Intergrable Models	**Exercises No. 1**

Exercise 1.1: Consider the KdV equation;

$$\partial_t u = u\partial_x u + \partial_x^3 u. \tag{1}$$

(a) Show that (1) is invariant under the translational symmetry,

$$x \to x + a, \qquad t \to t + a.$$

(b) Consider a transformation of the form,

$$x \to cx, \qquad t \to c^\alpha t, \qquad u \to c^\beta u.$$

Find the values of α and β for which the above transformation is a symmetry of (1).

(c) If $u(x,t)$ is a solution to (1), show that $u(x+vt,t)+v$, where v is a constant, is also a solution. What is the physical interpretation of this symmetry?

Exercise 1.2: Consider the following ansatz for a travelling wave solution to (1),

$$u = a\mathrm{sech}^2\left((x+ct)/b\right). \tag{2}$$

Derive the relations that a and b must satisfy if u solves (1).

Exercise 1.3: Consider a $2N$ dimensional phase space, with coordinates $(q_1,\ldots,q_N,p_1,\ldots,p_N)$. For any two differential functions, f and g on phase space, define the Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}. \tag{3}$$

(a) Let us combine the coordinates on phase space into a single variable y_μ, so that we have $y_\mu = q_\mu$ if $\mu < N+1$ and $y_\mu = p_\mu$ otherwise. Show that the Poisson bracket may then be written in the form,

$$\{f,g\} = \epsilon^{\mu\nu}\frac{\partial f}{\partial y_\mu}\frac{\partial f}{\partial y_\nu}, \tag{4}$$

where the Greek indices take values in the range 1 to $2N$ and $\epsilon^{\mu\nu} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Show that the Poisson bracket satisfies the Jacobi identity,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0. \tag{5}$$

(b) Consider a generalisation of the Poisson bracket, defined by

$$\{f,g\} = \omega^{\mu\nu}\frac{\partial f}{\partial y_\mu}\frac{\partial f}{\partial y_\nu}. \tag{6}$$

Show that in order for this generalisation of the Poisson bracket to still satisfy the Jacobi identity, $\omega^{\mu\nu}$ must satisfy,

$$\partial_\alpha\omega_{\mu\nu} + \partial_\mu\omega_{\nu\alpha} + \partial_\nu\omega_{\alpha\mu} = 0. \tag{7}$$