Intergrable Models	**Exercises No. 8**

Exercise 8.1 Recall the definition of the Wronskian in the first order formulation,

$$W(\omega, \phi) = \omega^T i\sigma_2 \phi = \omega_1\phi_2 - \omega_2\phi_1, \tag{1}$$

and also the definition of the Jost function coefficients a, b, $\tilde{a}$ and $\tilde{b}$,

$$f(x,\zeta) = a(\zeta)\tilde{g}(x,\zeta) + b(\zeta)g(x,\zeta), \qquad \tilde{f}(x,\zeta) = -\tilde{a}(\zeta)g(x,\zeta) + \tilde{b}(\zeta)\tilde{g}(x,\zeta). \tag{2}$$

Show that,

$$a(x,\zeta) = W(f,g), \qquad b(x,\zeta) = -W(f,\tilde{g}) \tag{3}$$

$$\tilde{a}(x,\zeta) = W(\tilde{f},\tilde{g}), \qquad \tilde{b}(x,\zeta) = W(\tilde{f},g) \tag{4}$$

Exercise 8.2: For a suitable choice of κ the non-linear Schrodinger equation is,

$$i\psi_t + \psi_x x + |\psi|^2\psi = 0. \tag{5}$$

(a) By making the ansatz $\psi = r(x-ct)\exp\left[i(\theta(x-ct) + \phi t)\right]$, where $r(\sigma)$ and $\theta(\sigma)$ are arbitrary functions and ϕ is a constant, show that if ψ solves (5) then r and θ satisfy,

$$\theta' = \frac{c}{2}(1 + As^{-1}), \tag{6}$$

$$(s')^2 = -2\left[s^3 - 2\left(\phi - \frac{1}{2}c^2\right)s^2 + Bs - A^2\right], \tag{7}$$

where A and B are arbitrary constants and $s = r^2$.

(b) Find a choice of A and B so that you reproduce the 1-soliton solution,

$$\psi = \eta\,\mathrm{sech}\left(\eta\frac{x-ct}{\sqrt{2}}\right)e^{i\left(\frac{1}{2}c(x-ct)+\phi t\right)}, \tag{8}$$

where $\eta^2 = 2(\phi - c^2/2)$.

Exercise 8.3: Given that under a gauge transformation, U, where U is an element of the gauge group,

$$\phi \to U\psi, \tag{9}$$

$$A_\mu \to UA_\mu U^{-1} - i(\partial_\mu U)U^{-1}, \tag{10}$$

show that $\mathcal{D}_\mu\psi \to U\mathcal{D}_\mu\psi$, where $\mathcal{D}_\mu \equiv \partial_\mu - iA_\mu$.

(a) Show that $[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi = -iF_{\mu\nu}\psi$ and give an expression for $F_{\mu\nu}$.

(b) Hence show that the trace of the field strength tensor is gauge invariant.