Exercise 12.1: Consider the Hamiltonian, $H = -\partial_x^2 + \alpha x^2 + \beta x^4 + \gamma x^6$, and define the following operators, $a^0 = 1$, $a^+ = x$ and $a^- = \gamma x^3 + \frac{\beta}{2\sqrt{\gamma}} x + \partial_x$

(i) Show that these operators satisfy the Heisenberg algebra,

$$[a^-, a^+] = a^0, \qquad [a^0, a^\pm] = 0. \tag{1}$$

Defining $a^-|0\rangle = 0$ and $|n\rangle = (a^+)^n|0\rangle$, show that $a^-|n\rangle = n|n-1\rangle$.

(ii) Show that the Hamiltonian may be written as,

$$H = (a^+)^2 \left[2\sqrt{\gamma} a^+ a^- + \left(\alpha - \frac{\beta^2}{4\gamma} + 3\sqrt{\gamma} \right) a^0 \right] + \frac{\beta}{2\sqrt{\gamma}} \left(2a^+ a^- + a^0 \right) - (a^-)^2 \tag{2}$$

(iii) Show that by taking a suitable limit, we may recover the harmonic oscillator Hamiltonian from the above result and that for the harmonic oscillator Hamiltonian,

$$H|n\rangle = \sqrt{\alpha}(2n+1)|n\rangle - n(n-1)|n-2\rangle \tag{3}$$

Exercise 12.2: Consider equation (3).

(i) Explain why the space $\Phi_n^p = \{|2i + p\rangle : i = 0, \ldots, n\}$ is invariant under the action of H.

(ii) Consider an element of Φ_n^p, $\phi = \sum_{j=0}^n \eta_{n-j}|2j+p\rangle$, where η_{n-j} are constants. Show that if ϕ satisfies $H\phi = E\phi$, i.e. it is an eigenvector, then $E = \sqrt{\alpha}(4n + 2p + 1)$ and the coefficients satisfy

$$-4\sqrt{\alpha}(m+1)\eta_{m+1} = \left(2(n-m) + p\right)\left(2(n-m) + p - 1\right)\eta_m. \tag{4}$$

Explain why this means $\phi = P_n(x)|0\rangle$, where $P(x)$ is a polynomial.

(iii) Recall the definition of a Hermite $H_k(x)$ polynomial as the solution to,

$$H_k''(x) - 2xH_k'(x) + 2kH_k(x) = 0 \tag{5}$$

and hence find an expression for $P_n(x)$ in terms of Hermite polynomials.

Exercise 12.3: Consider the operators,

$$a^0 = 1, \qquad a^+ = \alpha(x), \qquad a^- = \frac{1}{\alpha'(x)} \left(\partial_x - \beta(x) \right) \tag{6}$$

(i) Show that these operators satisfy the Heisenberg algebra.

(ii) Explain why the Hamiltonian must have the form,

$$H = \left[A_1(a^+)^2 + A_2 a^+ + A_3\right](a^-)^2 + \left[A_4(a^+) + A_5\right]a^- + A_6, \qquad (7)$$

where A_i are constants. Given that H must have the form $H = -\partial_x^2 + V(x)$, find expressions for α and β in terms of A_i.

(iii) Show that if $A_1 = A_3 = 1$, $A_2 = A_5 = 0$, $A_4 = -2$ and $A_6 = -1/4$ we obtain the Poschl Teller potential,

$$-\lambda(\lambda + 1)\mathrm{sech}^2(x), \qquad (8)$$

in the Schrodinger equation and give a value for λ.