Intergrable Models	**Exercises No. 5**

Exercise 5.1: Consider an eigenfunction $\phi(x)$ of the Hamiltonian $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}U(x)$, with eigenvalue λ. Define the operators

$$Af = \frac{1}{\sqrt{2}}\phi(x)\partial\left[\phi^{-1}(x)f\right], \quad (1)$$

$$A^+f = -\frac{1}{\sqrt{2}}\phi^{-1}(x)\partial\left[\phi(x)f\right]. \quad (2)$$

where $\phi^{-1}(x)$ denotes the reciprocal of $\phi(x)$. Show that,

(a) $A^+A = H - \lambda\mathbb{I}$

(b) $AA^+ = \tilde{H} - \lambda\mathbb{I}$, where $\tilde{H} = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\tilde{U}(x)$ and give an expression for $\tilde{U}$.

(c) Show that if $H_+\psi \equiv (H - \lambda\mathbb{I})\psi = \epsilon\psi$ then $H_-A\psi \equiv (\tilde{H} - \lambda\mathbb{I})A\psi = \epsilon A\psi$, and explain what it means for two operators to be isospectral.

Exercise 5.2: Show that $(H - \lambda\mathbb{I})\psi = 0$ implies $A\psi = 0$ or $A\psi$ is not normalisable and hence that $(\tilde{H} - \lambda\mathbb{I})$ does not have a nontrivial eigenfunction in correspondence with ψ.

Exercise 5.3 : Given the KdV equation $\partial_t u = u\partial_x u + \partial_x^3 u$ we can obtain the the mKdV equation,

$$\partial_t v = (v^2 - 6\lambda)\partial_x v + \partial_x^3 v, \quad (3)$$

via the Riccati transformation, $u + 6\lambda = v^2 + i\sqrt{6}\partial_x v$.

(a) Show that the Riccati transformation can be transformed into a Schrodinger equation by setting

$$v = i\sqrt{6}\frac{\partial_x\psi}{\psi}. \quad (4)$$

(b) Solve (4) for ψ and consider $\partial_t\psi$. Show that by using the mKdV equation for v, the relation (4) and the result of question (a), that $\partial_t\psi$ satisifies,

$$\partial_t\psi + \frac{1}{6}\psi\partial_x u + 4\lambda\partial_x\psi - \frac{1}{3}u\partial_x\psi = const.\psi \quad (5)$$