9. The GOE & GSE [chapter 12, LNV book

for jpdfs $\sim |\Delta_N(\{x\})|^\beta$ with $\beta = 1$ and $\beta = 4$ (GOE and GSE)

a modification of the OP approach can be used to compute all k-point correlation functions

OP: $\perp$ wrt scalar product $\langle f, g\rangle_2 = \int f(x) g(x) w(x)\, dx$ $\beta = 2$

for $\beta = 1, 4$ we introduce polynomials skew-orthogonal wrt 2 different antisymmetric products ($\neq$ scalar non pos. def.):

$$\langle f, g\rangle_4 = \int [f(x) g'(x) - f'(x) g(x)]\, w(x)\, dx = -\langle g, f\rangle_4 \quad \text{for } f, g \in \mathbb{R}$$

$$\langle f, g\rangle_1 = \iint f(x) g(y)\, \varepsilon(x-y)\, w(x) w(y)\, dx\, dy = -\langle g, f\rangle_1$$

$$\varepsilon(x) = \begin{cases} +\frac{1}{2} & x > 0 \\ -\frac{1}{2} & x < 0 \end{cases}$$

more details see Mehta

Appearance of the scalar product:

$\beta=4$: replace $\Delta_N(\{x\})^4$ in jpdf. see Ex 8.2

$$\Delta_N^4 = \prod_{j>i}(x_j-x_i)^4 = \det_{\substack{1\le i\le N\\ 0\le j\le 2N-1}}\left[\,x_i^{j}\quad j x_i^{j-1}\,\right] = \det_{\substack{1\le i\le N\\ 0\le j\le 2N-1}}\left[Q_j(x_i)\quad Q_j'(x_i)\right]$$

use e. de Bruijn formula 1955 (we will prove a similar formula)

$$\boxed{\int dx_1 \ldots \int dx_N \det_{\substack{1\le i\le 2N\\ 1\le j\le N}}\left[\varphi_i(x_j)\quad \psi_i(x_j)\right] = N!\,\mathrm{Pf}\left[\int\left(\varphi_i(x)\psi_j(x)-\varphi_j(x)\psi_i(x)\right)dx\right]}$$

proof Chap. 5 The Oxford Handbook of RMT, by M. Adler

$$\Rightarrow Z_N^{(\beta=4)} = \tilde{C}_N^{(\beta)} \int dx_1 \ldots \int dx_N \prod_{i=1}^{N}\omega(x_i)\det\left[Q_j(x_i)\quad Q_j'(x_i)\right]$$

$$= \tilde{C}_N^{\beta}\prod_{j=0}^{N-1}h_j^{(\beta=4)}\; N!\,\mathrm{Pf}\left[\int \frac{Q_i(x)}{\sqrt{h_i^{(4)}}}\frac{Q_j'(x)}{\sqrt{h_j^{(4)}}} - Q_i'(x)\,Q_j(x)\,dx\right]$$

∃ version of Dyson's Thm for this

where ($N=2s$ even), A $N\times N$, $A=-A^T$

$$\mathrm{Pf}[A] = \frac{1}{(2^s s!)}\sum_{P\text{ all}}(-)^P A_{i_1 i_2}\cdots A_{i_{2s-1} i_{2s}} \qquad \text{Pfaffian det.}$$

$N!$ perm of N indices $\sigma\in S_N$ $A_{\sigma(1)\sigma(2)}\ldots$

$$= \sum_{\substack{P\text{ all perm}\\ \text{s.th } i_1<i_2,\ i_3<i_4 \ldots i_{2s-1}<i_{2s}}}(-)^P A_{i_1 i_2}\cdots A_{i_{2s-1} i_{2s}} \Rightarrow \det A = (\mathrm{Pf} A)^2$$

Ex

example 2×2: $\mathrm{Pf}A = A_{12}$

4×4: $\mathrm{Pf}A = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$

note in $\det A = \sum_\sigma (-)^\sigma A_{1\sigma(1)}\cdots A_{N\sigma(N)}$ every index appears twice, here only once

If we choose the $Q_j(x)$ s.th. they are skew orthogonal

$$\langle Q_{2j}, Q_{2k+1}\rangle_4 = -\langle Q_{2k+1}, Q_{2j}\rangle_4 = h_j^{(\beta=4)}\delta_{kj}$$

$$\langle Q_{2j}, Q_{2k}\rangle_4 = \langle Q_{2j+1}, Q_{2k+1}\rangle_4 = 0$$

we have that $\mathrm{pf}[\langle Q_i, Q_j\rangle] = 1$ and we can construct a kernel from them

example: [Mehta ch 19] Laguerre weight $\omega(x) = x^a e^{-x}$ on $\mathbb{R}_+$

$$Q_{2j+1}(x) = -L_{2j+1}^{(a)}(x) + L_{2j}^{(a)}(x) \Rightarrow Q_{2j+1}'(x) = L_{2j}^{(a)}(x)$$

$$Q_{2j}(x) = L_{2j}^{(a)}(x) - L_{2j-1}^{(a)}(x) - \frac{(2j+a-1)}{(2j-1)}Q_{2j-2}(x)$$

$$\Rightarrow Q_{2j}'(x) = -L_{2j-1}^{(a)}(x) - \frac{(2j+a-1)}{(2j-1)}Q_{2j-2}'(x)$$

$$h_j^{(4)} = \frac{\Gamma(2j+a+1)}{(2j)!}$$

- the skew orthogonal polynomials enjoy a determinantal rep:

$$Q_{2j}(x) = \langle \det(x-H)\rangle$$

$$Q_{2j+1}(x) = \langle \det(x-H)(\mathrm{Tr}(H) + x + \text{const.})\rangle$$

as the odd polynomials are only defined up + const. $\cdot\, Q_{2j}(x)$ (skew-product!)

* all k-point correlation functions can be expressed in terms of a Pfaffian of a 2x2 matrix valued kernel expressed in terms of Q_j's.

* universality proof more difficult, see Deift, Gioev

<u>$\beta = 1$</u> is analogous, except we need another integration formula

- replace $\det x_i^{j-1} = \det P_j(x_i)$, remove absolute value of $|\Delta_N|^1$ by ordering eigenvalues

$$Z_N^{(\beta=1)} = \tilde{C}_N^{(\beta=1)} \int dx_1 \ldots dx_N \, \omega(x_1) \ldots \omega(x_N) \, |\Delta_N(\{x\})|^1$$

$$= N! \, \tilde{C}_N^{(\beta=1)} \underset{x_1 \leq x_2 \leq \ldots \leq x_N}{\int dx_1 \ldots \int dx_N} \prod_{j=1}^N \omega(x_j) \, \det \left[P_j(x_i) \right]$$

1. de Bruijn formula:

$$\boxed{\int\limits_{x_1 \leq \ldots \leq x_N} dx_1 \ldots \int dx_N \, \det_{1 \leq i,j \leq N} \left[\varphi_i(x_j) \right] = \mathrm{Pf} \left[\iint 2\varepsilon(x-y) \varphi_i(x) \varphi_j(y) \, dx \, dy \right]}$$

$\Rightarrow$ skew-product for $\beta = 1$ $\langle f, g \rangle_1$, same construction of SOP $\langle p_{2j}, p_{2k+1} \rangle_1 = h_j^{(\beta=1)} \delta_{jk}$ other's 0,

same rep of P's terms of $\langle \det (\ldots) \rangle$

Gap-probabilities and individual eigenvalue distr.

- So far we have only studied density correlation functions, $R_{N,k}^{(\beta)}$, i.e. the probability to find an eigenvalue at x: $R_{N,k=1}^{(\beta)}(x)$, or the probability to find 1 ev at x and 1 at y, $R_{N,k=2}^{(\beta)}(x,y)$ etc
- Now we will consider the probability that an interval I (e.g. $I=(-s,s)$) is empty of eigenvalues, also called <u>gap-probability</u>. First $\beta=2$:

$$A_I = \prod_{j=1}^{N} \int_{\mathbb{R}-I} dx_j \; P_N^{(\beta=2)}(x_1,..,x_N)$$

include to make orthonormal

$$= \frac{1}{N! \prod_{j=0}^{N-1} h_j} \prod_{j=1}^{N} \underbrace{\int_{\mathbb{R}-I} dx_j\, w(x_j)}_{\displaystyle = \int_{\mathbb{R}} dx_j\, \underbrace{(1-\chi_I(x_j))\, w(x_j)}_{\tilde{w}(x_j)}} \det_{1\le i,j\le N}\left[\tilde{P}_{j-1}(x_i)\right]^2$$

char. funct on I

$$= \frac{1}{N!} \prod_{j=1}^{N} \int_{\mathbb{R}} dx_j \det_{1\le i,j\le N}\left[\varphi_j(x_i)\right]^2 , \quad \varphi_j(x) = \tilde{w}(x)^{\frac{1}{2}} \tilde{P}_{j-1}(x)$$

Andréief formula $\varphi=\psi$

$$\overset{\text{Andréief}}{=} \det_{1\le i,j\le N}\left[\int_{\mathbb{R}} dx\, \varphi_i(x)\varphi_j(x)\right] = \det_{1\le i,j\le N}\left[\int_{\mathbb{R}} dx\,(1-\chi_I(x))\, w(x)\, \tilde{P}_i(x) \tilde{P}_j(x)\right]$$

$$A_I = \det_{1\le i,j\le N}\left[\delta_{ij} - \int_I dx\, w(x)\, \tilde{P}_i(x)\tilde{P}_j(x)\right]$$

→ Fredholm determinant

Andréief formula (integral identity) → was exercise 8.1

Let $\varphi_i(x)$, $\psi_i(x)$, $i=1,2,\dots N$ be 2 sets of integrable functions

Then
$$\int dx_1 \dots \int dx_N \det_{1\le i,j\le N}\left[\varphi_i(x_j)\right] \det_{1\le i,j\le N}\left[\psi_i(x_j)\right] = N! \det_{1\le i,j\le N}\left[\int dx\, \varphi_i(x)\psi_j(x)\right]$$

Proof: • induction: $N=1$ is trivial

• " assumption: holds for N

$N+1$: expand both det's on l.h.s. w.r.t. column containing x_{N+1}:

$$= \int dx_1 \dots \int dx_N \int dx_{N+1} \sum_{i=1}^{N+1} (-)^{N+1+i} \varphi_i(x_{N+1}) \det_{\substack{j\neq N+1\\ k\neq i}}\left[\varphi_k(x_j)\right] \sum_{l=1}^{N+1} (-)^{N+1+l} \psi_l(x_{N+1}) \det_{\substack{m\neq N+1\\ n\neq l}}\left[\psi_n(x_m)\right]$$

N funct. $\varphi_1 \dots \cancel{\varphi_k} \dots \varphi_{N+1}$ — N funct.

induction assumpt. for N

$$= \int dx_{N+1}\, N! \sum_{i,l=1}^{N+1} (-)^{i+l} \varphi_i(x_{N+1}) \psi_l(x_{N+1}) \det_{\substack{k\neq i\\ n\neq l}}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$$

• for fixed i: $\displaystyle\sum_{l=1}^{N+1} (-)^{i+l} \int dx\, \varphi_i(x)\psi_l(x) \det_{\substack{k\neq i\\ n\neq l}}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$

is the Laplace expansion wrt row "i" of

of $\displaystyle\det_{1\le k,n\le N+1}\left[\int dx\, \varphi_k(x)\psi_n(x)\right]$

$\Rightarrow$ we sum over $(N+1)$ such expansions

$$= (N+1)\, N! \det_{1\le k,n\le N+1}\left[\int dx\, \varphi_k(x)\psi_n(x)\right] \qquad \blacksquare$$

$$A_I = \prod_{j=0}^{N} (\lambda - \lambda_j) \, , \quad \lambda_j \text{ eigenvalues of matrix}$$

$$M_{ij} = \int_I dx\, w(x)\, P_i(x)\, P_j(x)$$

remark:

The eigenvalues λ_j are the same as the solutions of the integral eq.

$$\int_I dy\, w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} K_N(x,y)\, f(y) = \lambda f(x) \qquad [\text{see Mehta}]$$

k-th gap probability and k-th eigenvalue: for $\beta = 1, 2, 4$

$$\boxed{E_k^{(\beta)}(s) \equiv \frac{N!}{(N-k)!\,k!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_k \int_s^{\infty} d\lambda_{k+1} \ldots d\lambda_N \, P_N^{(\beta)}(\lambda_1 \ldots \lambda_N) \quad \text{kth gap prob.}}$$

Example: $E_{k=0}(s) = \int_s^{\infty} d\lambda_1 \ldots d\lambda_N \, P_N^{\beta}$ prob. that all ev are $\geq s$, $E_{k=0}^{(\beta)}(s=-\infty) = 1$

general: E_k " " k ev $\leq s$, $N-k \geq s$

$$\boxed{P_k^{(\beta)}(s) \equiv \frac{k \cdot N!}{k!(N-k)!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_{k-1} \int_s^{+\infty} d\lambda_{k+1} \ldots d\lambda_N \, P_N^{(\beta)}(\lambda_1, \ldots \lambda_{k-1}, \lambda_k = s, \lambda_{k+1} \ldots \lambda_N)}$$

if we order $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

$P_k^{(\beta)}(s)$ is the probability to find the k-th eigenvalue at $\lambda_k = s$

- it holds $\boxed{\frac{\partial}{\partial s} E_0^{(\beta)}(s) = -P_1^{(\beta)}(s)}$

and

$$\frac{\partial}{\partial s} E_k^{(\beta)}(s) = \frac{N!}{(N-k)!} \frac{k}{k!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_{k-1} \int_s^{\infty} d\lambda_{k+1} \ldots d\lambda_N \, P_N^{(\beta)} \Big|_{\lambda_k = s}$$