Intergrable Models **Exercises No. 3**

Exercise 3.1: Explain how a pair of quantities $(\rho(u(x)), j(u(x)))$ satisfying the continuity equation,

$$\partial_t \rho = \partial_x j, \tag{1}$$

give rise to a conserved quantity. Show that the KdV equation may be written in the form of a conservation law in which $\rho(u(x)) = \frac{1}{2}u(x)^2$.

Exercise 3.2: The change of variable,

$$u = v^2 + i\sqrt{6}\partial_x v, \tag{2}$$

is called the Riccati transformation. Show that if v satisfies the Modified KdV (MKdV),

$$\partial_t v = v^2 \partial_x v + \partial_x^3 v, \tag{3}$$

then u satisfies the KdV equation. Consider the converse; why might it not be true that if u satisfies the KdV, then v satisfies the MKdV?

Exercise 3.3: Consider the equation obtained from the KdV equation via the transform $u = \frac{1}{6}\epsilon^2 v^2 + v + i\epsilon\partial_x v$,

$$\partial_t v = \left(\frac{\epsilon^2}{6}v^2 + v\right)\partial_x v + \partial_x^3 v. \tag{4}$$

Furthermore, define $v_n(x)$ by $v = \sum_{n=0}^{\infty} \epsilon^n v_n(x)$.

(a) Splitting $v(x)$ into real and imaginary parts; $v(x) = y(x) + iz(x)$, show, by considering the relations between u and v, that $z(x)$ contains only odd powers of ϵ all of which have a coefficient which is a total derivative.

(b) Deduce the scaling dimension of v_n, v and ϵ.

Exercise 3.4 : Using the recursion relation between the conserved quantities H_n, we were able to show $\{H_n, H_m\}_1 = \{H_{n-1}, H_{m+1}\}_1$.

(a) Explain why $\{H_n, H_m\}_1 = \{H_{n-1}, H_{m+1}\}_1$ implies that $\{H_n, H_m\}_1 = 0$.

(b) Show that $\{H_n, H_m\}_2 = 0$ for all n and m.