Exercise 10.1: A conformal transformation $x' = x'(x)$ is one which changes the metric by an overall scale factor $\Lambda(x)$. Recall that in $d > 2$ dimensions the only conformal transformations are; translation, dilation, rotation and the special conformal transformation (SCT),

$$x'^{\mu} = \frac{x^{\mu}}{1 - 2b^{\nu}x_{\nu} + b^2x^2}. \tag{1}$$

For the SCT $\Lambda(x) = (1 - 2b^{\nu}x_{\mu} + b^2x^2)^2$. In a field theory with conformal symmetry (CFT), the field $\phi(x)$ transforms under conformal transformations as,

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d}\phi(x). \tag{2}$$

where $|\partial x'/\partial x|$ is the Jacobian of the transformation and Δ is called the conformal weight of ϕ. The correlation functions therefore transform as,

$$\langle\phi_1'(\mathbf{x}'_1)\phi_2'(\mathbf{x}'_2)\ldots\rangle = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta_1/d}\left|\frac{\partial x'}{\partial x}\right|^{-\Delta_2/d}\ldots\langle\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\ldots\rangle. \tag{3}$$

For the 2-point function $F(\mathbf{x}_1, \mathbf{x}_2) = \langle\phi_1'(\mathbf{x}'_1)\phi_2'(\mathbf{x}'_2)\rangle$ show that,

(a) translation invariance implies, $F(\mathbf{x}_1, \mathbf{x}_2) = f(\mathbf{x}_1 - \mathbf{x}_2)$.

(b) which together with rotational invariance implies, $F(\mathbf{x}_1, \mathbf{x}_2) = f(|\mathbf{x}_1 - \mathbf{x}_2|)$.

(c) which together with how (3) transforms under dilations implies, $F(\mathbf{x}_1, \mathbf{x}_2) = \frac{C}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}}$, where C is a constant.

(d) Show that under a SCT,

$$|\mathbf{x}'_1 - \mathbf{x}'_2| = \frac{|\mathbf{x}_1 - \mathbf{x}_2|}{(1 - 2b^{\nu}x_{1\mu} + b^2{x_1}^2)^{1/2}(1 - 2b^{\nu}x_{2\mu} + b^2{x_2}^2)^{1/2}} \tag{4}$$

Finally, apply the SCT to $F(\mathbf{x}_1, \mathbf{x}_2)$ and deduce that F is non-zero iff $\Delta_1 = \Delta_2$.

Exercise 10.2: We would like to understand the physical origin of the Liouville equation. For a curved surface, the curvature is measured by the Riemann tensor,

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc} \tag{5}$$

where g is the metric, a comma followed by a letter e.g ",c" represents the differentiation $\partial/\partial x^c$ and,

$$\Gamma^a_{bc} = \frac{1}{2}g^{af}\left(g_{bf,c} + g_{cf,d} - g_{bc,f}\right) \tag{6}$$

are called the Christoffel symbols. [1]

[1] It is interesting to compare the expression for Riemann tensor to the zero curvature condition we met earlier.

In two dimensions the metric g_{ab} may always be brought to the form $e^{\phi}\delta_{ab}$. Show that for such a metric,

$$\Gamma^a_{bc} = \frac{1}{2}\left(\phi_{,c}\delta^a_b + \phi_{,b}\delta^a_c - \phi^{,a}\delta_{bc}\right), \tag{7}$$

where $\phi^{,a} = \delta^{ab}\phi_{,b} = \phi_{,a}$. Hence show that the Ricci tensor, $R_{bc} = R^a_{bac}$ is,

$$R_{bc} = -\frac{1}{2}\partial_a\partial^a\phi\delta_{bc}. \tag{8}$$

Hence show that the Liouville field equation for Liouville theory, $\partial_\mu\partial^\mu\phi = \mu e^{\phi}$ may be written as $R[g] = -\mu$, where $R = g^{bc}R_{bc}$ is the Ricci scalar. What kind of system does the Liouville theory therefore describe [2]?

Exercise 10.3: Recall the NLSE $i\psi_t + \psi_{xx} + 2\psi|\psi|^2 = 0$. Show that the equations,

$$\psi_x + \phi_x = (\psi - \phi)\sqrt{4\lambda^2 - |\psi + \phi|^2} \tag{9}$$

$$\psi_t + \phi_t = i(\psi_x - \phi_x)\sqrt{4\lambda^2 - |\psi + \phi|^2} + \frac{i}{2}(\psi + \phi)\left(|\psi + \phi|^2 + |\psi - \phi|^2\right) \tag{10}$$

are an auto-Backlund transformation for the NLSE. (Hint, first consider differentiating the first equation with respect to x and combining with the second to get one equation. Then consider the compatibility condition between the two equations.)

[2] If you have studied GR, compare these equations to the field equations.