Intergrable Models	**Exercises No. 9**

Exercise 9.1: The Baker-Campbell-Hausdorff (BCH) theorem states,

$$e^{A}Be^{-A}=\sum_{n=0}^{\infty}\frac{1}{n!}B_n \tag{1}$$

where $B_n=[A,B_{n-1}]$ and $B_0=B$.

(a) Define $C(\tau)=e^{\tau A}Be^{-\tau A}$, and by computing the Taylor expansion of C prove the BCH theorem.

(b) Consider $C(\tau)=e^{-\tau A}e^{-\tau B}e^{\tau(A+B)}$. Show that,

$$\partial_\tau C=-\left(B-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\tau)^{n+m}}{n!m!}B_{n,m}\right)C \tag{2}$$

where $B_{n,m}=(B_n)_m$. Using this result, compare the Taylor expansion of $C(\tau)$ with the ansatz $C(\tau)=e^{\tau D_1}e^{\tau^2 D_2/2}e^{\tau^3 D_3/3!}\ldots$ and find an expression for D_1, D_2 and D_3.

Exercise 9.2: Consider the Wilson line defined by,

$$W_C(z(s),y)=P\left[\exp\left(i\int_0^s ds\frac{dx^\mu}{ds}A_\mu(x(s))\right)\right], \tag{3}$$

where C is a path form y to z and P denotes the matrices are path-ordered.

(a) By differentiating the expression for the Wilson line show that is satisfies the equation,

$$\frac{dx^\mu}{ds}\mathcal{D}_\mu W_C(z(s),y)=0, \tag{4}$$

where the covariant derivative $\mathcal{D}_\mu\equiv\partial_\mu-iA_\mu$ acts on the variable z.

(b) By multiplying the left side of (4) by $U(z)$ and the right by $U^{-1}(y)$ and using the fact that the solution of a first order differential equation is unique show that the Wilson line transforms under a gauge transformation as,

$$W_C(x,y)\to U(x)W_C(x,y)U^{-1}(y). \tag{5}$$

Hence deduce that the trace of a Wilson loop is gauge invariant.

Exercise 9.3: By considering the expansion of the exponential in the expression for W_C, show that,

(a) $W_C(y,x)=1+\int_x^y dw^\mu A_\mu+\int_x^y dw^\mu\int_x^w dz^\nu A_\mu A_\nu+\ldots.$

(b) Consider the Wilson lines $W_{C_1}(y,x)$ and $W_{C_2}(z,y)$, where C_1 and C_2 are two curves smoothly joined at y. Show, to second order, that $W_{C_1+C_2}(z,x)=W_{C_2}(z,y)W_{C_1}(y,x)$.

(c) Explain in general why $P[A]P[B]\neq P[AB]$, where P is the path-ordering operator.