Intergrable Models **Exercises No. 4**

Exercise 4.1: Beginning with the expression for u in terms of the Schrodinger wave function ψ,

$$u(x,t) = -6\left(\lambda + \frac{\partial_x^2 \psi}{\psi}\right), \tag{1}$$

show that the KdV equation may be written as,

$$\psi \partial_t \lambda - \partial_x \left[\psi^2 \partial_x \left(\frac{\partial_x^3 \psi - \partial_t \psi + (\frac{1}{2}u - 3\lambda)\partial_x \psi}{\psi}\right)\right] = 0. \tag{2}$$

Explain why this implies λ is independent of t.

Exercise 4.2: Consider the Gelfand-Levitan equation,

$$K(x,y) + B(x+y) + \int_x^\infty K(x,z)B(y+z) = 0 \tag{3}$$

where $B(x+y) = \sum_{n=0}^N c_n e^{-\kappa_n x} + \frac{1}{2\pi}\int_{-\infty}^\infty dk R(k) e^{ikx}$, $R(k)$ is the reflection coefficient and κ_n are the bound state energies for the bound states for the potential $u(x,0)$. Suppose $B(x+z)$ can be written in the form $B(x+z) = \sum_{n=1}^\infty X_n(x) Z_n(z)$, then

(a) show that there exists $L_n(x)$ such that $K(x,z) = \sum_{n=1}^\infty L_n(x) Z_n(z)$ and provide an expression for $L_n(x)$.

(b) Define $a_{nm}(x) = \int_x^\infty dy X_n(y) Z_m(y)$. Show that the solution to the Gelfand-Levitan equation may be written as,

$$K(x,y) = \sum_n \sum_m Z_n(y) A^{-1}_{nm} X_m(x) \tag{4}$$

where $A_{nm} = \mathbb{I}_{nm} + a_{nm}$.

(c) An initial condition $u(x,0)$ is called a reflectionless potential if $R(k) = 0$ for all k. In this case define $X_n(x) = c_n e^{-\kappa_n x}$ and $Z_n(x) = e^{-\kappa_n x}$ and show that (hint: use $\operatorname{tr}\log A = \log\det A$) ,

$$K(x,x) = \frac{d}{dx}\log\det A. \tag{5}$$

Hence give an expression for $u(x,t)$.

(d) Use the above expression to rederive the one-soliton solution of the KdV equation (hint: recall that the initial condition for the one-soliton solution has only one bound state. Does this method of solution generalise to other reflectionless potentials?

Exercise 4.3 :

(a) Give the definition of phase and group velocity.

(b) Show that the equation $\partial_t u = \partial_x^3 u$ has a non-trivial dispersion relation.

(c) Given an argument for why the phase velocity and group velocity of the soliton are the same.