| Intergrable Models | **Exercises No. 13** |
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**Exercise 13.1:** Recall the $XXX_{1/2}$ spin chain is defined by the Hamiltonian,

$$H = \sum_{\alpha,n} S_n^\alpha S_{n+1}^\alpha - \frac{1}{4} \tag{1}$$

where $S^\alpha$ is the spin operator in direction $\alpha$ acting on the Hilbert space of the $n$th vertex. The set of vertices are periodically identified; $n + N = n$.

(i) Recall the term ultralocality and explain how this affects the commutation of operators in the limit when the lattice spacing is very small.

(ii) The total spin in the direction $\alpha$ is given by $S^\alpha = \sum_n S_n^\alpha$. Show that $[H, S^\alpha] = 0$.

(iii) The Lax operator is defined as $L_{n,a} = \lambda I_n \otimes I_a + i \sum_\alpha S_n^\alpha \otimes \sigma^\alpha$, which acts on the space $h_n \otimes V$, where $h_n$ is the Hilbert space of the $n$th vertex and $V$ is known as the auxillary space. Show that it can be written as,

$$L_{n,a} = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix} \tag{2}$$

**Exercise 13.2:** Recall the permutation operator $P_{n,a} = \frac{1}{2}\left(I_n \otimes I_a + \sum_\alpha \sigma_n^\alpha \otimes \sigma^\alpha\right)$, which operators on the same space as $L_{n,a}$. Show,

(i) by writing $P_{n,a}$ in the form of a matrix, that given $a \otimes b \in h_n \otimes V$, $P_{n,a}(a \otimes b) = b \otimes a$.

(ii) We now enlarge the space $P_{n,a}$ acts on. In particular we define $P_{n,a_i}$ acting on $h_n \otimes V_1 \otimes V_2$, by saying that its affect on the space $V_j$ where $i \neq j$, is the identity. By considering the affect of $P_{n,a_i}$ on $a \otimes b \otimes c$, show that $P_{n,a_1} P_{n,a_2} = P_{a_1,a_2} P_{n,a_1} = P_{n,a_2} P_{a_1,a_2}$ and $P_{n,a_2} P_{n,a_1} = P_{a_1,a_2} P_{n,a_2} = P_{n,a_1} P_{a_1,a_2}$

**Exercise 13.3:** Recall the definition of the $R$-matrix, which acts on the space $h_n \otimes V_1 \otimes V_2$ (acting on $h_n$ trivially); $R_{a_1,a_2} = \lambda I_{a_1} \otimes I_{a_2} + i P_{a_1,a_2}$. Prove the following equation,

$$R_{a_1,a_2}(\lambda - \mu) L_{n,a_1}(\lambda) L_{n,a_2}(\mu) - L_{n,a_2}(\mu) L_{n,a_1}(\lambda) R_{a_1,a_2}(\lambda - \mu) = 0 \tag{3}$$

by expanding the terms.
Defining the monodromy matrix $T_{N,a} = L_{N,a} \ldots L_{1,a}$ show

$$R_{a_1,a_2}(\lambda - \mu) T_{N,a_1}(\lambda) T_{n,a_2}(\mu) = T_{n,a_2}(\mu) T_{n,a_1}(\lambda) R_{a_1,a_2}(\lambda - \mu) \tag{4}$$

Finally, show that by multiplying (4) with $R^{-1}$ from the right, we can write the RHS as, $\mathrm{tr}_{V_2} T_{N,a_2} \mathrm{tr}_{V_1} T_{N,a_1}$. Find also an expression for the LHS. Defining, $F(\lambda) = \mathrm{tr}_V T_{N,a}$, where the trace is taken over the auxilary space, use the previous results to show that $[F(\lambda), F(\mu)] = 0$.

Since $F(\lambda)$ is a polynomial in $\lambda$, this shows that the coefficients of this polynomial, which are operators on the Hilbert space of the chain, are all in involution. Furthermore, recall that in the lectures we showed that one of these operators is the Hamiltonian. Hence we have proved this system is integrable.