Introduction to Random Matrix Theory (RMT)

[IRTG 2235 – research area C]

· Winter term 2018/19 by Gernot Akemann

· Lectures Mondays 10^{15} – 11^{45} in D5-153
 Exercises Thursdays 15^{15} – 16^{00} in D01-295

Tutor: Tim R. Würfel

Contact GA: akemann@physik.uni-bielefeld.de, E5-129

· → based on the following literature:

these · Lecture notes (handwritten)

→ Mathematical Physics webpage → teaching → Intro to RMT
cf. notes from 2011 (Fleckberner)

· Introduction to Random Matrices by Livan, Novaes, Vivo (LNV)
Springer Lecture notes, 2018

available on arXive 1712.07903 ← refer to eqs. & figs
+ further literature tba

· Criteria for successful participation – appart from attending lectures

– oral exam after the end of semester

– exercises: active participation, showing
≥ 2 solutions of questions

the homework questions will be distributed during lecture

dates: 22.10. & 3.12. Tim will replace me

1. What is a random matrix and what are possible applications?

(Getting started)

1.1. setting up notation: random variables in 1 dim:

- a <u>random variable</u> (r.v.) X can take values from
  - a <u>discrete</u> set, e.g. $\in\{+1,-1\}$ Bernoulli, with equal prob.
    or dice $\in\{1,2,3,4,5,6\}$, ——, —
  - an <u>interval</u> $\sigma \subseteq \mathbb{R}$, with $g(x)$ its probability density function (pdf) s.th. $\int_a^b dx\, g(x)$ = probab. that $X \in [a,b] \subseteq \sigma$

- such pdf should be <u>normalized</u> $\int_\sigma dx\, g(x) = 1$
- <u>Average</u> (or <u>mean</u>) of X: $\langle X\rangle = \int_\sigma dx\, g(x)\, x$ = <u>1st moment</u>, kth $\langle X^k\rangle = \int_\sigma dx\, g(x)\, x^k$

→ <u>Variance</u> $Var(X) = \langle (X-\langle X\rangle)^2\rangle = \langle X^2\rangle - \langle X\rangle^2$, if $\langle X\rangle = 0$ centred ⇒ $Var X = \langle X^2\rangle$

[e.g. Gauß: $g(x) = \sqrt{\frac{a}{\pi}}\exp(-ax^2)$, $\langle X\rangle = 0$, $Var X = \langle X^2\rangle = \frac{1}{2a}$: $\mathcal{N}(0, \frac{1}{2a})$]

- <u>cumulative distribution</u> $F(x) = \int_{-\infty}^{x} dx\, g(x)$, $\begin{cases} \to 0 & x\to-\infty \\ \to 1 & x\to+\infty\end{cases}$

when doing numerics often <u>more useful</u> than doing histograms as this is free of the choice of a binning!

- for $n \geq 2$ random variables $X_1, \dots, X_n$ these are described by a <u>joint pdf (jpdf)</u> $g(x_1, \dots x_n)$.

⇒ they are <u>independent</u> iff $g(x_1,\dots,x_n) = \prod_{i=1}^{n} g(x_i)$, where

<u>probab.</u> $g(x_1,x_2,\dots,x_n)\, dx_1 \dots dx_n$ to find X_1 in $[x_1, x_1+dx_1]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 2 2 2 . .
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ n n n

<u>marginal pdf</u> $g(x_1) = \int dx_2 \dots dx_n\, g(x_1, \dots, x_n)$; $g(x_1)\,dx_1$ prob. that $X_1 \in [x_1, x_1+dx_1]$, indep. of all others.

$\Rightarrow$ change of variables: $x_i = x_i(\vec{y})$, $\vec{y} = (y_1, \ldots, y_n)$

$$\varrho(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \varrho(x_1(\vec{y}), \ldots, x_n(\vec{y}))\, |J(\vec{x}, \vec{y})|\, dy_1 \ldots dy_n$$

with Jacobian $J(\vec{x}, \vec{y}) = \det\left[\frac{\partial x_i}{\partial y_j}\right]_{i,j=1}^{n}$

1.2. What is a random matrix?

- the Gaussian Orthogonal Ensemble (GOE)

  take H a matrix $N \times N$, fill in entries H_{ij}, $\forall i,j = 1, \ldots, N$ that are $\in \mathcal{N}(0,1)$

  = normal Gauß' rand. var. with mean $\langle X \rangle = 0$ and $\operatorname{Var} X = 1$

  $\Rightarrow H \neq H^T$ in general, eq. (1.1), symmetrise $H_s = \frac{1}{2}(H + H^T)$ eq. (1.2)

  $\Rightarrow$ lin. algebra: H_s can be diagonalised with an

  orthogonal trafo $H_s = O \Lambda O^T$, $\Lambda = \operatorname{diag}(x_1, \ldots, x_N)$

  with $x_{1,\ldots,N}$ its real eigenvalues (ev), see fig. 1.1 for histogram of ev of many such H_s

Note: we will not consider RMT with complex eigenvalues in this lecture ($\to$ Ma phys, teaching: Topics in RMT), that is the above H

* constructing one such H_s gives one member of the ensemble:

$H_s \in$ GOE (below: prob. measure on the space of sym. matrices)

similarly construct GUE (unitary) from H_{ij} filling in Gauß' real & imag. parts,

with $H_{her} = \frac{1}{2}(H + H^\dagger)$ $\quad H^\dagger = \bar{H}^T$

- distribution of matrix elements

  indep. Gauß r.v. $\Rightarrow$ $$\boxed{\varrho(H) = \varrho(H_{11}, H_{12}, \ldots, H_{NN}) = \prod_{i,j=1}^{N} e^{-\frac{H_{ij}^2}{2}} \frac{1}{\sqrt{2\pi}}}$$

- the ev of $H \neq H^T$ are real or come in complex conjugate pairs

  as $\det(\lambda \mathbb{1}_N - H) = 0$, the characteristic equation, is an algebraic eq. of degree N with real coeff. This is ensemble is called real Ginibre ensemble (J. Ginibre JMP 1965) and was only solved in 2007.

* in the GOE $H_s = \frac{1}{2}(H + H^T)$, so only the N diagonal and $\frac{N(N-1)}{2}$ (lower or) upper triangular matrix elements $(H_s)_{ij}$ with $i<j$ are independent ($H_{sij} = H_{sji}$)

Exercise (more gen): the sum of 2 Gauss rv of equal variance is Gauss, with var. $\frac{1}{2}$ (or $a \to 2a$) (not true for product!)

$\Rightarrow (H_s)_{ij} = \frac{1}{2}(H_{ij} + H_{ij}^T)$ with $i<j$ are distributed according to $e^{-(H_s)_{ij}^2}/\sqrt{\pi}$ (no $\frac{1}{2}$)

$$\Rightarrow \underline{g(H_s)} = g(\underbrace{H_{s11}, H_{s12}, \ldots, H_{sNN}}_{N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2} \text{ real elements}}) = \prod_{i=1}^{N} \frac{e^{-\frac{1}{2}H_{sii}^2}}{\sqrt{2\pi}} \prod_{i<j}^{N} \frac{e^{-(H_s)_{ij}^2}}{\sqrt{\pi}} \quad (1.7)$$

diag and off diag have different variance!

$$= \frac{1}{2^{\frac{N}{2}} \pi^{\frac{N(N+1)}{4}}} \exp\Big[-\frac{1}{2}\sum_{i=1}^{N} (H_s)_{ii}^2 - \underbrace{\sum_{i<j} (H_s)_{ij}^2}_{\frac{1}{2}\sum_{i\neq j}} \Big]$$

$(H_s)_{ij} = (H_s)_{ji}$

* the exponent can be written as a trace:

$$\underline{\frac{1}{2}\sum_{i,j=1}^{N} (H_s)_{ij}^2} = \frac{1}{2}\sum_{i,j=1}^{N} (H_s)_{ij}(H_s)_{ji} = \underline{\frac{1}{2}\operatorname{Tr}[(H_s)^2]}$$

cf. for real Ginibre $g(H) \sim e^{-\operatorname{Tr} HH^T}$

which is invariant under diagonalisation, only depends on ev!

- The big goal of this course is to understand the statistics of ev of ensembles such as the GOE, GUE etc.

  → In contrast to the matrix elements the ev $x_1, \ldots, x_N$ of H_s will not be independent rv:

$$\boxed{g(x_1, x_2, \ldots, x_N) = \frac{1}{Z_{N,\beta}} \prod_{j<k}^{N} |x_k - x_j|^{\beta} \, e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2}} \quad (2.15)$$

← from $\operatorname{Tr} H_s^2$

↑ jacobian from $H_s = O \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} O^T$

as will be shown later, with

with normalisation $Z_{N\beta} = (2\pi)^{\frac{N}{2}} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1+\beta/2)}$

* this holds for selfadjoint $N \times N$ matrices Hs with its indep. matrix elements $\in \mathbb{R}\ (\beta=1)$, $\mathbb{C}\ (\beta=2)$, $\mathbb{H}\ (\beta=4)$.

So β counts the number of indep. real degrees of freedom per H_{ij}

• (2.15) makes sense for any $\beta>0$, $\exists$ tridiag matrix rep for those as well: β- ensembles [Dumitriu, Edelman JMP, 2002]

• in physics: partition function $Z = \sum_{\text{states}} e^{-\beta\mathcal{H}}$ $\quad \beta = \frac{1}{k_B T}$ inverse temperature, $\mathcal{H}$ Hamiltonian

so $\prod_{i<j} |x_j - x_i|^{\beta} = \exp[+\frac{\beta}{2}\sum_{i\neq j}^{N} \ln|x_j - x_i|]$

ev repell each other, like Coulomb interaction in 2dim

and they are confined by a harmonic potential $\exp[-\frac{1}{2}\sum_{i=1}^{N} x_i^2]$

$\Rightarrow$ behave like in fig 1.1 (Coulomb gas picture, more later)

• a popular quantity: level spacing distribution $P(s)$ quantifying the repulsion between adjacent ev

$\rightarrow$ for an $N=2$ GOE we get $\boxed{P_{GOE}(s) = \frac{s}{2} e^{-s^2/4}}$ $\rightarrow$ exercise

"Wigner surmise", for indep. identically distributed $s_1, \dots, s_N$ $\boxed{P_{iid}(s) = e^{-s} \text{ Poisson}}$

* surprise 1: this formula is an excellent approximation to the $P_{GOE}(s)$ for $N \gg 1$ large random matrices $\rightarrow$ next lect.

* surprise 2: $P_{GOE}(s)$ can be found in many real data: nuclear physics, quantum chaos, quantum field theory, Riemann ζ function, random graphs, ...

Statistical mechanics: T temperature, p pressure are scalar quantities, quantifying the behaviour for a large (10^{23}) number of particles

RMT : statistical approach to describe spectral properties (quantitatively) for operators / matrices / system with a large number of degrees of freedom (dof)

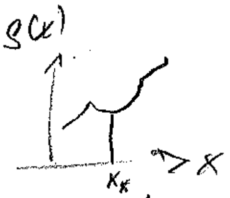
"many particle, strongly interacting, chaotic"

→ figures in the introduction of Guhr, Müller-Groeling, Weidenmüller

cond-mat/9707301 Phys. Rep. 1998 [*]

and M.L. Mehta Random Matrices, 3rd Ed. Elsevier 2004

* remark to the comparison with data:



- based on these one can define a $\rho_{data}(x)$
- it is easy to see that locally the average spacing $\sim \frac{1}{\rho(x_*)}$ at x_*

⇒ to compare with RMT we have to normalise this to unity = unfolding, cf. see p 30, [*]

* after fixing $\int_0^\infty ds\, p(s) = 1$, the normalisation and the 1st moment $\int_0^\infty ds\, p(s)\, s = 1$, the RMT prediction is parameter free!! (→ exercise)