Intergrable Models **Exercises No. 6**

Exercise 6.1: Recall that for the KdV equation, the scaling dimensions of the variables are $[x]=1$, $[u]=-2$, $[t]=3$.

(a) Deduce the scaling dimension of the Lax operators $L=\partial_x^2+\frac{1}{6}u$ and B and explain why $[L,B]$ must be a multiplicative operator.

(b) Give the most general operator that satisfies the scaling dimension and symmetries of B.

(c) Show that,

$$[\partial_x^n,u]=\sum_{k=1}^{n}\binom{n}{k}\partial^k u\partial^{n-k},\quad \left[\partial_x^2,u\partial_x\right]=\partial_x^2 u\partial_x+2(\partial_x u)\partial_x^2,\quad [u\partial_x,u]=u(\partial_x u) \tag{1}$$

(d) Using the results in (c), compute the commutator of the general operator from part (b) with L. By ensuring $[L,B]$ is a multiplicative operator, deduce an expression for B.

(e) Recall that the nth KdV equation has the form $\partial_t u=\partial_x\left(\frac{\delta H_{n+1}}{\delta u(x)}\right)$. Deduce the general form of the B operator which will give the next equation (after the KdV) in the KdV hierarchy.

Exercise 6.2: Define the Pauli matrices,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2}$$

Let $\sigma_\pm=\frac{1}{2}(\sigma_1\pm i\sigma_2)$ and show,

$$\sigma_\pm^2=0,\quad \sigma_3^2=I,\quad \sigma_\pm\sigma_3=\mp\sigma_\pm,\quad \sigma_\pm\sigma_\mp=\frac{1}{2}(I\pm\sigma_3) \tag{3}$$

Exercise 6.3 : Recall the first order formalism for the Lax pairs. We replace the Schrodinger wave-function by $\Phi=(\phi_1,\phi_2)$, and we make the ansatz,

$$L=(\sigma_3\partial_x-q(x,t)\sigma_++r(x,t)\sigma_-)\,,\quad B=(P(x,t)\sigma_++Q(x,t)\sigma_-+R(x,t)\sigma_3) \tag{4}$$

so that the Lax equations become,

$$L\Phi=-i\zeta\Phi,\quad \partial_t\Phi=B\Phi. \tag{5}$$

By considering $\partial_x\partial_t\Phi=\partial_t\partial_x\Phi$, show that for the Lax equations to be consistent we require,

$$\begin{aligned}\partial_x R &= qQ-rP & (6)\\ \partial_t r &= Q_x-2rR-2i\zeta Q & (7)\\ \partial_t q &= P_x+2qR+2i\zeta P. & (8)\end{aligned}$$