Intergrable Models **Exercises No. 2**

Exercise 2.1: A choice of coordinates (Q_i, P_i) on phase space are known as canonical if they satisfy the relations,

$$\{Q_i, Q_j\} = 0, \qquad \{P_i, P_j\} = 0, \qquad \{Q_i, P_j\} = \delta_{ij}, \tag{1}$$

where $\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$. Show that the expression for the Poisson bracket may also be written as,

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial Q_i}\frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial Q_i}\frac{\partial f}{\partial P_i}. \tag{2}$$

Exercise 2.1: Consider the generalisation of the Hamiltonian formalism necessary to describe a continuum system. The canonical Poisson bracket for the KdV system is given by,

$$\{u(x), u(y)\}_2 = \partial_x \delta(x - y). \tag{3}$$

Show that this Poisson bracket satisfies the following properties,

(a) $\{f, g\} = -\{g, f\}$,

(b) $\{fh, g\} = \{f, g\}h + f\{h, g\}$,

(c) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

Prove property (a) and (b) for the Poisson bracket given by,

$$\{u(x), u(y)\}_1 = \left[\partial_x^3 + \frac{1}{3}(\partial_x u(x) + u(x)\partial_x)\right]\delta(x - y). \tag{4}$$

Exercise 2.3: Consider the Hamiltonians,

$$H_1 = \int_{-\infty}^{\infty} (\frac{1}{3!}u(x)^3 - \frac{1}{2}(\partial_x u(x))^2), \tag{5}$$

$$H_2 = \int_{-\infty}^{\infty} \frac{1}{2}u(x)^2. \tag{6}$$

Show, using the explicit form for the brackets, that

(a) $\{H_1, H_2\}_2 = 0$,

(b) $\{H_i, H_j\}_1 = 0$ where i and j take values 0, 1 and 2.

Exercise 2.4 (Bonus): Consider imposing periodic boundary conditions on $u(x, t)$ such that we can expand it in a Fourier series,

$$u(x, t) = \sum_{n=-\infty}^{\infty} u_n(t)e^{inx} + \beta. \tag{7}$$

Find the Poisson brackets of the u_n with each other using the bracket $\{u(x), u(y)\}_1$. Compare this with the Virasoro algebra,

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0}cn(n^2 - 1) \tag{8}$$

where c is a constant. Can β be chosen so that they match (up to rescalings)?