Intergrable Models	**Exercises No. 11**

Exercise 11.1: Recall that $S^\nu_\mu = F_{\mu\nu} f^{\mu\nu}$, where $f_{\mu\nu}$ and $F_{\mu\nu}$ are the two symplectic forms. We define the Nijenhuis tensor to be,

$$N^\mu_{\alpha\beta} = S^\lambda_\alpha \partial_\lambda S^\mu_\beta - S^\lambda_\beta \partial_\lambda S^\mu_\alpha - S^\mu_\lambda(\partial_\alpha S^\lambda_\beta - \partial_\beta S^\lambda_\alpha). \tag{1}$$

Given that S varies with time as $\frac{dS^\nu_\mu}{dt} = S^\lambda_\mu U^\nu_\lambda - U^\lambda_\mu S^\nu_\lambda$, where $U^\beta_\alpha = \partial_\alpha \dot{y}^\beta$, show that,

(i) $\frac{d}{dt}\partial_\alpha S^\mu_\beta = \partial_\alpha \frac{d}{dt} S^\mu_\beta - U^\sigma_\alpha \partial_\sigma S^\mu_\beta$.

(ii) Hence that, $\frac{dN^\mu_{\alpha\beta}}{dt} = -U^\lambda_\alpha N^\mu_{\lambda\beta} - U^\lambda_\beta N^\mu_{\alpha\lambda} + U^\mu_\lambda N^\lambda_{\alpha\beta}$.

(iii) Show also that $N^\mu_{\alpha\beta}(S^{n-1})^\beta_\mu = S^\lambda_\alpha \partial_\lambda K_n - \partial_\alpha K_{n+1}$, where $K_n = \frac{1}{n}\mathrm{Tr} S^n$ are the conserved quantities of the time evolution. Show that a zero Nijenhuis tensor implies the conserved quantities are in involution.

Exercise 11.2: For the Toda lattice, S is a $2N \times 2N$ matrix with the block form,

$$S^\nu_\mu = \begin{pmatrix} B & A \\ -e & B \end{pmatrix} \tag{2}$$

where $A_{ij} = \delta_{i+1,j} e^{-(Q_{i+1}-Q_i)} - \delta_{i,j+1} e^{-(Q_{j+1}-Q_j)}$, $B_{ij} = P_i \delta_{ij}$ and $e_{ij} = \epsilon(j-i)$, where $\epsilon(n) = 1$ if $n > 0$, $\epsilon(n) = 0$ if $n = 0$ and $\epsilon(n) = -1$ if $n < 0$.

(i) Show that $\frac{1}{2}K_1 = \sum_{i=1}^N P_i$ and $\frac{1}{2}K_2 = \sum_{i=1}^N \frac{1}{2}P_i^2 + e^{-(Q_{i+1}-Q_i)}$.

(ii) Show that $\frac{1}{2}K_3 = \sum_{i=1}^N \frac{1}{3}P_i^3 + (P_i + P_{i+1})e^{-(Q_{i+1}-Q_i)}$

Can you identify these quantities?

Exercise 11.3: The following integral arises in the study of random matrices,

$$Z = \int_{-\infty}^{\infty} \prod_{i=1}^N d\lambda_i |\Delta(\lambda)|^2 e^{-\sum_{i=1}^N V(\lambda_i)}, \tag{3}$$

where λ_i is the ith eigenvalue of the random matrix, $V(\lambda) = \sum_{k=0}^\infty t_k \lambda^k$ and $\Delta(\lambda) = \prod_{1\le i<j\le N}(\lambda_j - \lambda_i)$. Consider, the quantity,

$$\det \lambda_i^{j-i} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots \\ 1 & \lambda_2 & \lambda_2^2 & \dots \\ 1 & \lambda_3 & \lambda_3^2 & \dots \\ . & . & . & \dots \end{vmatrix}. \tag{4}$$

Show via row and column operations it can be written,

$$\det \lambda_i^{j-i} = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \lambda_2 - \lambda_1 & (\lambda_2-\lambda_1)\lambda_2 & (\lambda_2-\lambda_1)\lambda_2^2 & \dots \\ 0 & \lambda_3 - \lambda_1 & (\lambda_3-\lambda_1)\lambda_3 & (\lambda_3-\lambda_1)\lambda_3^2 & \dots \\ . & . & . & \dots & \end{vmatrix}, \tag{5}$$

and hence demonstrate that $\Delta(\lambda) = \mathrm{det}\lambda_i^{j-i}$. If we introduce the orthogonal polynomials P_n, defined such that,

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = \delta_{n,m} h_n \tag{6}$$

show that,

(i) $\lambda P_n(\lambda) = P_{n+1} + s_n P_n(\lambda) + r_n P_{n-1}(\lambda)$, where $r_n = h_n/h_{n-1}$ and s_n is a constant.

(ii) Defining $h_n = e^{\phi_n}$, show $\partial_{t_1}\phi_n = -s_n$ and $\partial_{t_1} P_n = r_n P_{n-1}$.

(iii) By considering $\partial_{t_1}^2 e^{\phi_n}$, deduce that,

$$\partial_{t_1}^2 \phi_n = e^{\phi_{n+1} - \phi_n} - e^{\phi_n - \phi_{n-1}}, \tag{7}$$

i.e. the log of the polynomial norms satisfy the Toda lattice equation.