Intergrable Models **Exercises No. 7**

Exercise 7.1: Show that the Lax pair,

$$L = \partial_x + \frac{1}{36}u\sigma_+ - 6\sigma_-, \qquad B = \left(-\frac{u^2}{108} - \frac{1}{36}(\partial_x^2 u)\right)\sigma_+ + 2u\sigma_- + \frac{1}{6}\partial_x u\sigma_3, \quad (1)$$

give the KdV equation.

Exercise 7.2: Show that the mKdV equation may be obtained from the 1st order consistency conditions by making the choice,

$$r = q = \frac{-i}{\sqrt{6}}v(x,t) \quad (2)$$

$$R = -i\zeta\left(-4\zeta^2 + \frac{1}{3}v^2(x,t)\right). \quad (3)$$

Exercise 7.3: Consider the non-linear Schrodinger equation,

$$i\partial_t\psi = -\partial_x^2\psi + 2\kappa|\psi|^2\psi. \quad (4)$$

It can be obtained as the equation of motion for the Hamiltonian system,

$$H = \int_{-\infty}^{\infty} dx(\partial_x\psi)(\partial_x\psi^*) + \kappa|\psi|^4, \quad (5)$$

$$\{\psi(x), \psi^*(y)\}_2 = -i\delta(x-y), \qquad \{\psi(x), \psi(y)\}_2 = \{\psi^*(x), \psi^*(y)\}_2 = 0 \quad (6)$$

(a) Show that $H_2 = \int_{-\infty}^{\infty} \psi^*\partial_x\psi - \psi\partial_x\psi^*$ is conserved.

(b) Show that $\{,\}_2$ satisfies the Poisson bracket axioms.

(c) Show that the second Poisson bracket defined by,

$$\{\psi(x), \psi^*(y)\}_1 = -i(\partial_x^2 + 2\kappa|\psi|^2)\delta(x-y) \quad (7)$$

$$\{\psi(x), \psi(y)\}_1 = \{\psi^*(x), \psi^*(y)\}_1 = 0 \quad (8)$$

satisfies the necessary anti-symmetry property.

(d) Show that $\{H_2, H_1\}_1 = 0$.

Exercise 7.4: Recall the derivation of the non-linear Schrodinger equation and Sin-Gordon equation using the first order formalism. In both cases we did not use the consistency condition $q_t = P_x + 2qR + 2i\zeta P$. Show that this consistency condition is consistent with the choice,

(a) for the NLSE: $r = \sqrt{\kappa}\psi$, $q = \sqrt{\kappa}\psi^*$, $R = 2i\zeta^2 + i\kappa|\psi|^2$.

(a) for the Sin-Gordon: $r = -q = \frac{1}{2}\psi_x$, $P = Q = \frac{i}{4\zeta}\sin\psi$.