE, GSE: symmetry breaking transition)

- pdf for matrix elements

the matrices A, B have Gauss distr. , a, b funct of τ

$$P_1(A)=\exp[-a\,\mathrm{Tr}A^2],\quad P_2(B)=\exp[-b\,\mathrm{Tr}B^2]$$

$$\Rightarrow\ \underset{\tau}{P}(H)=\int dA\ P_1(A)\,P_2(H-A)$$

nontrivial coupling

$$=\int dA\,\exp[-(a+b)\mathrm{Tr}A^2-b\,\mathrm{Tr}H^2+2b\,\mathrm{Tr}\,HA]$$

partition function

$$Z=\int dH\,P(H)=\int dH\,dA\,\exp[-(a+b)\mathrm{Tr}A^2-b\,\mathrm{Tr}H^2+2b\,\mathrm{Tr}\,HA]$$

diag A, H

$$=\int\prod_{i=1}^{N}d\lambda_i\,e^{-b\lambda_i^2}\,|\Delta_N(\lambda)|^2\int\prod_{j=1}^{N}dx_j\,e^{-(a+b)x_j^2}\,|\Delta_N(x)|^\beta\int dU\,dV\,e^{+2b\mathrm{Tr}[U\Lambda U^\dagger VXV^\dagger]}$$

H-eigenvalues

- unitary d.o.f. and eigenvalues do not decouple!

  ⇒ use invariance of Haar measure $\int dU$ under $U \to VU$, $V^{-1} = V^\dagger$ (or orth., sympl.) and ⇒ $\int dV$ decouples

  Harish-Chandra-Itzykson-Zuber integral (HCIZ)

$$\boxed{\int\limits_{U(N)} dU \exp[zb \operatorname{Tr} U \Lambda U^\dagger X] = \frac{\det\limits_{1\leq j,k\leq N}\left[e^{zb\lambda_j x_k}\right]}{\Delta_N(\lambda)\cdot\Delta_N(x)}}$$

  ⇒ the resulting integral over $\int \prod dx_j$ can be performed using the de Bruijn or Andréief formulas (Lecture 3&4) yielding a det or Pfaffian depending on ev λ_i only $P_Z(\{\lambda\})$

  ⇒ the remaining jpdf of H-eigenvalues λ_i can be solved using Bi- or skew-orthogonal polynomials, as well as all correlation functions

  (example for a resulting kernel $K_N(\lambda_1,\lambda_2;t) \sim \int_0^1 dt\, e^{-ct^2} \cos(\lambda_1-\lambda_2)t$)

- the question of universality (non-Gauß distr. $\rho_1(A)\rho_2(B)$) is difficult due to a lack of generalisation of HCIZ

→ Brownian motion

- starting from $H = (1-e^{-2\tau})^{\frac{1}{2}} X + e^{-\tau} H^{(0)}$ it can be shown [Forrester book] that the following Fokker-Planck eq. holds on the level of H_{ij} or λ_j

$$\frac{\partial P_\tau(H)}{\partial \tau} = \sum_{\substack{\text{indep}\\ \text{matrix el}}} \left(\frac{\partial}{\partial H_{ij}} \left(H_{ij} \cdot P_\tau(H) \right) + \frac{1}{\beta} \frac{\partial^2 P_\tau(H)}{\partial H_{ij}^2} \right)$$

or

$$\boxed{\frac{\partial P_\tau(\{\lambda\})}{\partial \tau} = \frac{1}{\beta} \sum_{j=1}^{N} \left(\frac{\partial^2}{\partial \lambda_j^2} + \frac{\partial}{\partial \lambda_j} \frac{\partial W}{\partial \lambda_j} \right)} \qquad \boxed{\Delta}$$

with $\boxed{W = \frac{1}{2} \sum_{j=1}^{N} \lambda_j - \sum_{j<k} \log |\lambda_j - \lambda_k|}$, $\frac{\partial W}{\partial \lambda_j} = \lambda_j - \sum_{\substack{k=1\\ k\neq j}}^{N} \frac{1}{\lambda_j - \lambda_k}$

effective potential for eigenvalues $e^{-\beta W}$ in original 1-matrix model

- for small time intervalls $\delta\tau = \tau - \tau_0$ the displacement of eigenvalues become Gauss' random variables

$$\langle (\lambda_j(\tau) - \lambda_j(\tau_0)) \rangle = -\frac{\partial W}{\partial \lambda_j} \delta\tau, \quad \langle (\lambda_j(\tau) - \lambda_j(\tau_0))^2 \rangle = \frac{2}{\beta} \delta\tau$$

⇒ $\boxed{\Delta}$ is equiv. to Langevin eq:

$$\boxed{\frac{d\lambda_j(\tau)}{d\tau} = -\frac{\partial W}{\partial \lambda_j} + F_j(\tau)}$$

↑ random force of Brownian motion

$$\langle F_i(\tau) F_j(\tau') \rangle \sim \delta_{ij} \, \delta(\tau - \tau')$$